HANDBOOK OF BALLISTICS

BY

C. CRANZ

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VOLUME I

EXTERIOR BALLISTICS

BEING A THEORETICAL EXAMINATION OF THE MOTION OF THE PROJECTILE FROM THE MUZZLE TO THE TARGET

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TRANSLATED FROM THE SECOND GERMAN EDITION

LONDON:
PUBLISHED BY HIS MAJESTY'S STATIONERY OFFICE.
To be purchased through any Bookseller or directly from
H.M. STATIONERY OFFICE at the following addresses:
IMPERIAL HOUSE, KINGSWAY, LONDON, W.C. 2, and
28, ABINGDON STREET, LONDON, S.W. 1;
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1, ST ANDREW'S CRESCENT, CARDIFF;
23, FORTH STREET, EDINBURGH;
or from E. PONSONBY, LTD., 116, GRAFTON STREET, DUBLIN.

1921

Price £1. 10s. Net

Ps. 4228. Wt. 35025/3832 (47) 750 5/21 C.U.P.

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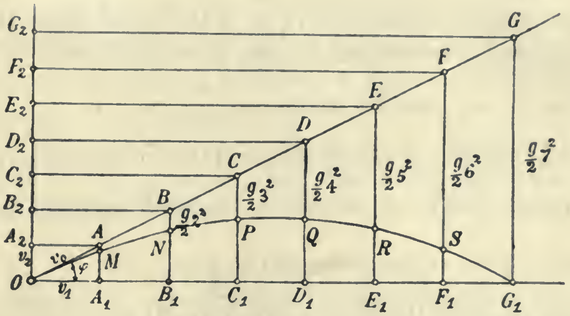
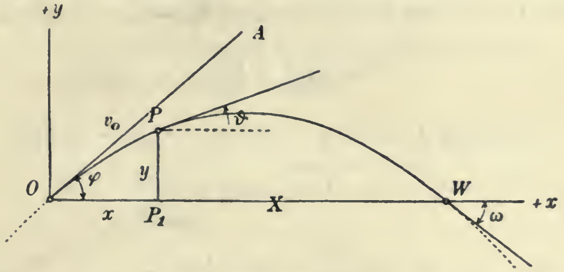
CHAPTER I

Motion of a projectile, neglecting the resistance of the air

§ **1.** The resistance of the air will be neglected in our preliminary investigation. Let the origin O be taken as the centre of the muzzle of the gun and the velocity of the shell at the point O be v_0, and let its direction make an angle ϕ with the horizon; ϕ is called the "angle of departure"; the vertical plane through the initial tangent is called the "plane of fire."



This plane of fire through O is taken as the plane of the coordinate axes; and the axes of x and y are the horizontal and vertical lines through O in this plane.



Suppose the centre of gravity of the body after t seconds, reckoned from O, to be at the point (xy), and to have a velocity v in the path in a direction making an angle θ with the horizontal. Then in the ascending branch of the trajectory, θ is positive, and diminishes to zero. At the vertex $\theta = 0$, while in the descending branch of the trajectory, θ is negative.

At the end of the trajectory, $\theta = \theta_e = 360 - \omega$, where ω is the "acute angle of descent"; and there $y = 0$, $x = X$, $v = v_e$, $t = T$.

Now $$\frac{d^2x}{dt^2} = 0, \text{ and } \frac{d^2y}{dt^2} = -g,$$

and initially $$\frac{dx}{dt} = v_0 \cos\phi, \; \frac{dy}{dt} = v_0 \sin\phi.$$

Therefore $$\left.\begin{aligned} x &= v_0 \cos\phi \, . \, t = v_1 t \\ y &= v_0 \sin\phi \, . \, t - \tfrac{1}{2}gt^2 = v_2 t - \tfrac{1}{2}gt^2 \end{aligned}\right\}, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

whence we get $$y = x \tan\phi - \frac{x^2}{4h \cos^2\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$\left(\text{where } h = \frac{v_0^2}{2g}\right)$; this is the equation of a parabola with vertical axis.

The vertex, with coordinates x_s, y_s, is the point where the tangent of the flight path is horizontal, so that y' or $\tan\theta = 0$.

Now $$\tan\theta = \tan\phi - \frac{x}{2h \cos^2\phi},$$

so that $x_s = 2h \cos\phi \sin\phi = h \sin 2\phi$; and thence from (2), $y_s = h \sin^2\phi$.

The average height y_m at which the shell is found is $\frac{1}{T}\int_0^T y\,dt$, or $\frac{1}{X}\int_0^X y\,dx$. Both values are equal to $\frac{2}{3}y_s$.

The total range of the shell is given by (2) as $X = 2h \sin 2\phi$.

The greatest range, for given initial velocity v_0 or given value of h, will then be obtained when $\sin 2\phi$ is greatest, that is when $\phi = \frac{1}{4}\pi$; this fact was verified approximately by Tartaglia from experiment.

The velocity v of the projectile after the time t is given by

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

$$= v_0^2 \cos^2\phi + (v_0 \sin\phi - gt)^2 = v_0^2 + g^2t^2 - 2v_2 gt;$$

and since $$y = v_2 t - \tfrac{1}{2}gt^2, \text{ and } h = \frac{v_0^2}{2g},$$

therefore $$v^2 = 2g\left(\frac{v_0^2}{2g} - v_2 t + \tfrac{1}{2}gt^2\right).$$

Therefore $$v^2 = 2g(h - y).$$

Thus the velocity of the body at the point (xy), or after the time t, is the same as if the body fell freely through the distance $h - y$.

The time of flight, that is, the time the body requires to reach the point (xy) is, according to (1), $t = \frac{x}{v_0 \cos \phi}$.

In particular, the time required to describe the horizontal range OW of the trajectory is

$$\frac{OW}{v_0 \cos \phi} = \frac{4h \sin \phi}{v_0} = \frac{2v_0 \sin \phi}{g} = T.$$

Substitute the value $v_0 \sin \phi = \frac{1}{2} gT$ in the second equation of (1), and it follows that

$$y = \tfrac{1}{2} gt (T - t).$$

When $t = \frac{1}{2} T$, then on account of the symmetry of the parabola about the vertex ordinate, y has the value y_s of the vertex height, and

$$y_s = \tfrac{1}{8} gT^2 = 1{\cdot}226 T^2 \quad (g = 9{\cdot}808 \text{ m/sec}^2).$$

This formula is frequently useful for motion in the air, and is called in Germany Haupt's formula, in England Sladen's formula; but it is nothing more than an approximation in actual practice.

Further relations can be obtained if we imagine a family of trajectories to be drawn, and study the common properties that connect the various trajectories.

§ 2. Family of trajectories for constant initial velocity.

An unlimited number of parabolic paths may lie in the same vertical plane, with the same point O as origin, and the same initial velocity. This series of curves is obtained when the angle of departure ϕ is made to assume a series of values.

First, let two parabolas of the series be taken, so related that both shall pass through the same point (xy).

We had before

$$y = x \tan \phi - \frac{x^2}{4h \cos^2 \phi}, \text{ and } \cos^2 \phi = \frac{1}{1 + \tan^2 \phi};$$

and with $\tan \phi = z$, we have

$$4hy + x^2 - 4hxz + x^2 z^2 = 0,$$

therefore $$z = \tan \phi = \frac{2h}{x} \pm \frac{1}{x} \sqrt{(4h^2 - 4hy - x^2)}. \quad \text{............(3)}$$

The double sign shows that the same target can be hit in two ways, with the same initial velocity v_0, and with the same value of $h = \frac{v_0^2}{2g}$. The two angles of departure, ϕ_1 and ϕ_2, are determined from (3); the one constitutes flat or direct fire, the other, curved or indirect fire.

A relation between ϕ_1 and ϕ_2 will be given later, but first we must consider more closely the two solutions of (3).

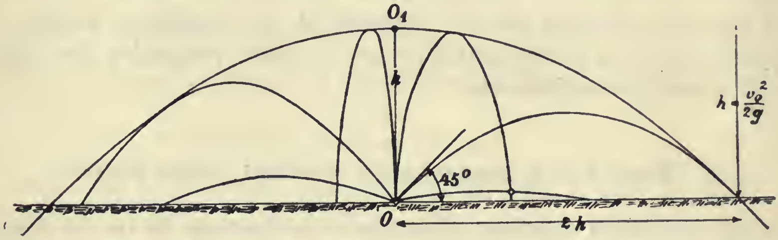
Obviously there are then two angles of fire ϕ, when the radical in (3) is real, that is, when $4h^2 > 4hy + x^2$, but when (xy) lies so that $4h^2 < 4hy + x^2$, there is no real angle of departure ϕ with which the point (xy) can be hit.

The complete plane thus splits into two parts; in the one part a point (xy) lies that can be hit in two ways; in the other part there is no point with this property.

The two parts are separated by the curve which has the equation

$$4h^2 = 4hy + x^2,$$

and this determines the locus of points in the plane that can be hit in one way only, and for which the direct and indirect fire coincide.



This curve is a parabola, with focus at O. Replace y by $h + y'$; then the equation of the curve becomes

$$4h^2 = 4h(h - y') + x^2, \text{ or } x^2 = 4hy',$$

and this proves that the curve is a parabola, with vertex at O_1, focus at O, and axis vertical in consequence.

Therefore this parabola

$$4h^2 = 4hy + x^2 \quad \text{.........................} (4)$$

represents the envelope of all the parabolic paths of the given family.

Take the equation of the parabola of flight (2) in the form

$$\frac{x^2}{\cos^2 \phi} - 4hx \tan \phi + 4hy = 0,$$

and differentiate with respect to ϕ; therefore

$$+\frac{2\cos\phi\sin\phi}{\cos^4\phi}x^2-\frac{4hx}{\cos^2\phi}=0;$$

or

$$\tan\phi=\frac{2h}{x};$$

hence we obtain

$$x^2\left(1+\frac{4h^2}{x^2}\right)-4h\frac{2h}{x}x+4hy=0,$$

or

$$x^2+4h^2-8h^2+4hy=0,$$

or

$$4h^2=4hy+x^2,$$

as before.

So far the treatment has been restricted to the movement in a vertical plane; but if we consider it in space with all possible angles of departure ϕ with same initial velocity v_0, then all the collective trajectories are enveloped by a paraboloid of revolution, with vertex at O_1 and focus at O.

Returning to the relations in one plane of fire, let us enquire as to what is the geometrical locus of the focus and vertex of all the parabolas of the family.

The original equation (2) of the parabola of flight

$$y=x\tan\phi-\frac{x^2}{4h\cos^2\phi}$$

can be written in the form

$$\left(x-\frac{v_1v_2}{g}\right)^2=-\frac{2v_1^2}{g}\left(y-\frac{v_2^2}{2g}\right),$$

where, as above, $v_1=v_0\cos\phi$, $v_2=v_0\sin\phi$.

From this form of the equation, the locus of the directrix of the parabola can be determined immediately.

For, since the double parameter of the parabola is given in the equation by $\frac{2v_1^2}{g}$, and the directrix is at a distance from the vertex equal to half the parameter, then the distance of the directrix from the axis of x is

$$y_s+\frac{v_1^2}{2g}$$

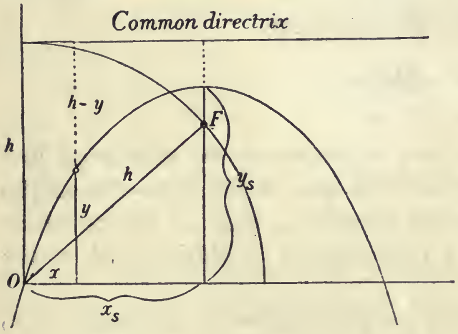
(where y_s is the ordinate of the vertex), or equal to

$$\frac{v_2^2}{2g} + \frac{v_1^2}{2g} = \frac{v_0^2}{2g} = h;$$

thus this distance is independent of ϕ, and we have the law:

All parabolas of the given family have a common directrix; its height above the level range is equal to the height h, reached by a body thrown vertically upward with the initial velocity v_0.

The velocity of the projectile at a given point (xy) of the flight path was found before to be equal to $\sqrt{[2g(h-y)]}$. This velocity is that which the body would possess, if it were allowed to fall freely from the directrix to any point of the path. (This can be seen to follow immediately from the law of *vis viva* or Kinetic Energy; because the kinetic energy of the shell of mass m at a given point of the path (xy) is $\frac{1}{2}mv^2$, and the loss of kinetic energy $\frac{1}{2}mv_0^2 - \frac{1}{2}mv^2$ is equal to the gain $mg \, . \, y$ in energy of position; and since $v_0^2 = 2gh$, therefore $v^2 = 2g(h-y)$.)



From this relation concerning the directrix, another follows concerning the locus of the focus of the parabolas of the family.

The directrix of every parabola is at a height h above the horizontal through O; the vertex of the parabola corresponding to the departure angle ϕ has the ordinate $y_s = h \sin^2\phi$; thence, since the vertex of a parabola is equidistant from the focus on one side and the directrix on the other side, the ordinate of the focus F is less than the vertex ordinate by $h - h\sin^2\phi$, or $h\cos^2\phi$.

It follows that the ordinate S_1F of the focus is equal to

$$h\sin^2\phi - h\cos^2\phi = -h\cos 2\phi;$$

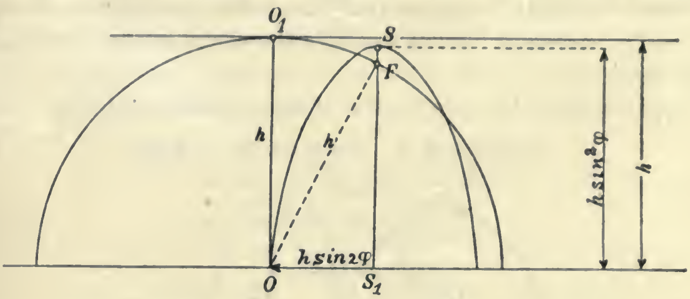
the abscissa OS_1 of the vertex was $OS_1 = h\sin 2\phi$; therefore

$$OF^2 = S_1F^2 + OS_1^2 = h^2\cos^2 2\phi + h^2\sin^2 2\phi = h^2.$$

The geometrical locus of the focus F of all parabolas of the family is thus a circle round O.

The point where this circle cuts the horizontal through O determines the focus corresponding to the parabola with the greatest range.

On the other hand, the geometrical locus of the vertex of all parabolas of the family is an ellipse, with the semi-axes h and $\frac{1}{2}h$,

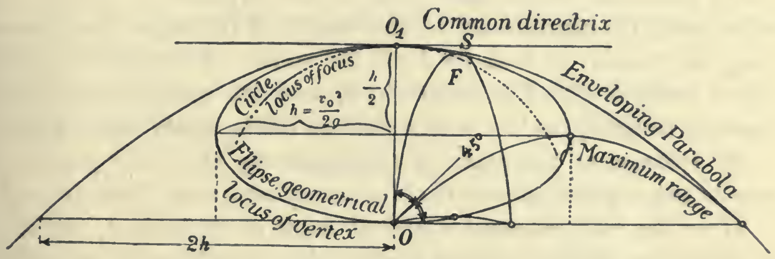


which touches the horizontal through O in O; because it was found for the coordinates x_s, y_s, of the vertex that

$$\frac{x_s}{h} = \sin 2\phi, \quad \frac{y_s}{h} = \sin^2\phi, \quad \frac{y_s - \frac{1}{2}h}{\frac{1}{2}h} = -\cos 2\phi.$$

By squaring and adding we have

$$\left(\frac{x_s}{h}\right)^2 + \left(\frac{y_s - \frac{1}{2}h}{\frac{1}{2}h}\right)^2 = 1.$$



Moreover the geometrical locus of the points of intersection of the initial tangent of the different parabolas with the axis of the corresponding parabola is a circle, the centre of which lies on the common directrix of the parabolic family, and touches the axis of x at the origin O. For such a point of intersection $x = h \sin 2\phi$ and $y = x \tan\phi$, whence

$$x^2 + (y - h)^2 = h^2.$$

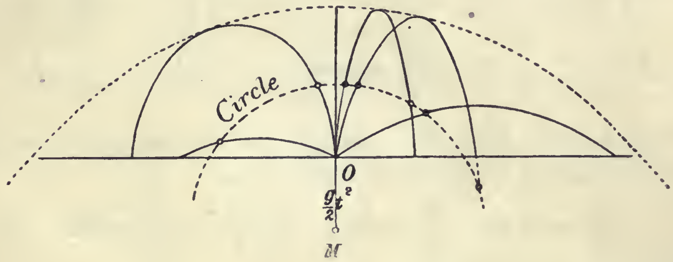
We can also consider the question: What is the geometrical locus of all points arrived at in the same time, when projected with the same initial velocity v_0, with all possible departure angles ϕ?

We suppose then that several shells are fired from O simultaneously, under all possible angles of departure, but with the same initial velocity. At a given instant, the shells will be found to lie on a certain surface; what is the nature of this surface?

Since everything is symmetrical about the vertical line through O, it is merely necessary here to consider the shells in the vertical plane of the figure.

After t seconds the coordinates of such a shell would be

$$x = v_0 \cos\phi \, . \, t, \quad y = v_0 \sin\phi \, . \, t - \tfrac{1}{2} g t^2,$$



and this gives

$$x^2 + (y + \tfrac{1}{2} g t^2)^2 = (v_0 t)^2.$$

On the plane of the figure, this is the equation of a circle; its radius $(= v_0 t)$ is proportional to the time, and its centre drops down along the y-axis; at first, for $t = 0$, the centre of the circle is at O, after t seconds it is $\frac{1}{2} g t^2$ below O; and so the centre of the circle descends vertically, as if it were a particle falling freely under gravity.

By rotation of the plane of the trajectory about the y-axis, we have as the required geometrical locus a sphere, with the radius $v_0 t$.

If we fire with constant v_0 and an angle of elevation ϕ_1, which in shooting over sloping ground denotes the angle between the initial tangent and the line of slope, the locus of the point of intersection of the line of sight and the trajectory is the parabola

$$y = x \cot\phi_1 - \frac{g x^2}{2 v_0^2 \sin^2 \phi_1}.$$

This is a parabolic trajectory with initial velocity v_0 and an angle of departure equal to the complement of ϕ_1.

For different angles ϕ_1 with equal v_0, there is a family of such parabolas.

This complete family is obviously identical with the family of parabolas with constant v_0, considered in this article. The application of these facts will be given in § 4.

§ 3. Family of trajectories with constant angle of departure.

A. Let us now consider a rifle or a gun clamped at a constant inclination, and fired with different initial velocities.

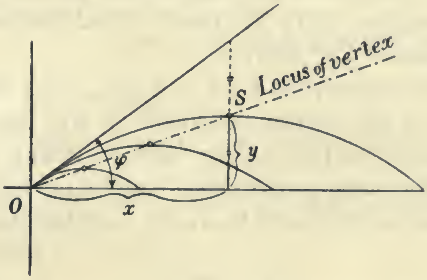
The geometrical locus of the vertex of all trajectories of this second family is a straight line.

For at the vertex S,

$$x = h \sin 2\phi,\ y = h \sin^2 \phi;$$

therefore

$$\frac{y}{x} = \frac{\sin^2 \phi}{\sin 2\phi};\ \frac{y}{x} = \frac{1}{2} \tan \phi.$$

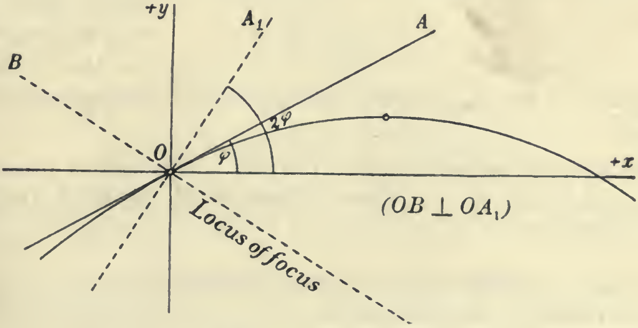


This line bisects the vertical distances between the axis of x and the initial tangent.

The geometrical locus of the focus is a straight line.

The coordinates of the focus were

$$x = h \sin 2\phi,\quad y = -h \cos 2\phi,$$



and by elimination of h it follows that

$$\frac{y}{x} = -\cot 2\phi = \tan\left(\tfrac{1}{2}\pi + 2\phi\right),$$

therefore the locus is a straight line.

Now let us suppose that we always fire with the same departure angle ϕ but with different initial velocities $v_0 = \sqrt{(2gh)}$; we can enquire, what is the locus of the projectiles after a given number of seconds?

The locus of these bodies after t seconds is determined through the coordinates

$$x = v_0 \cos\phi \,.\, t, \quad y = v_0 \sin\phi \,.\, t - \tfrac{1}{2}gt^2,$$

and then by elimination of v_0,

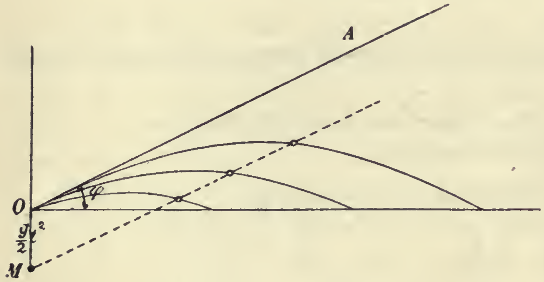
$$y = x \tan\phi - \tfrac{1}{2}gt^2.$$

This is the equation of a straight line parallel to the constant direction of departure OA; its point of intersection M with the y-axis is at a distance $\frac{1}{2}gt^2$ below O.

Therefore the geometrical locus in space of the shells is a conical surface, parallel to the surface described by OA: the vertex M falls from O downwards, just like a heavy particle falling freely.

Finally, the following propositions may be proved:

Consider the straight line $OM_1M_2M_3 \ldots$, cutting the separate parabolas of the family in M_1, M_2, M_3, ...; then the tangents to the parabolas at M_1, M_2, M_3, ... will be parallel, and the times of flight to reach these points will be proportional to the velocity at these points of the trajectories, and also to the corresponding initial velocity.



Draw another straight line $ON_1N_2N_3 \ldots$ through O, meeting the parabolas in $N_1, N_2, N_3, \ldots$; then the lines M_1N_1, M_2N_2, ... are parallel to each other.

B. Family of parabolic trajectories with invariable horizontal component of the initial velocity; $v_0 \cos\phi = \text{const.} = \kappa$.

The locus of the vertex is the parabola $y = \dfrac{gx^2}{2\kappa^2}$.

(For the coordinates of the vertex were

$$x_s = \frac{v_0^2}{g} \sin\phi \cos\phi = \frac{\kappa^2}{g} \tan\phi, \qquad y_s = \frac{v_0^2}{2g} \sin^2\phi = \frac{\kappa^2}{2g} \tan^2\phi;$$

and by elimination of ϕ, it follows that

$$y_s = \frac{gx_s^2}{2\kappa^2}.\Big)$$

The locus of the focus is the same parabola, but parallel and displaced vertically downward by $\frac{\kappa^2}{2g}$.

(For the coordinates of the focus were

$$x_f = h \sin 2\phi = \frac{\kappa^2}{g} \tan \phi, \quad y_f = -h \cos 2\phi = -\frac{\kappa^2}{2g}(1 - \tan^2 \phi),$$

$$y_f + \frac{\kappa^2}{2g} = \frac{gx_f^2}{2\kappa^2}.\Big)$$

Further, the locus of the points which will be reached in the same time t, is represented by the vertical line $x = v_0 t \cos \phi = \kappa t$.

Finally, the locus of the points where the slope of the tangent is the same is a parabola.

For according to the preceding

$$x = \frac{v_0^2}{g} \cos^2 \phi (\tan \phi - \tan \theta) = \frac{\kappa^2}{g} (\tan \phi - \tan \theta),$$

$$y = \frac{v_0^2}{2g} \cos^2 \phi (\tan^2 \phi - \tan^2 \theta) = \frac{\kappa^2}{2g} (\tan^2 \phi - \tan^2 \theta),$$

and this gives, by elimination of ϕ,

$$y = x \tan \theta + \frac{gx^2}{2\kappa^2}.$$

C. Family of parabolas with constant vertical component of the initial velocity, $v_0 \sin \phi = \text{const.} = m$; or, with constant time of flight T, or with constant vertex height y_s.

The locus of the vertex is the horizontal line $y_s = \frac{m^2}{2g}$.

The locus of the focus is the parabola

$$y = \frac{m^2}{2g} - \frac{gx^2}{2m^2}.$$

For
$$x_f = \frac{v_0^2}{g} \sin \phi \cos \phi = \frac{m^2}{g} \cot \phi,$$

$$y_f = -\frac{v_0^2}{2g} (\cos^2 \phi - \sin^2 \phi) = -\frac{m^2}{2g} (\cot^2 \phi - 1).$$

Thence it follows by the elimination of $\cot \phi$.

Finally the geometrical locus of the points reached in the same time t is the straight line

$$y = mt - \tfrac{1}{2} gt^2; \text{ since } y = v_0 t \sin \phi - \tfrac{1}{2} gt^2.$$

Corresponding results can be deduced for the family of parabolas (with the same origin O) which pass through the same target, and further for those parabolas in the same plane of fire which touch a given straight line, and so forth. The proof depends on elementary or projective geometry. The work of Fr. Külp, to which we direct attention, provides examples of this kind.

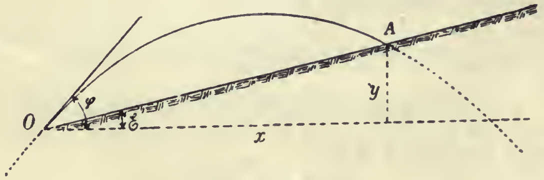
§ 4. Fire over sloping ground.

The preceding problems are now to be generalised: The surface of the ground (assumed horizontal before) may now be taken to make any angle of slope E with the horizontal through the point of departure, and the angle of departure ϕ is then to be calculated.

What is the length, measured on the sloping plane, of the range attained with given initial velocity? What is the corresponding time of flight? Under what conditions will the greatest range be attained?

The equation of the sloping plane being $y = x \tan E$, we have in addition

$$x = v_1 t, \quad y = v_2 t - \tfrac{1}{2} g t^2 = x \tan E;$$



from these three equations x and y are to be eliminated, if we wish to determine the time t, which elapses before the shot reaches the sloping plane.

We have
$$v_2 t - \tfrac{1}{2} g t^2 = v_1 t \tan E,$$

and
$$t = \frac{2v_2}{g} - \frac{2v_1}{g} \tan E = \frac{2(v_2 - v_1 \tan E)}{g};$$

now $v_1 = v_0 \cos\phi$, $v_2 = v_0 \sin\phi$, and therefore

$$t = \frac{2v_0}{g} \frac{\sin(\phi - E)}{\cos E}.$$

Further,

$$x = v_1 t = v_0 t \cos\phi = \frac{2v_0^2}{g} \frac{\cos\phi \sin(\phi - E)}{\cos E},$$

and thence the range OA over the plane of slope is

$$OA = \frac{x}{\cos E} = \frac{2v_0^2}{g} \frac{\cos\phi \sin(\phi - E)}{\cos^2 E}.$$

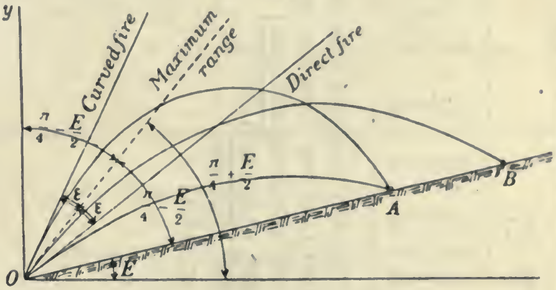
What is the value of the departure angle ϕ, with given initial velocity v_0, and given slope E of the ground, for which the range OA is a maximum?

The expression $\cos\phi \sin(\phi - E)$ is to be differentiated with respect to ϕ. This gives

$$-\sin\phi \sin(\phi - E) + \cos\phi \cos(\phi - E) = 0,$$
$$\tan(\phi - E) = \cot\phi = \tan(\tfrac{1}{2}\pi - \phi),$$

and so we have

$$\phi - E = \tfrac{1}{2}\pi - \phi,$$
$$\phi = \tfrac{1}{2}(\tfrac{1}{2}\pi + E).$$



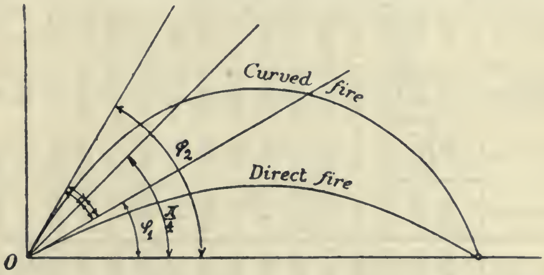
The angle in this case between the initial tangent and the vertical through O is equal to

$$\tfrac{1}{2}\pi - \tfrac{1}{4}\pi - \tfrac{1}{2}E = \tfrac{1}{4}\pi - \tfrac{1}{2}E;$$

and on the other hand, the angle between the inclined plane and the vertical is $\tfrac{1}{2}\pi - E$: so that the direction of projection must bisect the angle between the inclined plane and the vertical, if the range is to be a maximum, measured on the inclined plane.

If we fire with two angles of departure, of which one is smaller and the other is larger, by the same amount ϵ, than the angle of maximum range of fire, then both shots will strike the inclined plane in the same point A.

The greater of the two departure angles is $\tfrac{1}{4}\pi + \tfrac{1}{2}E + \epsilon$; the other



is $\tfrac{1}{4}\pi + \tfrac{1}{2}E - \epsilon$; it follows from the above that in the first case the range is

$$= \frac{2v_0^2}{g} \frac{\cos(\tfrac{1}{4}\pi + \tfrac{1}{2}E + \epsilon)\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}E + \epsilon)}{\cos^2 E},$$

and in the second case the range is

$$= \frac{2v_0^2}{g} \frac{\cos(\tfrac{1}{4}\pi + \tfrac{1}{2}E - \epsilon)\sin(\tfrac{1}{4}\pi - \tfrac{1}{2}E - \epsilon)}{\cos^2 E},$$

and the two expressions have the same value.

Angle of tangent elevation $\phi_1 =$ ↓	Range over the sloping ground for an angle of slope $E = \ldots$ → $E=-20°$	$E=-10°$	$E=-5°$	$E=0°$	$E=+5°$	$E=+10°$	$E=+15°$	$E=+20°$	$E=+25°$	$E=+30°$	$E=+35°$	$E=+40°$	$E=+45°$	$E=+50°$	$E=+55°$	$E=+60°$	$E=+65°$	$E=+70°$	$E=+75°$	$E=+80°$	$E=+85°$	$E=+87°$
3°	567	536	527	522·6	522·3	526	534	546	562	585	615	652	700	762	843	950	1098	1308	1624	2115	2404	0
5°	953	895	878	868·3	865	868·1	878	895	919	952	995	1050	1120	1210	1325	1473	1669	1928	2259	2519	0	
10°	1937	1790	1743	1710	1690	1682	1687	1703	1732	1774	1830	1902	1992	2101	2231	2376	2516	2578	2259	0		
15°	2920	2658	2568	2500	2451	2419	2402	2401	2414	2440	2479	2530	2588	2647	2691	2680	2516	1928	0			
20°	3873	3473	3329	3214	3123	3054	3003	2967	2944	2931	2924	2914	2891	2831	2691	2376	1669	0				
25°	4768	4209	4002	3830	3688	3570	3470	3384	3307	3232	3149	3044	2891	2646	2231	1473	0					
30°	5576	4844	4566	4330	4127	3949	3789	3640	3492	3333	3149	2914	2588	2101	1325	0						
35°	6274	5360	5005	4698	4427	4182	3952	3726	3492	3232	2924	2530	1992	1210	0							
40°	6840	5740	5306	4924	4580	4260	3952	3640	3307	2931	2479	1902	1120	0								
45°	7258	5972	5458	5000	4580	4182	3789	3384	2944	2440	1830	1050	0									
50°	7513	6050	5458	4924	4427	3949	3470	2967	2414	1774	995	0										
55°	7600	5972	5306	4698	4127	3570	3003	2401	1732	952	0											
60°	7513	5740	5005	4330	3688	3054	2402	1703	919	0												
65°	7258	5360	4566	3830	3123	2419	1687	895	0													
70°	6840	4844	4002	3214	2451	1682	878	0														
75°	6274	4209	3329	2500	1690	868·1	0															
80°	5576	3473	2568	1710	865	0																
85°	4768	2658	1743	868·3	0																	
90°	3873	1790	878	0																		

The geometrical locus of the point of impact A on the sloping ground, for constant angle of elevation ϕ_1 and the same initial velocity v_0, but with variable angle of slope E, is the parabola

$$y = x\tan(\tfrac{1}{2}\pi - \phi_1) - \frac{gx^2}{2v_0^2\cos^2(\tfrac{1}{2}\pi - \phi_1)} = x\cot\phi_1 - \frac{gx^2}{2v_0^2\sin^2\phi_1}.$$

This follows forthwith, when we eliminate E between $x = OA\cos E$ and $y = x\tan E$.

These relations may be illustrated by means of the foregoing Table. The range in it is given as

$$OA = \frac{2v_0^2}{g}\frac{\sin\phi_1\cos(\phi_1+E)}{\cos^2 E},$$

on the assumption that $\frac{2v_0^2}{g} = 10{,}000$ metres, or $v_0 = 221$ m/sec, and that

$$E = -20,\ -10,\ -5,\ 0,\ +5,\ +10, \ldots \text{ up to 87 degrees},$$

with angles of the tangent sight

$$\phi_1 = 3,\ 5,\ 10,\ 15,\ 20, \ldots \text{ up to } 90^\circ.$$

We learn first from the actual numbers of the Table that for the same slope E, the range is a maximum, when the initial tangent of the path of flight bisects the angle between the slope of the ground and the vertical; for example, with the slope angle $E = -20^\circ$, the maximum of 7600 m occurs when

$$\phi_1 = \tfrac{1}{2}(90+20) = 55^\circ.$$

In the Table the vertical limits of the ranges, which are attained with the appropriate angle of elevation ϕ_1 on horizontal ground ($E=0$), are made prominent by thicker lines of division.

These ranges may be called the sighting ranges. On the tangent scales of guns and rifles, besides the sighting angle ϕ_1 (in degrees), the corresponding sighting range in metres is often engraved.

Thus for example in an actual case the reading "sighting 2500" is equivalent to "sighting angle 15°": or the reading "sighting 5000" is equivalent to "sighting angle 45°."

Now if there is a target at a distance of 5000 m from the gun, on ground at a slope $E = 20^\circ$, and if the sight is set at 5000 m to hit the mark, that is with a tangent elevation $\phi_1 = 45^\circ$, then this setting does not take into account the slope of the ground.

We might imagine the trajectory to turn about the muzzle of the gun, and through the angle of slope, for example 20°, as if the trajectory were a rigid curve.

A reference to the figure on p. 4 will show that this is not actually the case. In this "swinging of the trajectory" there is in fact an error.

In the present examples, slope of ground $E = +20^\circ$, tangent elevation $\phi_1 = 45^\circ$; then the range will not be 5000 m, but 3384 m. With slope of ground $E = -20^\circ$, and the same tangent elevation $\phi_1 = 45^\circ$, the range is 7258 m instead of 5000 m; so that the shell goes too far.

In all elevations ϕ_1 greater than a certain angle between 15° and 20° (viz., 16° 43′, as shown later) the "swinging of the trajectory" strikes short with positive ground slope, and strikes beyond with negative slope: and the error of range is greater for shooting downhill than for shooting uphill, other things being equal.

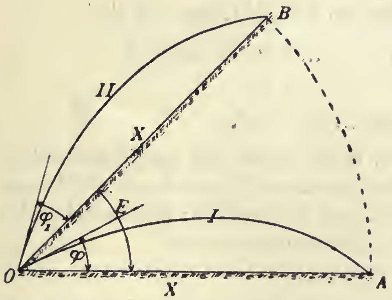
For tangent elevation less than 16° 43′, the relations are somewhat more complicated. For example, take the sighting 522·6 m, on a tangent elevation 3°.

Let the ground slope increase from nothing to +87°. For $E=0$ the sighting range will be 522·6 m; for $E=87°$ (vertical fire) the range is zero.

Consequently the range diminishes a little at first, and then increases considerably, finally diminishing again very quickly to zero.

Since the fluctuation is a continuous one, there must be two ground slopes between the slope $E=0°$ and $E=87°$ in existence, for which the range 522·6 m will be obtained. For these two values ($E_1=5° 57'$ and $E_2=86° 51'$), the "swinging of the trajectory" is quite right; at least there is no error in the range attained.

In this way a certain region is obtained of positive ground slope, in which no short range lies, but, on the other hand, too long a range (framed in outline in the Table).



The limits of this region are those angles of slope, for which the "swinging of the trajectory" gives the same range on the sloping ground as on the horizontal.

The question then arises: For what slope E is the range OB on the incline the same as the range OA on horizontal ground, if the same tangent elevation is employed for the two cases?

The condition requires

$$\frac{2v_0^2}{g}\frac{\cos(E+\phi_1)\sin\phi_1}{\cos^2 E}=\frac{v_0^2}{g}\sin 2\phi;$$

or employing the principle of "swinging the trajectory" $(\phi_1=\phi)$

$$2\cos(E+\phi)\sin\phi=\cos^2 E\sin 2\phi.$$

Therefore $$\cos^3 E-\cos^2 E+\cos E\tan^2\phi+\tan^2\phi=0.$$

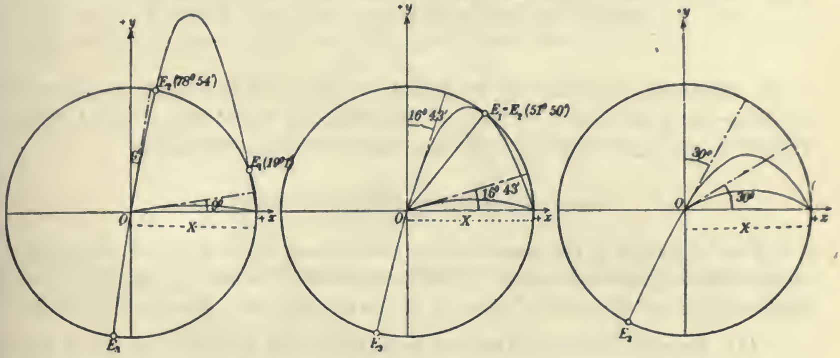
When positive angles of slope are considered, this equation is satisfied by

1.	$\phi=0$,	$E_1=0$,	$E_2=90°$;
2.	$\phi=3°$,	$E_1=5° 57'$,	$E_2=86° 51'$;
3.	$\phi=6°$,	$E_1=12° 12'$,	$E_2=83° 13'$;
4.	$\phi=9°$,	$E_1=19° 1'$,	$E_2=78° 54'$;
5.	$\phi=12°$,	$E_1=26° 47'$,	$E_2=73° 26'$;
6.	$\phi=16°$,	$E_1=42° 26'$,	$E_2=60° 42'$;
7.	$\phi=16° 43'$,	$E_1=E_2=51° 50'$.	

For example, if the sight of 3° is employed, the range is short if $E<5° 57'$, or if $E>86° 51'$; and long if E lies between 5° 57′ and 86° 51′. This error diminishes as ϕ increases from zero up to 16° 43′.

A. N. Obermayer in 1901 gave also a simple geometrical relation between the corresponding angles of ground slope E_1 and E_2. The parabolic trajectory is drawn for the given v_0 with the departure angle $90°-\phi$ from the horizon, and this corresponds in the example above to $90-3=87°$; the circle is drawn with centre O and radius equal to the corresponding range on the horizontal; and the two points are marked in the first quadrant where the parabola and the circle intersect, and the radii are drawn.

The inclination to the horizon of these two radii ($5°57'$ and $86°51'$ in the given example) are the ground slopes E_1 and E_2 for which the swinging of the trajectory holds good.



At the same time, as already pointed out above, this particular parabola is the geometrical locus of the extremity of all sloping ranges of equal initial velocity, and equal tangent elevation $\phi'=\phi$, so that from the relative position of the circle and parabola we can determine for given slope E whether the shell will range under or over.

The circle $x^2+y^2=X^2$ and the parabola

$$y=x\cot\phi-\frac{gx^2}{2v_0^2\sin^2\phi}$$

intersect in two points in the first quadrant (as in the figure) which are either both real and distinct, or real and coincident, or else imaginary.

§ 5. Examples.

1. The following short table shows how with small initial velocity and relatively great weight of shell the formulae give reasonable approximations to the real flight of the shell.

The French 22 cm mortar, pattern 1887, is chosen as an example; the smallest charge 1·135 kg, $v_0=90$ m/sec; the greatest charge 6·126 kg, $v_0=230$ m/sec; weight of shell 118 kg.

The range X, time of flight T, height of vertex y_s, angle of descent ω, and final velocity v_e are given in the Table, calculated from the formulae for a vacuum; the corresponding results in practice are given in brackets.

v_0	ϕ	X	T	y_s	ω	v_e
230	66° 22′	3961 (3200)	43·0 (40·7)	2263 (2017)	66° 22′ (70° 2′)	230 (206)
,,	35° 0′	5067 (4300)	26·9 (25·9)	887 (820)	35° 0′ (39° 23′)	230 (195)
90	65° 15′	628 (600)	16·7 (16·6)	340 (336)	65° 15′ (66° 31′)	90 (88)
,,	34° 2′	766 (750)	10·3 (10·2)	129 (127)	34° 2′ (35° 1′)	90 (88)

2. Since the curvature in the actual trajectory at the point of departure ($x=0$, $y=0$) is the same as that of the parabolic trajectory of equal initial velocity v_0 and equal angle of departure, where the curvature radius

$$\rho_0=\frac{v_0^2}{g}\sec\phi,$$

it is often allowable in the immediate neighbourhood of the muzzle to replace the actual trajectory with advantage by the parabola with the same v_0 and ϕ, or in the neighbourhood of the point of descent by the parabola with the same v_e and ω.

(*a*) Suppose an armour plate is to be pierced at a distance of 100 m from the muzzle of a mortar. At what point of the plate must the axis of the piece be aimed, in order that the desired point of impact shall be struck? Assume there is no error in the angle of departure. The point aimed at must be

$$\tfrac{1}{2}g\left(\frac{100}{200}\right)^2=1{\cdot}23\text{ m}$$

above the desired point to be struck.

(*b*) Determination of the error of departure on the same principles, by a comparison of the actual and calculated points of impact, on a target at a given distance from the muzzle.

(*c*) If the total range is X, the danger zone for a target of height h is

$$\frac{X}{2}\left(1-\sqrt{1-\frac{4h}{X\tan\omega}}\right).$$

3. Is it possible to throw a stone from the top of the pyramid of Cheops beyond the base?

The height of the pyramid is 137·2 m; the length of a side of the square base is 227·5 m; so that the angle of slope $ABC=50°\,20'$.

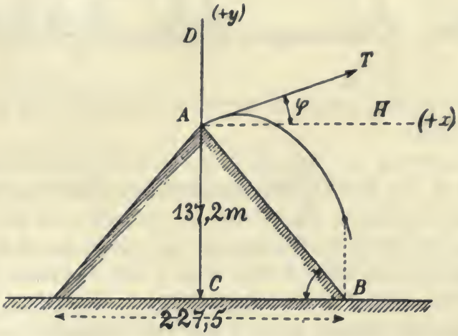
To obtain the greatest range, the stone must be thrown from the top A in a direction AT bisecting the angle BAD between the inclined plane AB and the upward vertical AD, so that

$$\angle DAT=\tfrac{1}{2}\angle DAB=\tfrac{1}{2}(90°+50°\,20')=70°\,10':$$

and the angle of projection ϕ is thus $=19°\,50°$.

The initial velocity v_0 of a throw by hand is assumed as 24 m/sec (mean of 30 experiments with different persons); the equation of the path is

$$y = x \tan \phi - \frac{gx^2}{2v_0^2 \cos^2 \phi}.$$



The question is then, how great is y when x has the value $\frac{1}{2}(227\cdot5)$ m? Then we shall have

$$y = 113\cdot7 \tan 19° \, 50' - \frac{113\cdot7^2 \times 9\cdot81}{2 \times 24^2 \cos^2 19° \, 50'} = -83\cdot4 \text{ m}$$

(with $v_0 = 22$ m/sec, y will be $= -107\cdot0$ m, and with $v_0 = 20$ m/sec, $y = -138\cdot1$ m).

The answer is then that it is possible.

4. At what angle of projection ϕ with the horizon must a body be thrown, so that it may strike at right angles a plane inclined at an angle E (perpendicular to the plane of the trajectory)?

Result:

$$\tan(\phi - E) = \tfrac{1}{2} \cot E.$$

5. What is the difference between the times of flight to the same mark of two shells fired from the same point with initial velocities v and v', and angles ϕ and ϕ'?

The difference is

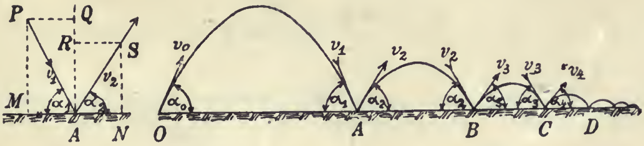
$$\frac{2}{g} \cdot \frac{vv' \sin(\phi - \phi')}{v \cos \phi + v' \cos \phi'}.$$

6. One shell strikes the foot of a tower on a horizontal plane through the gun after t seconds. A second shell with another charge and double elevation strikes the top of the tower after t' seconds. The distance of the tower is

$$\tfrac{1}{2} g t^2 \sqrt{\frac{t'^2 + t^2}{t'^2 - t^2}}.$$

7. Ricochet Fire. Suppose a spherical shell is projected from O over a horizontal plane with velocity v_0 and angle of departure a_0. Let the elasticity be e ($e = 0$ for completely inelastic bodies, $e = 1$ for perfectly elastic bodies).

The shell strikes the ground at A at the same angle a_0, and with the same velocity v_0; it starts afresh to describe a parabola (but with smaller departure angle a_2 and smaller initial velocity v_2), strikes the ground again at B a second ime, and so on (see figure).



How great is the total range up to the nth impact, and what is the corresponding time of flight?

By Newton's Law for the vertical impulse of two elastic masses m and M, one of the masses M (the Earth) being assumed as infinitely great compared with the other, it is easy to deduce the velocity v_2 with which a ball springs up, if it falls vertically on the ground with velocity v_1. It is found to be ev_1, where e denotes the elasticity of the ball. If then such a ball is thrown aslant at an acute angle a_1 with the horizontal ground surface (as in the above figure), it is only necessary to resolve the striking velocity into two components at right angles; in the horizontal direction there is no impulse, and assuming that friction may be neglected, then the horizontal component of the velocity remains unaltered, $MA = AN$, or

$$v_1 \cos a_1 = v_2 \cos a_2.$$

On the other hand there is direct impact in the vertical direction, so that

$$AR = e \, . \, AQ, \text{ or } v_2 \sin a_2 = ev_1 \sin a_1.$$

In this way we know the direction a_2 and the magnitude v_2 of the velocity with which the rebounding ball leaves the surface; for it follows from the two equations that $\tan a_1 : \tan a_2 = 1 : e$; thence a_2 is known, and then v_2.

(These considerations are to be applied again as often as the ball rebounds, as at A, B, C.)

Denote by a_n the acute angle at which the ball is moving immediately before the nth rebound, and by v_n the corresponding velocity. Further let W_n denote the range up to the nth rebound, measured from O; t_n the time elapsed. In the horizontal direction we have for the different impacts

$$v_0 \cos a_0 = v_1 \cos a_1 = v_2 \cos a_2 = \dots = v_n \cos a_n.$$

On the contrary, in the vertical direction

$$v_2 \sin a_2 = ev_1 \sin a_1 = ev_0 \sin a_0 \text{ (because } a_1 = a_0, \text{ and } v_1 = v_0);$$

so also $$v_3 \sin a_3 = ev_2 \sin a_2 = e^2 v_1 \sin a_1 = e^2 v_0 \sin a_0;$$

and generally $$v_0 \cos a_0 = v_n \cos a_n, \quad v_0 \sin a_0 = \frac{v_n \sin a_n}{e^{n-1}}.$$

Therefore

$$\tan a_n = e^{n-1} \tan a_0, \quad v_n^2 = v_0^2 (e^{2n-2} \sin^2 a_0 + \cos^2 a_0). \quad \text{............(I)}$$

Then the velocity of the ball before the nth rebound is calculated from the initial condition a_0, v_0, and the elasticity e.

What is the time elapsed up to the nth rebound?

The first arc OA will be described in time

$$t_1 = \frac{2v_0}{g} \sin a_0;$$

the second arc in time

$$t_2 - t_1 = \frac{2v_2}{g} \sin a_2 = \frac{2ev_0 \sin a_0}{g} = et,$$

and so on. The time to the nth rebound will then be

$$t_n = \frac{2v_0 \sin a_0}{g}(1 + e + e^2 + e^3 + \ldots + e^{n-1}) = \frac{2v_0 \sin a_0}{g} \cdot \frac{1 - e^n}{1 - e}. \quad \ldots\ldots\text{(II)}$$

The ranges OA, AB, BC, ... will be

$$OA = v_1 t_1 \cos a_1 = v_0 t_1 \cos a_0, \quad \text{where} \quad t_1 = \frac{2v_0}{g} \sin a_0,$$

$$AB = v_2(t_2 - t_1) \cos a_2, \quad \text{where} \quad t_2 - t_1 = \frac{2ev_0}{g} \sin a_0, \quad v_2 \cos a_2 = v_0 \cos a_0,$$

$$AB = \frac{2v_0^2}{g} e \cos a_0 \sin a_0, \text{ and so forth.}$$

The whole range W_n from O up to the nth point of rebound is thus

$$= \frac{2v_0^2}{g} \sin a_0 \cos a_0 (1 + e + e^2 + \ldots + e^{n-1}) = \frac{v_0^2}{g} \sin 2a_0 \frac{1 - e^n}{1 - e} \ldots\ldots\text{(III)}$$

This expression (III) is sufficient, either to calculate W_n, when a_0, v_0 and e are known, or else to determine the elasticity e from v_0, a_0, and W_n.

Theoretically the ball will keep on striking the ground, always describing smaller and smaller parabolas. But although the number of these arcs, described by the ball, is infinite, still the aggregate range is finite, and so also is the total time during which the ball is in movement.

In fact, for $n = \infty$ (since e is a proper fraction and the limit of $e^n = 0$), we have

$$t = \frac{2v_0}{g} \sin a_0 \cdot \frac{1}{1 - e}, \qquad W = \frac{v_0^2}{g} \sin 2a_0 \cdot \frac{1}{1 - e}.$$

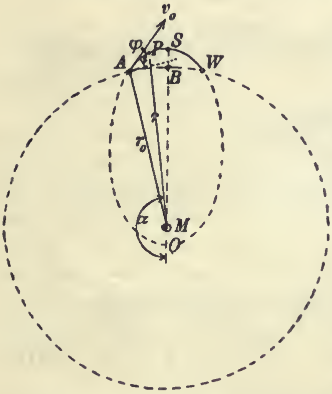
The difference from motion in a single arc, with the same initial velocity v_0 and same departure angle a_0, is that in the ricochetting the range and time of flight are increased in the ratio $1 : 1 - e$; where e is the elasticity of the ball.

[Ricochet Fire was known already in the 16th century; and this kind of fire was first introduced systematically by Vauban in 1688. This method was in use till about the middle of the 18th century: Lieutenant Paul Jacobi wrote in 1756 an extensive work on ricochetting and the rules by which the best results would be obtained. The mathematical theory was developed by Bordoni, 1816, and Otto, 1841.]

Consult § 75 on ricochetting over water with partial penetration. In water $a_2 < a_1$. On the contrary it often appears, according to the nature of the ground, that $a_2 > a_1$; this was the case in the experiments carried out by F. Krupp on sandy ground according to the procedure of F. Neesen. In such cases other assumptions must be made. The assumptions made in example 7 hold only for the case where the tangential friction on impact may be neglected (compare also for instance B. Keck, *Lectures on Mechanics*, Hannover 1901, Volume II, page 160).

§ 6. Trajectory in a vacuum, taking into account the decrease of gravity with the height and the convergence of vertical lines from the curvature of the Earth.

Take the centre M of the Earth (as in the figure) as the pole in a system of polar coordinates; and let any arbitrary point P have the polar coordinates MP = radius vector r, and $\angle OMP$ = polar angle α; the direction OM of the polar axis from which the polar angle α is measured may be left undetermined at first.

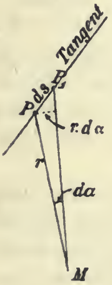


At A, the point of departure of the shell, let $r = r_0$ = radius of the Earth, 6,370,300 m; let the initial velocity be v_0, and the angle of departure ϕ.

According to Newton's Law of Gravitation, the acceleration of gravity $= g\dfrac{r_0^2}{r^2}$, or $\dfrac{\mu}{r^2}$. Moreover, $r^2\dfrac{d\alpha}{dt}$ is a constant, C, along the whole path of flight; the value of $r\dfrac{d\alpha \,.\, r}{dt}$ at the point A is given by $C = r_0 v_0 \cos\phi$; since here $r\,d\alpha = \cos\phi\, ds$ (where ds is the element of arc), and $\dfrac{ds}{dt} = v_0$, we have

$$r^2\frac{d\alpha}{dt} = C = r_0 v_0 \cos\phi. \quad \text{.....................(1)}$$

Further, in the motion of the shell along its path,



$$\frac{dv}{dt} = -\frac{\mu}{r^2}\frac{dr}{ds}, \quad \text{or} \quad v\frac{dv}{dr} = -\frac{\mu}{r^2};$$

and integrating from A to P,

$$v^2 - v_0^2 = -2\mu\int_{r_0}^{r} r^{-2}dr = +2\mu\left(\frac{1}{r} - \frac{1}{r_0}\right),$$

or

$$v^2 = q + \frac{2\mu}{r},$$

where

$$q = v_0^2 - \frac{2\mu}{r_0}. \quad \text{..................(2)}$$

Since

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\alpha}{dt}\right)^2,$$

and
$$\frac{dr}{dt}=\frac{dr}{d\alpha}\frac{d\alpha}{dt}=\frac{dr}{d\alpha}\frac{C}{r^2}, \text{ from (1)},$$

equation (2) may be written in the form

$$q+\frac{2\mu}{r}=\left(\frac{dr}{d\alpha}\right)^2\frac{C^2}{r^4}+\frac{C^2}{r^2},$$

or,

$$d\alpha=\frac{\frac{C}{r^2}dr}{\sqrt{\left(q+\frac{2\mu}{r}-\frac{C^2}{r^2}\right)}}=\frac{-d\frac{\frac{C}{r}-\frac{\mu}{C}}{\sqrt{\left(q+\frac{\mu^2}{C^2}\right)}}}{\sqrt{\left\{1-\left[\frac{\frac{C}{r}-\frac{\mu}{C}}{\sqrt{\left(q+\frac{\mu^2}{C^2}\right)}}\right]^2\right\}}}.$$

This is the differential equation of the path of the shell, with r and α as the two variables. Integration gives

$$\alpha-\gamma=\cos^{-1}\frac{\frac{C}{r}-\frac{\mu}{C}}{\sqrt{\left(q+\frac{\mu^2}{C^2}\right)}}$$

or
$$r=\frac{p}{1+\epsilon\cos(\alpha-\gamma)}, \qquad\qquad (3)$$

where γ denotes an integration constant, and where

$$p=\frac{C^2}{\mu},\quad \epsilon=\sqrt{\left(1+\frac{qC^2}{\mu^2}\right)}.$$

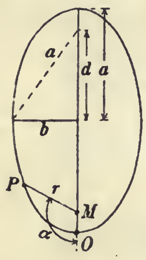
This equation (3) shows that the flight path is a conic section. To determine the integration constant γ, we know that

$$r=\frac{p}{1+\epsilon\cos\alpha}$$

is the polar equation of a conic section, in which the parameter

$$p=\frac{b^2}{a}=\frac{a^2-d^2}{a}=a-\epsilon d=a-\epsilon^2 a=a(1-\epsilon^2),$$

[a and b the two semi-axes, a the one that contains the focus, d the distance between centre and focus of the conic section, ϵ = the eccentricity].



One focus M is here the pole of the system of polar coordinates, and the polar angle α is measured from that vertex O of the major axis that lies nearest to the focus M; with $\epsilon < 1$ we have an ellipse; with $\epsilon = 0$, a circle; $\epsilon = 1$ a parabola; $\epsilon > 1$ an hyperbola.

When the integration constant γ is zero, the polar axis OM of the polar coordinate system is the line joining the perihelion O to the centre of the earth. We have then

$$r = \frac{p}{1 + \epsilon \cos \alpha}, \qquad (4)$$

$$v^2 = v_0^2 - \frac{2\mu}{r_0} + \frac{2\mu}{r}, \qquad (5)$$

where
$$p = \frac{C^2}{\mu}, \quad \epsilon = \sqrt{\left(1 + \frac{qC^2}{\mu^2}\right)},$$

and
$$q = v_0^2 - \frac{2\mu}{r_0}, \quad \mu = gr_0^2, \quad C = r_0 v_0 \cos \phi, \quad r_0 = 6{,}370{,}300 \text{ m}.$$

Since r_0, ϕ, v_0, are known, and also C, μ, q, ϵ, p, we are in a position to determine for any value of α the corresponding distance r of the shell from the centre of the Earth from (4), and from (5) the corresponding velocity v in the trajectory; the time of flight is obtained then from the integration of

$$dt = \frac{r^2 d\alpha}{C}.$$

The trajectory is an ellipse when $\epsilon < 1$, that is when

$$1 + \frac{C^2}{\mu^2}\left(v_0^2 - \frac{2\mu}{r_0}\right) < 1, \quad \text{or} \quad v_0 < \sqrt{\frac{2\mu}{r_0}}.$$

Now
$$\sqrt{\frac{2\mu}{r_0}} = \sqrt{[2 \,.\, (9{\cdot}81)\, 6{,}370{,}300]} = 11{,}050 \text{ m/sec},$$

so that the path is an ellipse so long as v_0 remains $<$ 11,050 m/sec.

This elliptic orbit is a circle in the special case when $\epsilon = 0$, or

$$1 + \frac{r_0^2 v_0^2 \cos^2 \phi}{\mu^2}\left(v_0^2 - \frac{2\mu}{r_0}\right) = 0.$$

If $\dfrac{r_0 v_0^2}{\mu} = z$, this equation becomes

$$z^2 - 2z = -\frac{1}{\cos^2 \phi}, \quad z = \frac{r_0 v_0^2}{\mu} = 1 \pm \sqrt{\left(1 - \frac{1}{\cos^2 \phi}\right)};$$

and this expression is real only when $\cos\phi = \pm 1$, $\phi = 0$ or π; in that case $\frac{r_0 v_0^2}{\mu} = 1$,

$$v_0 = \sqrt{\frac{\mu}{r_0}} = 7900 \text{ m/sec.}$$

Thus under the specified assumptions the trajectory with any ordinary initial velocity v_0 is an ellipse; it is a parabola when $v_0 = 11{,}050$ m/sec; with still greater initial velocity it would be an hyperbola.

The trajectory can be a circle only in the special case when the shell is fired horizontally, with an initial velocity of about 7900 m/sec.

To calculate the range AW and the vertex ordinate BS of an elliptic trajectory from A, we should proceed as follows: first calculate the polar angle $\alpha = \angle OMA$, which corresponds to the point A, from the relation $r_0 = \frac{p}{1 + \epsilon \cos\alpha_0}$, which holds if A is to lie on the trajectory.

Twice the supplement of this angle is the angle AMW; from this and r_0, the range AW can be calculated.

Further, the vertex ordinate BS of the trajectory $= MS - r_0$, in which MS denotes the maximum value of r; this is when the denominator in $r = \frac{p}{1 + \epsilon\cos\alpha}$ assumes its least value, that is when $\cos\alpha = -1$: thus $r_{\max} = \frac{p}{1-\epsilon}$ (and $r_{\min} = \frac{p}{1+\epsilon} = MO$, and thus the major axis of the ellipse is given by $r_{\max} + r_{\min}$). Thus

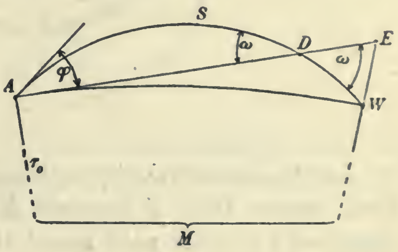
$$BS = \frac{p}{1-\epsilon} - r_0.$$

Numerical example: $v_0 = 820$ m/sec, $\phi = 44°$, $r = 6{,}370{,}300$ m. Then $\epsilon = 0{\cdot}99445$, $\alpha_0 = 179° \, 41' \, 23''{\cdot}26$;

$$\frac{AW}{2\pi r_0} = \frac{2(180 - \alpha_0)}{360°} = \frac{0{\cdot}31011}{180}.$$

Thence the range $AW = 68{,}958$ m; vertex height $BS = 16{,}620$ m.

On the other hand, in the parabolic path with the same v_0 and ϕ, the range will be 68,500 m, and the vertex height 16,538 m.



Of the three influences considered here, the curvature of the Earth's surface is the most important, as it makes an alteration of range

$$68958 - 68500 = 458 \text{ m.}$$

For if we draw the horizontal straight line ADE through A in the plane of the diagram perpendicular to r_0, and ASD the parabolic trajectory with its corresponding range AD, in which $\omega = \phi$ denotes the acute angle of descent, it is obvious at once from the figure that AD is less than the range AW.

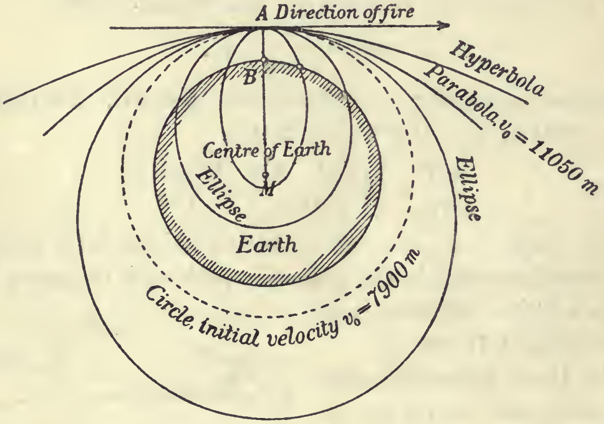
The difference of the two ranges can be calculated approximately from considerations similar to those which are usual in the calculations of a horizontal range: for $AE^2 = EW(EW + 2r_0)$, and, approximately, $AD^2 = DE \tan\omega \,.\, 2r_0$; DE is nearly equal to the difference in question $AW - AD = \sim \dfrac{AD^2}{2r_0 \tan\omega}$. This gives in the present case 387 m.

If similar calculations are carried out for ranges such as can occur in practice, it will always be found that the three influences, curvature of the Earth, convergence of the vertical, decrease of g with the height need not be considered.

It is interesting to discuss the trajectories if a shell is fired from the same point O, always in the same direction, but with increasing initial velocities v_0.

a. Horizontal Fire.

It is assumed in the figure that shells are fired horizontally from an elevation A in the neighbourhood of the surface of the Earth.



As the initial velocity v_0 increases, so the ellipse broadens; the focus moves from A towards M; with $v_0 = 7900$ m/sec the shell describes a circular path round the Earth, in perpetual motion, and the movable focus comes into coincidence with the fixed focus M. In this case the shell flies round always at the same distance from the ground.

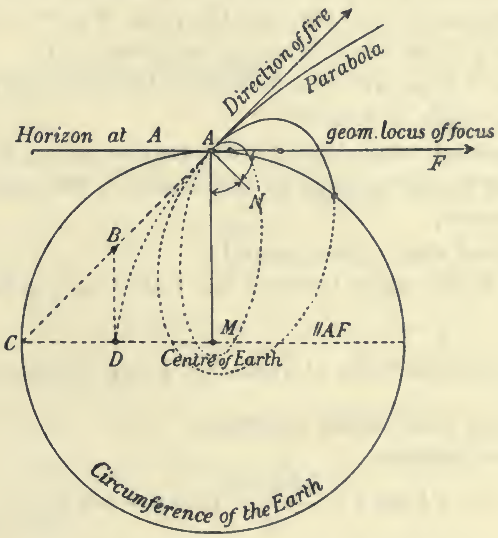
When the initial velocity increases still more, the shell at first goes further from the Earth's surface, in an ellipse, but returns again to A the point of departure. At the opposite side of the Earth the shell recedes more and more from the Earth's surface; the movable focus goes back beyond M, on the prolongation of the line AM.

With the velocity $v_0 = 11{,}050$ m/sec the shell does not come back to A; the ellipse is changed into a parabola; the movable focus has receded to infinity.

As soon as this initial velocity 11,050 m/sec is exceeded, the trajectory is an hyperbola, the branch of which through A approximates more and more to the horizontal direction; the movable focus then approaches A on the line AM produced backward.

b. Inclined Fire.

When the shots are made, with a given departure angle, with increasing velocities, the trajectory is again at first an ellipse. The point



of descent on the Earth's surface lies further and further from A. One focus of the ellipses lies constantly at M, the Earth's centre; the other focus moves along a straight line AF.

This holds, because if the line AN is drawn through A perpendicular to the line of fire, the angle NAF is made equal to the angle MAN.

A circular trajectory is not possible in this case.

With an initial velocity of 11,050 m/sec the ellipse changes into a parabola; the shot never returns to Earth again; the movable focus F has moved off to infinity in the direction AF.

To find the vertex of this parabola, draw through M, the centre of the Earth, a parallel to AF, and bisect at B the line AC between the initial point A and the point C where the initial tangent meets this parallel, and draw the perpendicular BD on CM; the foot D of this perpendicular is the vertex of the parabola. (In other words, D is the mid-point of MC.)

§ 7. Collection of the formulae for motion of a projectile in a vacuum, with constant acceleration of gravity g:

v_0 = initial velocity;

ϕ = initial angle of departure, or the angle between the initial tangent of the path and the horizontal;

g = acceleration of gravity, given in Table 2 in Volume IV;

x, y = coordinates of the shell after t seconds, referred to a rectangular coordinate system through the point O;

v = velocity of the shell at any given point (xy);

θ = angle of slope with the horizon at the point (xy);

ω = acute angle of descent;

X = horizontal range through the muzzle; x_s, y_s the coordinates of the vertex, v_s the velocity at the vertex, t_s the time of flight to reach the vertex;

E = angle of slope of the ground;

$\phi_1 = \phi - E$ the angle between the initial tangent and the slope of ground;

$h = \frac{v_0^2}{2g}$, for which refer to Table No. 1 a in Volume IV.

1. For any point of the trajectory.

Horizontal distance

$$x = v_0 t \cos\phi = \frac{v_0^2 \cos^2\phi}{g}(\tan\phi - \tan\theta)$$

$$= \frac{v_0^2}{2g}\sin 2\phi \pm \frac{v_0}{g}\cos\phi\,\sqrt{(v_0^2\sin^2\phi - 2gy)}$$

$$= \frac{v_0^2}{2g}\sin 2\phi \pm \frac{v_0^2}{g}\cos^2\phi\,\sqrt{\left(\frac{v^2}{v_0^2\cos^2\phi} - 1\right)};$$

Vertical height

$$y = x\tan\phi - \frac{gx^2}{2v_0^2\cos^2\phi} = v_0 t\cos\phi - \tfrac{1}{2}gt^2 = \tfrac{1}{2}gt\,(T-t)$$
$$= \frac{v_0^2 - v^2}{2g} = \frac{v_0^2}{2g}\cos^2\phi\,(\tan^2\phi - \tan^2\theta) = x\left(1 - \frac{x}{X}\right)\tan\phi;$$

Slope of the tangent

$$\tan\theta = \tan\phi - \frac{gt}{v_0\cos\phi} = \tan\phi - \frac{gx}{v_0^2\cos^2\phi} = \pm\frac{1}{v_0\cos\phi}\sqrt{(v_0^2\sin^2\phi - 2gy)};$$

Time of flight

$$t = \frac{x}{v_0\cos\phi} = \frac{v_0^2}{g}\cos^2\phi\,(\tan\phi - \tan\theta) = \frac{v_0}{g}\sin\phi \pm \frac{1}{g}\sqrt{(v_0^2\sin^2\phi - 2gy)}$$
$$= \frac{v_0}{g}\sin\phi \pm \frac{v_0}{g}\cos\phi\sqrt{\left(\frac{v^2}{v_0^2\cos^2\phi} - 1\right)};$$

Velocity

$$v = \frac{v_0\cos\phi}{\cos\theta} = \sqrt{(v_0^2 - 2gy)} = v_0\cos\phi\sqrt{\left[1 + \left(\tan\phi - \frac{gt}{v_0\cos\phi}\right)^2\right]}$$
$$= v_0\cos\phi\sqrt{\left[1 + \left(\tan\phi - \frac{gx}{v_0^2\cos^2\phi}\right)^2\right]}.$$

2. Vertex.

Horizontal distance

$$x_s = \frac{v_0^2}{2g}\sin 2\phi = h\sin 2\phi;$$

Vertical height

$$y_s = \frac{v_0^2}{2g}\sin^2\phi = h\sin^2\phi = \frac{gx_s^2}{2v_0^2\cos^2\phi} = \tfrac{1}{2}x_s\tan\phi$$
$$= \tfrac{1}{2}gt_s^2 = \tfrac{1}{8}gT^2 = 1{\cdot}23T^2;$$

Time of flight

$$t_s = \frac{v_0}{g}\sin\phi = \frac{x_s}{v_0\cos\phi} = \frac{1}{g}\sqrt{(x_s\tan\phi)} = \tfrac{1}{2}T;$$

Velocity $v_s = v_0\cos\phi$;

Average height of flight: $y_m = \tfrac{2}{3}y_s = 0{\cdot}816T^2$;

Greatest height (with $\phi = 90°$) $= \dfrac{v_0^2}{2g} = h$.

3. Point of fall.

Range

$$X = \frac{v_0^2}{g}\sin 2\phi = 2h\sin 2\phi = v_0 T\sqrt{\left(1 - \frac{g^2T^2}{4v_0^2}\right)}$$
$$= \tfrac{1}{2}gT^2\cot\phi = 2x_s;$$

Maximum X (for $\phi = 45°$) $= \frac{v_0^2}{g} = 2h$;

Time of flight

$$T = \frac{2v_0}{g}\sin\phi = \frac{X}{v_0\cos\phi} = \sqrt{\left(\frac{2X}{g}\tan\phi\right)} = \sqrt{\frac{8y_s}{g}}$$

$$= \frac{1}{g}[\sqrt{(v_0^2 + gX)} \pm \sqrt{(v_0^2 - gX)}];$$

Velocity, $v_e = v_0$; acute angle of descent, $\omega = \phi$.

4. Angle of departure to hit a given mark (a, b) with given v_0:

$$\tan\phi = \frac{2}{a}\left\{\frac{v_0^2}{2g} \pm \sqrt{\left[\frac{v_0^2}{2g}\left(\frac{v_0^2}{2g} - b\right) - \frac{a^2}{4}\right]}\right\}$$

$$= \frac{2}{a}\left\{h \pm \sqrt{\left[h(h-b) - \frac{a^2}{4}\right]}\right\};$$

(+ for high angle, − direct fire).

5. Initial velocity to reach the mark (a, b) with given departure angle ϕ:

$$v_0 = \sqrt{\left[\frac{ga\cos E}{2\sin(\phi - E)\cos\phi}\right]}, \text{ where } \tan E = \frac{b}{a}.$$

6. Range W on sloping ground at angle of slope E:

$$W = \frac{2v_0^2}{g}\cdot\frac{\sin(\phi - E)\cos\phi}{\cos^2 E} = \frac{2v_0^2}{g}\frac{\sin\phi_1\cos(\phi_1 + E)}{\cos^2 E};$$

Time of flight

$$T_\omega = \frac{2v_0}{g}\cdot\frac{\sin(\phi - E)}{\cos E} = \frac{2v_0}{g}\frac{\sin\phi_1}{\cos E}.$$

7. Parabola enveloping all trajectories with given v_0:

$$y = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

CHAPTER II

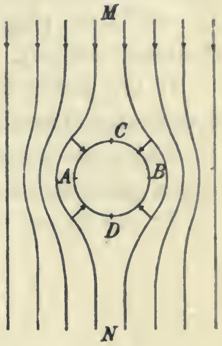
On air-resistance

I. Air-resistance to an elongated shell on the assumption that its axis lies in the direction of motion of the centre of gravity.

§ 8. General considerations.

Consider a sphere $ABCD$ at rest, in a current of air or fluid moving with given velocity, and assume that the air is frictionless.

The stream lines along which the separate air particles are moving will diverge at the front, and on the rear will converge again.



(This last can be shown through the familiar experiment of holding before the mouth a cylindrical flask about 15 cm in diameter, and behind it a burning candle; the candle can be blown out.)

We arrive therefore at the following conclusions:

On the front side ACB of the sphere a thrust will be exerted in the direction MN of the current; and an equal thrust on the rear side of the sphere, in the opposite direction.

The resultant total thrust on the sphere is zero; the sphere experiences no push.

The same will hold when the air is at rest, and the sphere is moving in the direction NM.

On the front side ACB of the sphere work is done, and the air particles alter their velocity and direction.

On the rear side ADB, the direction and velocity will be the same again; the work done is restored.

In a similar way, the pier or pile of a bridge standing in water, or a rudder moving through the water will feel no resistance.

This result of theoretical Hydromechanics for a frictionless fluid stands however in well-known contradiction to experience.

In reality the nature of the air motion round the shell is not so simple.

In the first place there is friction of the air particles among themselves and against the rigid body.

Friction acts so as to break up the air on the rear side of the body, and eddies are formed.

These are shown clearly behind a stick moved through water, or in air full of smoke through which a body is moving.

Secondly, wave-motion comes in. Mach showed by photography the waves and eddies following a flying bullet and made an important advance in the theory of air resistance to a bullet.

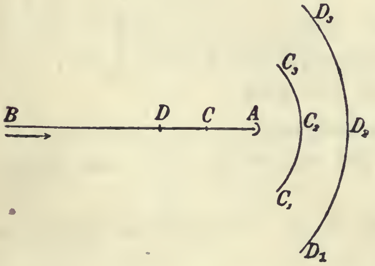
Take a long cylindrical tube, and suppose a plug to move in it with a velocity of 167 m/sec, and follow up this motion at small intervals of time.

A condensation of the air is set up in front of the plug; and this condensation is propagated with the velocity of sound, viz., 334 m/sec. In rear of the plug a wave of rarefaction is propagated backward with the same velocity.

If the piston is supposed to move with uniform velocity, a condensation and rarefaction is made anew in every small element of time.

The same happens when a shell moves in free air; only the air waves going forward and backward spread out in spherical form.

In the figure the shell is represented as a rod AB moving in the direction BA with velocity 167 m/sec.



At the front end A a spherical wave of condensation is started; its radius is zero.

A moment ago the point of the shell was at C, and the condensation air wave that started at this moment from the point has spread out with double the shell velocity, that is at 334 m/sec.

The radius $CC_2 = CC_1 = CC_3$ of the wave-surface $C_1C_2C_3$ is thus double as great as the advance CA.

Earlier still, the point of the shell was at D ($DC = CA$); and the condensation wave sent out is now the sphere $D_1D_2D_3$, of radius $DD_2 = 2DA$; and so on.

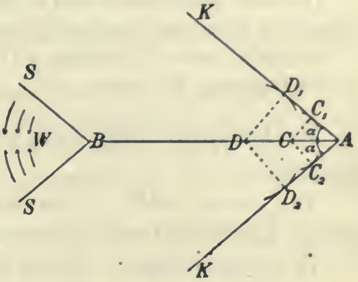
We see then that the condensation air waves must travel faster than the shell.

So also with the rarefaction waves that proceed from the rear end of the shell, and are propagated backward in a spherical form with double the shell velocity.

But if the shell velocity v is increased and at last becomes greater than the normal sound velocity s (for instance if $v = 668$ m/sec), the condensation air waves set up by the point of the shell cannot go faster than the shell but accompany it as conical waves.

Suppose again that the shell is regarded as an infinitely thin rod AB, and that the point of the shell is now at A.

A moment ago, the point was at C, and from C a condensation spherical wave is propagated with the sound velocity of 334 m/sec, and the wave-surface is now a sphere with radius $CC_1 = CC_2 =$ half the advance CA of the shell. Previously the point of the shell was at D; and the spherical wave produced at this time is a sphere which has spread out to a radius $DD_1 = DD_2$; and so on.



The wave-surfaces of the group of elementary waves, proceeding from the point of the shell at the different positions, will thus be enveloped by the cone KK (called the head wave), of which the angle of opening is 2α.

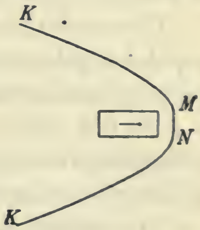
Similarly for the conical stern wave SS.

Mach supposes that $\sin \alpha = \frac{s}{v}$.

For in the same time as the head wave spreads out from D to D_1 with the sound velocity s, the head of the shell advances from D to A.

Actually the head wave, in consequence of the finite cross-section, is flattened in front. At this point MN, $\alpha = 90°$, and so $v = s$; the condensation of the air is propagated here with the velocity v, owing to the motion of the shell.

(The fact is that the velocity of sound is only in ordinary circumstances equal to about 334 m/sec; this is modified by the temperature of the air, and the velocity of sound s can assume a higher value according to the strength and nature of the air waves; this has been proved by Mach by a series of experiments; compare § 87.)



This region MN in which the surface of the head wave runs forward very nearly flat and perpendicular to the axis of the shell, depends on the breadth and flatness of the head of the shell.

If then the velocity of the shell is to be determined by means of the equation $\sin \alpha = \frac{s}{v}$, from the angle α of the head wave, this must be measured from the straight part of the boundary of the wave.

Out of 20 photographs of a flying bullet, which were produced in the ballistic laboratory, the velocity of the bullet was given by the mean value of 893·0 m/sec, with a probable deviation from the mean of 1·1 °/$_\circ$, while with the Boulengé Chronograph a mean velocity was measured of 888·3 m/sec.

The same result was found with 20 rounds of the bullet M. 88, with a mean velocity of 655 m/sec; against a simultaneous Boulengé record 640 m/sec.

So also 20 shots of the bullet of the Mauser pistol; mean 454 m/sec; measured by the Boulengé, 467 m/sec.

Evidently the head wave and also the tail wave must make closed surfaces.

The vertex MN of the head wave lies nearer the head of the bullet, as the velocity increases.

In the new infantry bullet, with sharp point and very great velocity, the head wave seems to begin somewhat behind the point, so that the point of the bullet pierces air at rest; only immediately in front of the point there exists a strong condensation of the air.

At the rear end of the bullet, eddies of air are formed. These eddies can be seen by photography extending many metres behind the bullet.

All these effects are revealed in water also, with ships, the piles of a bridge, and so forth; only here the head wave is made up of a large number of parts of waves, which does not appear to be the case with the air wave of the bullet, as given by microscopic examination.

The reason for this may well lie in the fact that in water no shock can arise with a single wave of elevation, while in the air there is a single wave of condensation.

For water the analogy to sound velocity is the velocity depending on the depth of water, with which the corresponding water waves propagate themselves.

We can observe that as a ship is moved through the water with increasing velocity, the crest of the head wave moves from the bow of the ship and retreats further towards the midship.

The analogy between a ship on water and a bullet goes further still:

Denote the resistance of a ship for any velocity v by $W(v)$, and plot graphically the value of $\frac{W(v)}{v^2}$ as a function of v determined by observation; a curve is obtained which, according to Schütte, Lang and Lorenz, assumes a shape similar to that of the analogous function for air resistance (see § 10).

The curve has an inflexion in the neighbourhood of the velocity of propagation of the water waves.

The same holds for the resistance W to a bullet for the different velocities v in the air: an inflexion lies in the corresponding curve in the neighbourhood of the normal sound-velocity s.

H. Lorenz supposes that a resonance phenomenon occurs here, such as is well known in other cases.

Consider the transference of energy between two swinging pendulums; or a otating motor, suspended as a pendulum; the sympathetic vibration of tuning forks and strings or the resonance of electric waves; the swinging of the body of a ship with the masses of machinery moving periodically in time with the ship-vibration; and so forth.

The energy of the bullet would be transferred to the air chiefly at a velocity which coincides very nearly with the natural velocity of propagation of the air-waves.

N. Mayevski appears to have determined the fact empirically for bullets, that the coefficient $K=\frac{W}{v^2}$ experiences a rapid increase in the neighbourhood of the sound-velocity s.

A. Indra sought to elucidate these facts by supposing that the energy of the bullet was consumed in the progressive generation of new head-waves.

Nevertheless it is not clear why the coefficient K diminishes again somewhat when the bullet velocity increases still further, although bullet-energy is expended through wave-making at all velocities. These explanations of the inflexion are not complete.

In the first place the inflexion, when it does exist, is not at $v=334$, but more nearly $v=480$ m/sec; and it will be found that especially with bullets of pure cylindrical shape there is no inflexion at all. The true reasons for the existence of the inflexion will be given later.

The analogy between ship- and bullet-motion shows that the form of the rear end of the bullet should receive more attention than hitherto has been the case.

Moreover it is evident that the movement of the bullet in air is quite as complicated as that of a ship on water, and consequently these simple laws of measurement will not apply entirely: Assuming that the long axis of the bullet lies in the direction of motion of the centre of gravity, and that the bullet flies like an arrow, the air-

3—2

resistance W to the bullet may be taken as being proportional to the following magnitudes:

(*a*) to the cross-section of the bullet, perpendicular to the axis,

$$R^2\pi \ (\text{m}^2),$$

(*b*) to the air-density δ, that is, the weight of a cubic metre of air, calculated from the temperature, pressure, and humidity of the air.

Generally the density δ is taken relatively to an arbitrarily assumed normal air-density δ_0, for example, $\delta_0 = 1{\cdot}206$, or $\delta_0 = 1{\cdot}220$, kg/m³,

(*c*) to a coefficient i depending on the shape of the bullet ($1000i = n$ is commonly called the form-value),

(*d*) to a certain function of the velocity $f(v)$, so that the air-resistance

$$W\,(\text{kg}) = R^2\pi \, . \, \frac{\delta}{\delta_0} \, . \, i \, . \, f(v).$$

Assumptions (*a*), (*b*), (*c*) are merely conventional, and seem reasonable.

A priori it is not likely that the air-resistance, as depending on calibre $2R$, air-density δ, i and v, should involve these four variables as factors of a product. Assumption (*a*) asserts that the air-resistance on unit area of the cross-section of the shell is the same for the same shape and velocity of the shell, whether a large or small calibre of the shell is considered.

In his experiments of 1848, Didion had already discovered the fact that the air-resistance to a bullet of small cross-section is relatively greater than that to a bullet of large cross-section; and he has sought to take this fact into account numerically by multiplying $R^2\pi$ by the factor

$$\left(0{\cdot}74 + \frac{0{\cdot}047}{0{\cdot}05 + 2R}\right).$$

(On the other hand, later in 1860, he abandoned this assumption.)

Some suppose, from their experiments on air-resistance with bullets of very different calibres, that proportionality does exist between air-resistance W and cross-section $R^2\pi$, and apply the results obtained from artillery shell to infantry bullets.

In reality this simple relation does not exist closely enough.

The new theory brought out by H. Lorenz on ship-resistance, which is a generalisation that was verified in its extension to the air-resistance of shells, leads equally to the same result, that a small cross-section experiences a relatively greater resistance than larger cross-section.

According to the experiments on the Tower Bridge (J. W. Barry) a large area of 100 m² experiences only one-sixth part of the specific resistance of an area of 0·1 m².

The experiments of the firm of Fr. Krupp in 1912 with shells of different form gave among other things the following results:

The resistance on the cross-section, estimated on the square metre, and with an air-density of 1·22 kg/m³, is

(*a*) for cylindrical shell:

	for $v=400$	500	600	700	800 m/sec
of 6·5 cm calibre	1·4	$2{\cdot}5_8$	3·8	$5{\cdot}1_5$	6·6 kg/m²
„ 10 „	$1{\cdot}2_9$	2·2	3·3	4·7	6·3 „

(*b*) for ogival shell, struck to 3 calibre radius:

	$v=550$	650	750	850	m/sec
for 6 cm calibre	1	1·3	$1{\cdot}5_8$	$1{\cdot}9_4$	kg/m²
„ 10 „	0·98	$1{\cdot}2_5$	$1{\cdot}5_2$	$1{\cdot}8_5$	„
„ 28 „	0·62	$0{\cdot}8_1$	$1{\cdot}0_1$	$1{\cdot}2_5$	„
„ 30 „	—	—	0·9	$1{\cdot}0_6$	„

The air-resistance distributed over $R^2\pi$ is thus in fact, as Didion found, less in the greater calibre than in the smaller calibre, and moreover independent of the velocity. This influence of the calibre comes consequently into consideration.

The ordinary assumption then does not hold, that the air-resistances to two equally large elements of surface of the head of the shell, equally inclined to the axis of the shell, at equal velocities are equal, whatever the distance of the surface element from the axis of the shell.

This appeared likely from the experiments of Mach. In his photographic work he determined experimentally the deviation of light by the penetration of different strata of air in the immediate neighbourhood of the bullet.

With rifle bullets of 11 mm calibre and 520 m/sec velocity (for which in Krupp's experiments the air-resistance would be one extra atmosphere) Mach found the following: in the vertex of the head wave the density of the air corresponds to about 3 atmospheres; 4·5 mm behind the vertex, 12 mm from the axis of the bullet, and

3 mm distant from the edge of the head wave, about 1·7 atmospheres of density; 7·5 cm behind the vertex, 9 cm from the axis of the bullet and 7·5 cm distant from the edge of the wave, about 1·6 atmospheres.

The fact that the air-resistance to the shell increases and decreases exactly in proportion to the air-density cannot be proved experimentally in a way free from objection.

Assumptions (*c*) and (*d*) contain in themselves the assertion that, with equal calibre and equal air-density, but different shape of the shell, the character of the air-resistance remains unaltered, and so in shells of different shape it is only the ordinate of the curve $f(v)$ that alters.

This hypothesis that the influence of the shape of the shell can be expressed by a single factor i, has not been verified theoretically or experimentally.

In any case i is not constant, but is a function of other quantities, on which it really depends, although it alters only slowly with these quantities.

In general most of the difficulties over the research with respect to air-resistance relate to the dependence of this resistance W on the velocity v of the centre of gravity of the shell for equal calibre, equal air density and the same shape of the shell.

§ 9. Theoretical considerations of the law of air-resistance.

Newton's considerations seem to rest on the following ideas.

If the velocity of the shell v is increased, then not only is the acceleration of the air particles increased but also the mass of air set in motion: the resistance of the air is thus much greater.

Consider a cylindrical shell at rest, of the cross-section $AB = F$ m² against which the air flows in the direction of the axis with velocity v m/sec.

Let the velocity v be represented by the vector $AC = BD$. The air particles, which at first were to be found in the line CD, soon arrive at AB, and lose their velocity by impact against the shell. The weight of a cubic metre of air being δ kg, this mass $\frac{Fv\delta}{9{\cdot}81}$ of air is discharged against the shell in one second.

The alteration of velocity in one second is v and the retardation is v also.

The thrust experienced by the cross-section AB from the impinging air is the product of the mass and the retardation of the colliding volume of air, and so is

$$\frac{Fv\delta}{9{\cdot}81}\cdot\frac{v-0}{1}=\frac{F\delta v^2}{9{\cdot}81}.$$

Consequently the resistance in still air to a moving cylinder is proportional to the cross-section F (in m²), the air-density δ (in kg/m³) and the square of the velocity v (in m/sec):

$$W\,(\text{kg})=\frac{F\delta v^2}{9{\cdot}81}.$$

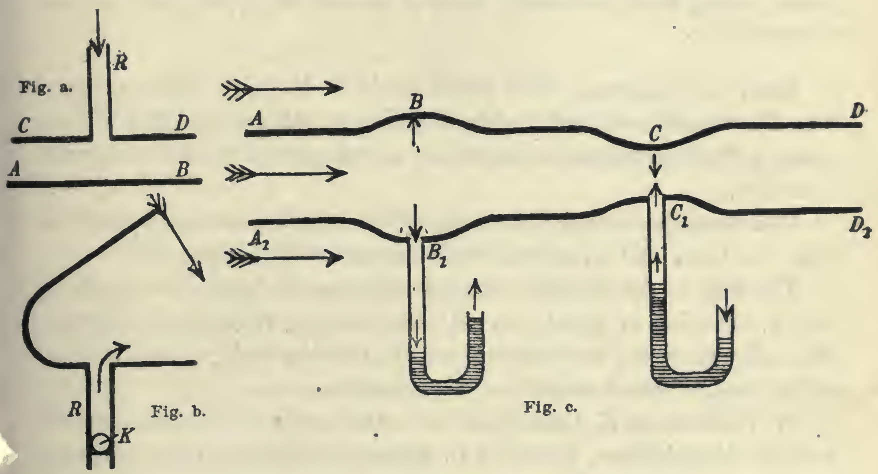
(The formula used in practice for the wind pressure W on a perpendicular plane surface F by a wind velocity v m/sec is in fair agreement, in that

$$W=0{\cdot}122\,Fv^2.)$$

It must be added that the effects produced by the air flowing away from the stern of the body, friction, wave-making, and all that occurs on the rear of the body are left out of account.

The Newtonian theory may be studied in the light of the following experiments.

Place a plane circular plate AB (figure a), about 30 cm in diameter, on one



scale of a balance. Superpose a fixed parallel plate CD, about 20 cm in diameter, having in the centre a circular hole which carries a tube R, through which air can be blown in the direction of the arrow.

If CD is 20 to 10 cm above AB, the movable plate AB will be driven away by the blast of air.

On the other hand if the distance between the plates AB and CD is about 1 cm, the lateral outflow of the air between the plates creates a defect of pressure, and the plate AB will be lifted.

When in the apparatus of figure *b* we blow downwards in the direction of the arrow, a ball on a small stick across the tube R will be drawn out, and fly in the direction of the curved arrow.

Figure *c* shows a tube $ABCDA_1B_1C_1D_1$, provided with an enlargement at BB_1, and a contraction at CC_1. A stream of air is blown through it in the direction of the longer arrows.

Inside the tube the air stream experiences at BB_1 an increase of cross-section, and so a diminution of velocity; the consequence is an excess of pressure (represented by the smaller arrows). At CC_1 there is a decrease of cross-section and an increase of velocity; a drop of pressure ensues (a sucking action in the manometer tubes).

The reverse is the case outside the tube.

The pure Newtonian impulse pressure theory cannot then suffice when, for example, we require to know the thrust experienced by the roof of a building in a horizontal gale of wind.

On the lower part of the roof it is true there will be in general an increase of thrust due to the impulse pressure; but higher up a negative pressure can occur from the flowing of the air over the ridge of the roof, and so a tilting moment is possible. In fact a lifting of the roof has sometimes been observed in a storm.

In a similar way it is possible with shell in flight for this sucking action to be present, arising from the external shape of the fuse, the driving band, the base of the shell, etc.

Many investigators, more particularly O. Mata, in 1895, assumed that the transference of the shell-energy to the surrounding air was solely a thermodynamical effect, or an alteration of an isothermal condition.

But that this is only one side of the explanation appears from what has been said already about wave and eddy making.

The first to consider the creation of air waves by the moving body was A. Schmidt, of Stuttgart. He assumed the resistance to become discontinuous when the velocity v of the moving body was made equal to the normal sound-velocity s of the medium.

P. Vieille and E. Ökinghaus are others who have discussed the laws of air-resistance, based on B. Riemann's theory of the propagation of air-waves of finite amplitude, by considering special shells with a flat head-surface perpendicular to the axis, and assuming that the head wave accompanying the shell may be taken as flat.

The propagation-velocity of the change of density of the air must thus be assumed to be the same as the shell velocity.

Vieille's relation between air pressure p and the velocity of the shell is

$$v^2 = \frac{gp_0}{2\delta}\left\{2k + (k+1)\frac{p-p_0}{p_0}\right\},$$

where δ is the weight in kg of 1 m³ of the surrounding air, p_0 the normal atmospheric pressure in kg/m², p the air pressure produced by the motion in kg/m², $k = 1{\cdot}41$ the ratio of the two specific heats, $g = 9{\cdot}81$.

For example, for $p_0 = 10333$, $\delta = 1{\cdot}206$, and $p = 15{\cdot}64$ atmospheres $= 15{\cdot}64 \times 10333$, we find $v = 1200$ m/sec.

In the following table the resistance W is given as calculated by him, and finally the temperature calculated by him for velocities above 1200.

Velocity v in m/sec	Resistance W in atmospheres		Temperature Centigrade
	calculated	observed	
400	1·58	1·25	—
800	6·85	6·23	—
1200	15·64	15·01	680
2000	43·8	—	1741
4000	175·6	—	7751
10000	1098	—	48490

The corresponding formula of E. Ökinghaus is

$$W = W_0 k\left[1 - \left(\frac{W_0}{W}\right)^{\frac{1}{k}}\right]\left(\frac{v}{340}\right)^2 + W_0.$$

Here W is the air-resistance for the velocity v, W_0 the special value of the air-resistance for $v = 340$ m/sec, k the ratio of the specific heats of air $= 1{\cdot}41$.

Since Riemann's theory holds only for plane waves, these developments apply mainly to shells with flat heads.

Nevertheless it seems as if a revision of the Riemann theory of air reaction might lead to results.

The theory of peripheral motion, enunciated lately by W. Lanchester for the motion of gliding flyers in the air, especially for surfaces like a bird's-wing, cannot well be of use in ballistics. In this theory a uniform streaming motion of the air is supposed, together with a circular motion of proper direction and strength.

So too the comprehensive calculations carried out by Colonel P Haupt and based on the kinetic gas-theory, are not of great importance for ballistics.

The complicated air-resistance is not satisfactorily represented by his calculations, and they have led to results differing from the latest experimental records.

On the other hand a reference may be made to two important theoretical researches on air-resistance by H. Lorenz and A. Sommerfeld. H. Lorenz has sought to express mathematically the complicated motion about a shell in flight, and thereby he has been led to the following relation for the air-resistance. He denotes by v the shell-velocity, by $R^2\pi$ the greatest cross-section of the shell, s the velocity of sound, l the length of the shell; then

$$W = k_1 R^2 \pi v^2 + k_2 l v + \frac{k_3 R^2 \pi v^4 + k_4 l v^3}{\sqrt{[(s^2 - v^2)^2 + k_5 l^2 v^2]}};$$

where k_1, k_2, k_3, k_4, k_5 are constants, of which k_1 and k_3 depend only on the shape of the shell, the others on the shape and the nature of the surface.

This expression for W, determined theoretically, against which many considerations may be urged, shows that W is not proportional to the cross-section $R^2\pi$, but that the specific resistance, $W : R^2\pi$, is the greater, the smaller the cross-section becomes.

Further according to this law W is not proportional to a single form-coefficient i, but five coefficients appear in the form of the expression, in such a way that the whole air-resistance function $f(v)$ depends on the shape and nature of the surface of the shell.

Finally the factor $W : v^2 = k$, in a graphical representation has an inflexion in the neighbourhood of the sound-velocity, and otherwise agrees with Siacci's results.

Consequently the law of Lorenz appears to be a suitable basis for considering questions relating to air-resistance.

H. Lorenz has shown lately how the coefficients arising in his formula can be calculated, based on the results of experiment.

It will be worth while then to determine if the Lorenz theory covers sufficiently the results of the latest experiments on air-resistance, and if it is capable of practical application.

A. Sommerfeld assumes the air-resistance W to be composed of the frictional resistance W_1 (in its extended sense), which he considers to be proportional to v^2, and of the wave-resistance W_2.

For this last he obtains an expression by employing the analogy of the electromagnetic field, as it exists when an electron moves in it with velocity exceeding the velocity of light.

Denoting by s the velocity of sound, then for $v < s$, we have $W = W_1 = a_3 v^2$; but for $v > s$, we have

$$W = W_1 + W_2 = av^2 + A\left(1 - \frac{s^2}{v^2}\right).$$

The curve $W : v^2$ agrees with the latest empirical results.

The inflexion in the curve is explained because in the neighbourhood of $v = s$, the wave-resistance W_2 increases; but this resistance increases with v at a smaller rate than the frictional resistance W_1; and the ratio $W_1 + W_2 : W_1$ diminishes again with increasing v.

The shape of the shell is not considered by Sommerfeld.

On the whole then it can be asserted that no completely satisfactory and universal law of air-resistance has been enunciated.

On this account we are compelled in ballistics to confine ourselves mainly to experimental results.

The expression for the air-resistance for any given calibre, any air-density and any form of shell must be a function of the velocity (and perhaps also of the acceleration), which represents the following elements.

1. Suction and eddy resistance.

With increasing velocity a region of attenuated air is formed behind the shell (similar to what is observed in water with a moving plate).

The corresponding energy is dissipated partly as heat and partly as energy of motion.

The resistance depends essentially on the shape of the shell. In the neighbourhood of the velocity of sound it is seen to increase rapidly; but further on, it approximates more and more to a fixed limiting value, given by an absolute vacuum behind the shell.

2. Wave resistance. This arises at all projecting parts of the shell, especially at the head; yet this only starts when the velocity exceeds the sound-velocity.

It increases with the velocity, and finally becomes proportional to the square of the velocity.

The energy absorbed in this part of the resistance disappears as wave energy in the form of sound.

3. Frictional resistance. This appears to be relatively small in the case of shell in use.

4. Finally a complete law of air-resistance must give information as to the manner in which air-resistance alters when the axis of the

shell, assumed hitherto to lie in the direction of motion of the centre of gravity, makes a given angle with this direction.

From the preceding it follows among other things that the shape of the head of the shell is of importance, especially at high velocities (over 300 m/sec), further that the shape of the rear end of the shell is important at low velocities (under 300 m/sec), but that this influence approaches a limit with increasing velocity.

§ 10. Laws of air-resistance obtained experimentally, and the corresponding experiments.

(*a*) With very small velocity v of the moving body, as for instance in slow pendulum oscillations, the air-resistance according to M. Thiesen is proportional to the first power of v. This is of no importance for our purposes.

(*b*) For velocities up to about 30 m/sec the quadratic law of air-resistance is universally employed: the air-resistance W against a cylindrical body at velocity v m/sec, with a plane end surface of $R^2\pi$ m², in air of density δ kg/m³ is $W = kR^2\pi\delta\frac{v^2}{1\cdot22}$.

For the value of the numerical function k Poncelet and Didion took $k = 0\cdot081$;

F. le Dantec	0·080;	P. C. Langley	0·085;
Ch. Renard	0·085;	Canovetti	0·090;
J. Weissbach	0·093;	J. Smeaton	0·122;
F. v. Lössl	0·106;	O. Lilienthal	0·125;
E. J. Marey	0·125;		

G. Kirchhoff (from theoretical calculation) 0·055.

For throwing a stone, the value would be about

$$W(\text{kg}) = 0\cdot08R^2\pi\delta\frac{v^2}{1\cdot22}.$$

(*c*) The velocities of shell range from $v = 50$ m to $v = 1500$ m/sec.

Some 27 empirical laws have been proposed, of which the most part take the form of $W = av^n$, or $W = av^m + bv^n + \ldots$.

Since Mayevski, the whole range of velocity is consequently divided into "zones" such that from one zone to the other either a or n or both would be altered in the law

$$W = av^n.$$

The calibre of the shell will be denoted by $2R$ metres; the weight of the shell by P (kg); the air-density by δ (kg/m³); the velocity of the shell by v (m/sec); the air-resistance by W (kg); the retardation due to air-resistance, that is the ratio $\frac{Wg}{P}$, by $cf(v)$. Here $f(v)$ denotes the factor depending on the velocity v in the expression for the retardation of the shell.

1. Didion's law, based on the experiments of the Metz Committee of 1839–40, using the ballistic pendulum, as well as the experiments of the Metz Committee of 1856–58, using the Navez apparatus; for spheres, with $i = 1$,

$$W = 0{\cdot}027 \,.\, R^2\pi\delta i \frac{v^2}{1{\cdot}208}\left(1 + \frac{v}{435}\right);$$

and

$$c = \frac{0{\cdot}027 \,.\, R^2\pi\delta g i}{1{\cdot}208P}; \quad f(v) = v^2\left(1 + \frac{v}{435}\right).$$

2. St Robert, in Italy; according to the Metz experiments of 1839–40; for spheres, holding good with $i = 1$,

$$W = 0{\cdot}0387 \,.\, R^2\pi\delta i \frac{v^2}{1{\cdot}206}\left[1 + \left(\frac{v}{696}\right)^2\right].$$

3. N. Mayevski, in Russia, according to Russian and English experiments 1868–9;

(*a*) for spheres, with $i = 1$:

$$W = 0{\cdot}012 \,.\, R^2\pi\delta i \frac{v^2}{1{\cdot}206}\left[1 + \left(\frac{v}{186}\right)^2\right], \quad \text{for } 0 < v < 376 \text{ m/sec},$$

$$W = 0{\cdot}061 \,.\, R^2\pi\delta i \frac{v^2}{1{\cdot}206}, \quad \text{for } 376 < v < 530 \text{ m/sec},$$

(*b*) with $i = 1$, for elongated shell with "ogival" point, struck to a radius of 1 to 1·5 calibres (ogival point means a longitudinal section like the pointed window of a church):

$$W = 0{\cdot}012 \,.\, R^2\pi\delta \frac{v^2}{1{\cdot}206}\left[1 + \left(\frac{v}{488}\right)^2\right], \quad 0 < v < 280 \text{ m/sec},$$

(11)

$$W = 0{\cdot}026 \,.\, R^2\pi\delta \frac{v^6}{1{\cdot}206}, \quad 280 < v < 360 \text{ m/sec},$$

$$W = 0{\cdot}044 \,.\, R^2\pi\delta \frac{v^2}{1{\cdot}206}, \quad 360 < v < 510 \text{ m/sec}.$$

4. Hélie gave the following values for spheres, based on the French experiments:

$$W = \kappa (2R)^2 \delta \frac{v^2}{g},$$

where

for v = 50 m/sec,	κ = 0·130	for v = 300 m/sec,	κ = 0·269
100	132	320	293
120	135	340	316
140	139	360	337
160	146	380	353
180	154	400	367
200	166	420	376
220	181	440	382
240	200	460	386
260	221	500	389
280	244	$v > 500$	390

5. F. Bashforth (England) according to some experiments of 1866–70: for elongated shells with ogival point rounded to a radius of about 1·5 calibres, for $i = 1$,

$$W = mR^2\pi\delta i \frac{v^3}{1{\cdot}206},$$

where

m = 0·000068	0·000075	0·000082	0·000090
for v = 600 to 550	550 to 500	500 to 460	460 to 419 m/sec

m = 0·000094	0·000084	0·000060
for v = 419 to 375	375 to 330	330 to 50 m/sec

To make the results hold good for the former Krupp normal shell, struck to a radius of 2 calibres, i must be taken = 0·896, according to Siacci.

6. Hojel (Holland), according to Dutch experiments of 1884, as well as Krupp's experiments: with $i = 1$, for elongated shell with ogival head of a radius of 2 calibres,

$$W = (2R)^2 \frac{1000\,\delta i}{9{\cdot}81 \times 1{\cdot}206} mv^n,$$

where

for v = 140 to 300 m/sec,	$m = \overset{(6)}{0{\cdot}084535}$,	n = 2·5
300 „ 350	$\overset{(11)}{0{\cdot}05423}$	5
350 „ 400	$\overset{(8)}{0{\cdot}051381}$	3·83
400 „ 500	$\overset{(4)}{0{\cdot}07483}$	1·77
500 „ 700	$\overset{(3)}{0{\cdot}05467}$	1·91

7. Mayevski in 1881, and N. Sabudski (Russia); from $v = 550$ m/sec based on the experiments of Krupp 1875–81, the English experiments of Bashforth 1866–70, and the Russian experiments of Mayevski 1868–69. With $i = 1$, for ogival shell rounded to radius of 2 calibres,

$$W = mR^2\pi\delta i \frac{v^n}{1\cdot206};$$

where

for $v =$ (0) to 240 m/sec,	$m = 0\cdot0140$,	$n = 2$
240 „ 295	$0\cdot\overset{(4}{0}5834$	3
295 „ 375	$0\cdot\overset{(9)}{0}6709$	5
375 „ 419	$0\cdot\overset{(4)}{0}9404$	3
419 „ 550	0·0394	2
550 „ 800	0·2616	1·7
800 „ 1000	0·7130	1·55

8. Laws of Chapel (1874), Vallier (1894), Scheve (1907); based especially on Krupp's experiments and Dutch experiments with ogival shell of 2 calibre radius: for $v > 330$ m/sec,

$$W = \frac{R^2 . 10000\delta i}{9\cdot81} \cdot \frac{0\cdot125}{1\cdot206}(v - 263);$$

for v between 330 and 300 m/sec,

$$\left.\begin{aligned} W &= \frac{R^2 . 10000\delta i}{9\cdot81} \cdot \frac{0\cdot\overset{(11)}{0}21692}{1\cdot206} v^5 \\ v < 300 \text{ m/sec, } W &= \frac{R^2 . 10000\delta i}{9\cdot81} \cdot \frac{0\cdot\overset{(5)}{0}33814}{1\cdot206} v^{2\cdot5} \end{aligned}\right\}.$$

According to Vallier, the coefficient i should be unity for 2 calibre radius of the ogival head, or for the semi-ogival angle $\gamma = 41^\circ\cdot5$. Otherwise i will be slightly variable if $v > 330$ m/sec; so that

$$i = \frac{\gamma[v - (180^\circ + 2\gamma)]}{41\cdot5(v - 263)}.$$

For $v < 330$ m/sec,

$$i = 0\cdot67, \quad 0\cdot72, \quad 0\cdot78, \quad 1\cdot10$$

for values of $\gamma = 31^\circ, \quad 33^\circ\cdot6, \quad 36^\circ\cdot9, \quad 48^\circ\cdot2.$

In the notation $W = \dfrac{R^2 . 10000\delta i}{9\cdot81} \cdot \dfrac{f(v)}{1\cdot206}$, and $$\frac{f(v)}{v^2} = K(v), \quad \frac{f(v)}{v^4} = K'(v),$$

the values are as follows:

v	$10^7 K(v)$	$10^{12} K'(v)$	v	$10^7 K(v)$	$10^{12} K'(v)$
150	415	1844	390	1043	687
160	430	1680	400	1070	669
170	441	1526	420	1108	628
180	452	1395	440	1143	590
190	466	1293	460	1165	550
200	478	1195	480	1178	511
210	491	1113	500	1185	474
220	502	1037	525	1187	431
230	515	974	550	1186	393
240	526	913	575	1180	357
250	535	856	600	1170	325
260	546	808	650	1145	272
270	558	765	700	1115	228
280	564	719	750	1082	192
290	578	687	800	1049	164
300	586	651	850	1016	140
310	645	667	900	983	121
320	708	671	950	952	106
330	769	706	1000	921	92
340	831	720	1050	892	81
350	888	724	1100	865	72
360	936	722	1150	838	64
370	977	714	1200	813	57
380	1013	702	1250	790	51

Therefore the function $K(v) = \frac{f(v)}{v^2}$ has its maximum at $v = 525$ m/sec.

9. Law of Siacci 1896. This combines all the experimental researches carried out so far, and holds for velocities up to

$$v = 1200 \text{ m/sec.}$$

Retardation $$= \frac{(2R)^2 1000 i\delta}{P \times 1{\cdot}206} f(v),$$

or $$W = 338 R^2 \delta i f(v),$$

where $$f(v) = 0{\cdot}2002\,.\,v - 48{\cdot}05 + \sqrt{[(0{\cdot}1648\,.\,v - 47{\cdot}95)^2 + 9{\cdot}6]}$$
$$+ \frac{0{\cdot}0442 v\,(v - 300)}{371 + \left(\frac{v}{200}\right)^{10}}.$$

The curve of the function $10^6 \frac{f(v)}{v^2}$ is given graphically in figure *b* on page 53. The curve has an inflexion at $v=340$ m/sec, and a point of maximum value at 500 m/sec.

With $i=1$ the law will hold for ogival shell with head of ogive from 0·9 to 1·1 calibre long.

If the law is to be applied to normal shell with 2 calibre for radius of rounding or 1·3 calibre for length of head, then according to Siacci $i = 0{\cdot}896$ is to be taken; but 0·865 is better.

10. The Krupp tables hold for a 2 calibre radius of rounding, and an air-density of 1·206 kg/m³.

The numbers, according to W. Gross, can only be considered approximately correct up to $v = 300$ m/sec, because at low velocity the errors of measurement have important effects.

The most reliable are the numbers which lie between 350 and 600 m/sec. Beyond these the vibration of the shell has a disturbing effect.

11. The latest and the most exact results of measurement are collected together in the following table of air-resistance given by the firm of Fr. Krupp (see pp. 51 and 52). This table is given:

(*a*) for the Krupp normal shell, that is, shell with 2 calibre radius of rounding of the ogival point, and with a flattening at the front to 0·36 calibre; here the air-resistance (kg) is

$$W = \frac{R^2\pi\delta i f(v)}{1{\cdot}22},$$

$R^2\pi$ is the cross-section in cm² (not in m²), δ the density of the air in kg/m³; $i=1$ for Krupp's normal shape; for shell of 3 calibre rounding radius, and 0·36 calibre of flattening,

$$\frac{1}{i} = 1{\cdot}3206 - \frac{58{\cdot}2}{v} - 0{\cdot}0001024v;$$

for shell of 5·5 calibre rounding radius and 0·36 calibre flattening,

$$\frac{1}{i} = 1{\cdot}4362 - \frac{73{\cdot}4}{v} - 0{\cdot}0001128v;$$

for shell of 3 calibre rounding radius and 0·26 calibre flattening,

$$\frac{1}{i} = 1{\cdot}1959 - \frac{40{\cdot}6}{v} + 0{\cdot}0001467v;$$

for shell of 3 calibre rounding radius and a sharp point,

$$\frac{1}{i} = 1{\cdot}1311 - \frac{47{\cdot}7}{v} + 0{\cdot}0003166v;$$

for bullets of the form of the S-bullet,

$$\frac{1}{i} = 1{\cdot}410 - \frac{122{\cdot}68}{v} + 0{\cdot}0005915v.$$

In Table A the numerical values of $10^6 \frac{f(v)}{v^2}$ are given as $10^6 K$.

For example, with $R^2\pi = 1$ cm², with $\delta = 1{\cdot}22$ kg/m³, the air-resistance per 1 cm² of the cross-section of an ogival shell of 2 calibre radius of rounding and 0·36 calibre of flattening of the point, at $v = 500$ m/sec

$$= (500)^2 \times 3{\cdot}998 \times 10^{-6} = 0{\cdot}999 \text{ (kg)}.$$

The experiments showed that the curve $f(v) : v^2$ has an inflexion at about $v = 480$ m/sec, and that at high velocity it appears to approach asymptotically a horizontal line; that is the quadratic law holds again at high velocity.

But the experiments showed further that the former assumption did not hold of an air-resistance W proportional to a single factor of shape, independent of the velocity v.

Speaking more strictly, for every form of shell another form of the air-resistance function $f(v)$ is required.

Ritter von Eberhard has now reached the point of splitting up with sufficient accuracy the air-resistance into two parts, of which one factor i depends on the velocity v and on the shape, and the other $f(v)$ depends on the velocity alone.

These values of i, or rather of $\frac{1}{i}$ are given above for many shapes of the shell, with the exception of the purely cylindrical shell.

(*b*) for cylindrical shell

$$W = \frac{R^2\pi\delta f(v)}{1{\cdot}22},$$

and the numbers for $10^6 \frac{f(v)}{v^2}$, denoted by $10^6 K$ for brevity, for such shell, are set down in Table B.

On p. 53, the results in figure *a* are for Krupp 10 cm normal shell and for cylindrical artillery shell: in figure *b* the results are for infantry bullets, and at the same time the results of Charbonnier and Siacci are given.

Figure *c* shows the variation of the resistance W itself, for the values of $\delta = 1{\cdot}22$ kg/m³, and $R^2\pi = 1$ cm².

The corresponding curve is seen among other things to be capable

TABLE A. $10^6 K$ for 10 cm Krupp normal shells.

v m/sec	$10^6 K$	v m/sec	$10^6 K$	v m/sec	$10^6 K$	v m/sec	$10^6 K$	v m/sec	$10^6 K$
150	1·190	334	2·566	525	3·976	785	3·520	1045	3·292
155	1·190	336	2·654	530	3·970	790	3·514	1050	3·289
160	1·191	338	2·739	535	3·963	795	3·507	1055	3·287
165	1·191	340	2·822	540	3·956	800	3·502	1060	3·284
170	1·191	342	2·902	545	3·949	805	3·496	1065	3·282
175	1·191	344	2·979	550	3·941	810	3·491	1070	3·279
180	1·192	346	3·051	555	3·933	815	3·485	1075	3·277
185	1·192	348	3·115	560	3·925	820	3·480	1080	3·275
190	1·193	350	3·174	565	3·916	825	3·474	1085	3·273
195	1·194	352	3·231	570	3·907	830	3·469	1090	3·271
200	1·195	354	3·286	575	3·899	835	3·463	1095	3·269
205	1·196	356	3·337	580	3·890	840	3·458	1100	3·267
210	1·198	358	3·384	585	3·881	845	3·453	1105	3·265
215	1·200	360	3·427	590	3·871	850	3·448	1110	3·263
220	1·203	362	3·468	595	3·862	855	3·443	1115	3·262
225	1·207	364	3·506	600	3·852	860	3·438	1120	3·260
230	1·212	366	3·541	605	3·841	865	3·433	1125	3·259
235	1·218	368	3·574	610	3·830	870	3·428	1130	3·257
240	1·225	370	3·605	615	3·818	875	3·423	1135	3·256
245	1·233	372	3·633	620	3·807	880	3·418	1140	3·255
250	1·243	374	3·659	625	3·796	885	3·413	1145	3·253
255	1·255	376	3·682	630	3·784	890	3·409	1150	3·252
260	1·270	378	3·703	635	3·773	895	3·404	1155	3·251
265	1·288	380	3·722	640	3·761	900	3·400	1160	3·250
270	1·309	385	3·761	645	3·750	905	3·395	1165	3·249
275	1·334	390	3·792	650	3·740	910	3·391	1170	3·248
280	1·363	395	3·819	655	3·729	915	3·386	1175	3·247
282	1·376	400	3·843	660	3·719	920	3·382	1180	3·247
284	1·390	405	3·864	665	3·709	925	3·378	1185	3·246
286	1·405	410	3·883	670	3·700	930	3·374	1190	3·245
288	1·421	415	3·900	675	3·690	935	3·369	1195	3·245
290	1·439	420	3·916	680	3·681	940	3·365	1200	3·244
292	1·458	425	3·931	685	3·672	945	3·361	1205	3·244
294	1·478	430	3·943	690	3·664	950	3·357	1210	3·243
296	1·500	435	3·955	695	3·655	955	3·353	1215	3·243
298	1·524	440	3·965	700	3·647	960	3·349	1220	3·243
300	1·551	445	3·973	705	3·638	965	3·345	1225	3·242
302	1·580	450	3·981	710	3·630	970	3·341	1230	3·242
304	1·613	455	3·987	715	3·622	975	3·338	1235	3·242
306	1·648	460	3·992	720	3·614	980	3·334	1240	3·241
308	1·687	465	3·995	725	3·606	985	3·330	1245	3·241
310	1·730	470	3·997	730	3·598	990	3·326	1250	3·241
312	1·779	475	3·999	735	3·590	995	3·323	1255	3·241
314	1·832	480	4·000	740	3·583	1000	3·320	1260	3·240
316	1·888	485	4·000	545	3·575	1005	3·316	1265	3·240
318	1·947	490	4·000	750	3·568	1010	3·313	1270	3·240
320	2·010	495	3·999	755	3·561	1015	3·310	1275	3·240
322	2·079	500	3·998	760	3·553	1020	3·307	1280	3·240
324	2·152	505	3·996	765	3·547	1025	3·304	1285	3·240
326	2·229	510	3·992	770	3·540	1030	3·301	1290	3·240
328	2·308	515	3·987	775	3·533	1035	3·298	1295	3·240
330	2·391	520	3·982	780	3·527	1040	3·295	1300	3·240
332	2·478								

TABLE B. $10^6 K$ for 10 cm cylindrical shells.

v_0 m/sec	$10^6 K$	v_0 m/sec	$10^6 K$	v_0 m/sec	$10^6 K$	v_0 m/sec	$10^6 K$	v_0 m/sec	$10^6 K$
100	4·160	285	5·652	440	8·381	630	9·318	940	10·117
110	4·173	290	5·780	450	8·448	640	9·356	960	10·144
120	4·190	295	5·919	460	8·512	650	9·393	980	10·168
130	4·209	300	6·071	470	8·573	660	9·430	1000	10·189
140	4·232	305	6·243	480	8·632	670	9·466	1020	10·207
150	4·260	310	6·430	490	8·689	680	9·501	1040	10·224
160	4·295	315	6·608	500	8·744	690	9·535	1060	10·238
170	4·337	320	6·779	510	8·796	700	9·568	1080	10·249
180	4·387	325	6·935	520	8·846	720	9·631	1100	10·258
190	4·443	330	7·074	530	8·895	740	9·692	1120	10·264
200	4·510	340	7·305	540	8·943	760	9·747	1140	10·268
210	4·589	350	7·495	550	8·989	780	9·800	1160	10·270
220	4·680	360	7·648	560	9·033	800	9·850	1180	10·270
230	4·783	370	7·777	570	9·076	820	9·897	1200	10·270
240	4·898	380	7·888	580	9·118	840	9·941	1220	10·270
250	5·025	390	7·987	590	9·159	860	9·982	1240	10·270
260	5·168	400	8·076	600	9·200	880	10·020	1260	10·270
270	5·335	410	8·161	610	9·240	900	10·055	1280	10·270
275	5·430	420	8·239	620	9·279	920	10·087	1300	10·270
280	5·536	430	8·312						

of representation for a long distance by a straight line $W = av - b$: this justifies the laws of Chapel, Vallier, and Scheve.

Finally the figure d shows how the true K curve would be replaced by the lines (1), (2), (3), (4), when the assumption is made throughout of the quadratic law of air-resistance $f(v) = c_1v^2$ (Newton and others), the cubic law $f(v) = c_2v^3$ (Bashforth, England), or the biquadratic law $f(v) = c_3v^4$ (B. Piton-Bressant, France), or the binomial law $f(v) = c_4v^2(1 + bv)$ (Didion, France).

Description of experiments.

1. The most important of the experiments which were applied to the determination of the laws of air-resistance, were the following:

(*a*) Research of the Metz Committee, Didion-Morin-Piobert 1839–40, chiefly with round shell; velocity 200 to 600 m/sec: measuring apparatus, the ballistic pendulum. This research was repeated in 1856–58 in Metz, with the help of the electrical chronograph of Navez.

(*b*) English experiments of Bashforth in the years 1866–70 with shell of various calibre (7·6 to 22·9 cm) with the height of head 1·12 calibre, length of shell 2·54, and with velocity $v = 230$ to $v = 520$ m/sec.

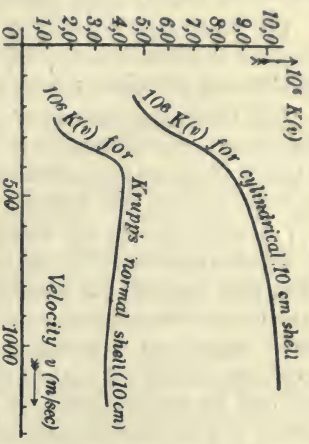


Fig. a.

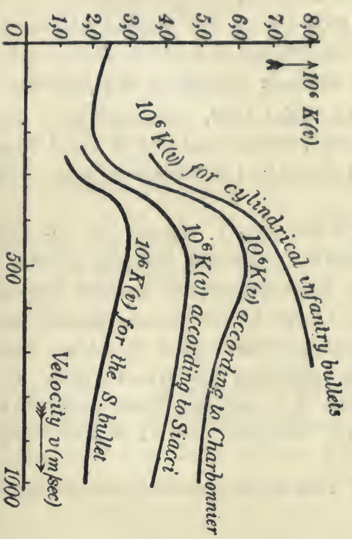


Fig. b.

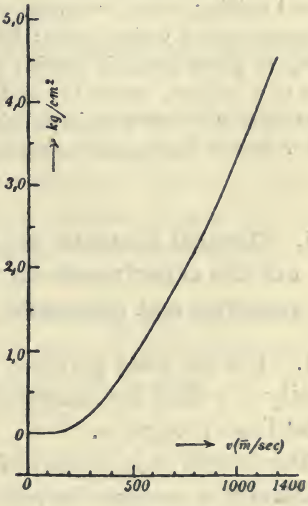


Fig. c.

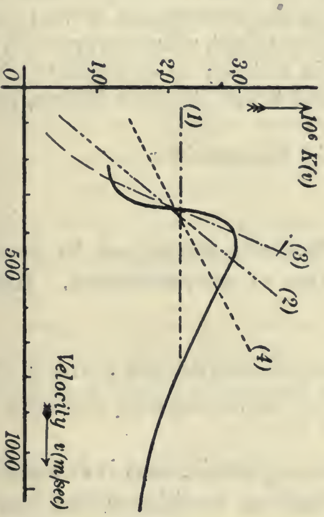


Fig. d

(*c*) Russian experiments, of N. Mayevski at St Petersburg in the year 1869, with shells of different calibre, different height of head (mostly 0·9 calibre) and different length of shell (mostly 2·01 calibre); $v=172$ to $v=409$ m/sec.

(*d*) F. Krupp's experiments of 1879–96, on the Meppen shooting range, with shells of different calibre, different length (2·8 to 4 calibre), different height of head (1·31 and 1·0 calibre; mostly 1·3 calibre); $v=150$ up to 910 m/sec.

(*e*) Dutch experiments of W. C. Hojel 1884, with shell of 8 to 40 cm calibre, length of shell from 2·5 to 4 calibre, with a head of 1·31 and 1·33 calibre; the velocity from $v=138$ to 660 m/sec; individual experiments were made with high velocity (up to 1500 m/sec).

(*f*) Simultaneous experiments of the firm F. Krupp (O. von Eberhard) with artillery shell and of K. Becker and C. Cranz with infantry bullets, 1912.

The first research was carried out with the help of a spark chronograph, over many short ranges, of about 50 m, at Essen; with large calibres by measurement of initial and final velocity on short ranges of 2 to 3 km. The other research was carried out in the ballistic laboratory with 8 mm bullets of various forms and by two methods; by means of a ballistic kinematograph and by means of a spark chronograph with photographic record; length of range 15 to 20 m.

Consult the account published in No. 69 of the *Artillerist-Monatshefte* of the year 1912.

2. For small velocities (from 30 m/sec upward) numerous measurements are in existence. They were carried out by means of falling bodies or with vertical guiding wires, or curved paths; by the measurement of wind pressure by manometers of various sorts; by experiments with whirling apparatus, where a body of given shape is carried round in a circle; by experiments with the beam of a balance, where the body is fixed to one side of a balance, and the whole balance is drawn up in the air.

For details consult the account given by Finsterwalder.

§ 11. General remarks on the methods employed in carrying out the experiments on the laws of air-resistance. Critical remarks, and proposals.

1. For the most part the horizontal components v_1 and v_2 of the velocity of a shell are measured at the beginning and end of a horizontal line of length a.

The length a is chosen of such magnitude, that it is considered satisfactory to assume the path of flight as rectilinear, but unavoidable errors in the measurement of v_1 and v_2 make it desirable that a should not be too short.

The diminution of the energy of the shell is then taken as due to a definite mean value W of the air-resistance.

This magnitude W, calculated from $P\frac{v_1^2 - v_2^2}{2g} = Wa$, is then said to be the air-resistance corresponding to the velocity

$$v = \tfrac{1}{2}(v_1 + v_2).$$

Thus $W(v)$ is obtained, and on the assumption of a law $W = cv^n$, the constants c and n are determined by the method of Least Squares.

This method becomes more free from objection, the smaller the length a is chosen. If we only desire to know the dependence of the air-resistance on the velocity v, then all the other quantities, viz., the weight of the shell, the shape of head, length of shell, rifling, etc., must be kept constant.

The law so obtained of the function $W(v)$ will hold, strictly speaking, only for shell, of which the calibre, shape of head, length of body, velocity of rotation, etc., differ little from the corresponding quantities in the shell employed; because the resistance is not exactly proportional to the cross-section and to a single coefficient depending on the shape of the head.

Suppose the effect of the shape of the point is only examined for a definite velocity v; then all the other quantities must remain unaltered, and the shape of the point is varied.

As a matter of fact such a method of determining the air-resistance does not seem to have been used.

Vibration of the shell seems frequently to have taken place, of which the amplitude has not been measured closely. Moreover the length of the measurement line was formerly chosen of such a length (6000 m and more) that the straight-line path of flight and a constant average value of the resistance W over this line cannot be assumed.

In such cases it is supposed sometimes that the horizontal component of the air-resistance for a given velocity v is identical with the air-resistance for the horizontal component of v, or that

$$f(v)\cos\theta = f(v\cos\theta).$$

Finally sometimes in the calculation of the air-resistance, based on the measurement of the velocity of the shell at the ends of a trajectory very much curved, an approximate method of calculation is used.

Thus on the basis of an uncertain theory the path of flight and the air-resistance W would be calculated; then tables for calculation would be drawn up; and these tables would then be employed to calculate the path of flight in some other case.

Finally the calculated results would probably be compared with the results of actual fire. And so one uncertain theory is checked by another.

If the construction of the laws of air-resistance is to be rational, all theory must be excluded, and the research so directed, that the law of kinetic energy, or some other law of Mechanics equally applicable, may be employed in a pure form.

2. F. Bashforth employed the following systematic procedure.

Near the muzzle of the gun several screens were set up at a small equal distance Δx behind each other. The first screen was at a distance x from the muzzle; denote the velocity of the shell by v, and the air-resistance by $W(v)$.

A shell is fired horizontally through the screens; and then by means of the Bashforth chronograph the time differences Δt, Δt_1, Δt_2, ... are measured, during which the shell flies from the first screen to the second, from the second to the third, and so on. It is required to determine thence the resistance W, that is the product of the mass $\frac{P}{g}$ and the retardation $-\frac{dv}{dt}$ of the shell. Now

$$v = \frac{dx}{dt}, \quad \frac{1}{v} = \frac{dt}{dx}:$$

thence by differentiation with respect to x, we get

$$-\frac{1}{v^2}\frac{dv}{dx} = \frac{d^2t}{dx^2}, \quad \text{or} \quad \frac{dv}{dt} = -v^2 \frac{dx}{dt}\frac{d^2t}{dx^2} = -v^3 \frac{d^2t}{dx^2},$$

so that

$$W = -\frac{P}{g}\frac{dv}{dt} = +\frac{P}{g} v^3 \frac{d^2t}{dx^2}.$$

In the series of the measured time intervals Δt, Δt_1, Δt_2, ..., Δt is the first term; in the corresponding 1, 2, 3, ... difference series, let $\Delta'' t$, $\Delta''' t$, ... be the first term.

Thus by a fundamental theorem of Finite Differences

$$\frac{1}{v} = \frac{dt}{dx} = \frac{1}{\Delta x}[\Delta t - \tfrac{1}{2}\Delta'' t + \tfrac{1}{3}\Delta''' t - \tfrac{1}{4}\Delta^{(4)} t \ldots],$$

$$\frac{d^2t}{dx^2} = \frac{1}{(\Delta x)^2}[\Delta'' t - \Delta''' t + \tfrac{11}{12}\Delta^{(4)} t - \tfrac{5}{6}\Delta^{(5)} t \ldots]$$

$$= \frac{1}{(\Delta x)^2}[\Delta'' t_{(x-\Delta x)} - \tfrac{1}{12}\Delta^{(4)} t_{(x-2\Delta x)} + \tfrac{1}{90}\Delta^{(6)} t_{(x-3\Delta x)} \ldots].$$

Thence
$$W = -\frac{P}{g}\frac{dv}{dt}$$
$$= +\frac{P}{g}\frac{v^3}{(\Delta x)^2}[\Delta''t - \Delta'''t + \tfrac{11}{12}\Delta^{(4)}t - \tfrac{10}{12}\Delta^{(5)}t + \tfrac{137}{180}\Delta^{(6)}t \ldots].$$

In this way Bashforth calculates from his measurements the air-resistance W corresponding to the various velocities v.

However rational this method may be, a doubt must be expressed whether the apparatus, as constructed, can measure the time intervals with sufficient accuracy.

Moreover there is not enough known concerning the resistance which the separate screen wires experience, and this with a large number of wires might influence the measurements to a considerable extent.

3. Let us now consider a shell fired in such a way that the shell itself shall trace photographically the whole path.

For this purpose shells emitting smoke can be employed by day (compare, for instance, the patent of Semple and the corresponding memoir of H. Rohne).

The later research of F. Krupp shows practically a fairly clear visible continuous line on the plate, or if the smoke makes a close determination of the path impossible on this method, the shell may be bored at the side on Neesen's method, and made visible by a light. In this case the firing must be done at night, and a sharp punctuated line will show the trajectory.

In this way the height y of flight of the shell is obtained for any horizontal distance x. Thence from the differentiations y', y'', and y''' with respect to x, we can calculate for any distance x,

(*a*) the velocity v by means of
$$v = \sqrt{g}\,\frac{\sqrt{(1+y'^2)}}{\sqrt{(-y'')}},$$

(*b*) the angle θ of slope of the tangent from
$$v\cos\theta = \sqrt{\frac{g}{-y''}},$$

(*c*) the time of flight t by the equation
$$dt = dx\sqrt{\frac{-y''}{g}},$$

(*d*) the retardation due to the air-resistance
$$\tfrac{1}{2}g\,\frac{y'''\sqrt{(1+y'^2)}}{y''^2}.$$

Thus the retardation and the air-resistance can be obtained for a large number of different values of x and consequently of v.

The advantage lies in the fact that the greater part of the table of air-resistance is obtained at once from fundamental principles from a single experiment.

The procedure rests on similar considerations, lately employed in England by C. F. Close, and developed further in its mathematical aspect by G. Greenhill and C. E. Wolff.

Shots are made with the same weapon for numerous angles of elevation ϕ and range X. If the principle of tilting the trajectory is applied to the individual paths of flight (compare § 5, example 6, and §§ 38 to 40), a number of points are obtained on the path of the longest trajectory; and these can be given by their polar coordinates. Thence, as shown already above, the retardation due to air-resistance can be calculated for each point, and with it the air-resistance as a function of the velocity.

Nevertheless in the employment of this method of calculation, an assumption has been introduced which contradicts more or less the actual flight of a rotating elongated shell. The calculation above holds strictly only when the long axis of the bullet lies exactly in the tangent of the path, that is, when the bullet flies like a well-delivered arrow.

But with rotating elongated bullets the oscillations of precession must occur, because the direction of the tangent of the path in the course of the flight makes a gradually increasing angle with the initial tangent of the path. In consequence of this, the long axis must lie askew to the tangent of the path, even when there is no oscillation of nutation present.

The actual air-resistance is thus exerted against a bullet placed askew, while the calculation employs the assumption of the normal position of the bullet.

This is an assumption that generally underlies methods of approximation in the solution of special ballistic problems.

By this method, the air-resistance to a bullet cannot be obtained closely enough in the case where the axis remains continually in the tangent of the path, because the relation is not known between the resistance of a bullet askew and the resistance of a bullet placed normally, so long as the instantaneous angle of this position is not known.

Perhaps, however, this procedure might provide means of determining this relation, by the employment of two lights in orifices, and thereby measuring the angle of this skew position.

It must be noticed specially in the method of Close, that the relation between ϕ and X is represented by an approximate mathematical formula; and that the underlying errors are magnified by the threefold differentiation; further that the tilting of the trajectory is the cause of an error. The individual cases must then be examined to see if these errors are so small that they may be neglected.

On these grounds perhaps the following procedure is to be preferred: the barrel of the rifle is clamped in a vertical position and shielded above. The bullet is again provided in Neesen's method with a side light. The shots are made at night, and fired off by electricity. At a convenient distance from the rifle a photographic camera is set up, and in it a drum about 120 cm high in the field of view of the muzzle is rotated about a vertical axis with known velocity.

A bromide-silver band is placed on the drum. In the vertical upward flight of the bullet a dotted spiral line is shown on the rotating band of the drum. The axis of ordinates is given by a corresponding shot with the drum stationary, and the abscissa axis by artificial illumination of the muzzle of the rifle while the drum is rotating.

In this way the abscissae of the separate points of the curve give the corresponding time of flight t, and the ordinates the corresponding height y.

By differentiation, the velocity, y', and the acceleration, y'', are given as functions of t; and thence the air-resistance

$$W = -P\frac{y''}{g} - P.$$

Thus by this method, in principle at least, and by appliances free from objection, the whole table of air-resistance can be obtained from the maximum initial velocity downwards.

It is evident that the error of the objective must be determined by alignment on a distant horizontal line.

In vertical fire there are no oscillations of precession. On the other hand care must be taken that rifle and bullet are chosen so that no oscillations of nutation are present.

In carrying out such work many difficulties in detail would be encountered, which must be overcome. Whether the procedure is feasible and would provide useful results, is not at all certain.

II. ON THE INFLUENCE OF THE SKEW POSITION OF THE SHELL ON THE DIRECTION OF MOTION OF THE CENTRE OF GRAVITY.

§ 12. Let κ denote the resistance that a plane surface of 1 sq cm experiences, moving with given velocity v m/sec, in a direction perpendicular to itself in still air.

Then let it be assumed that the resistance of a plane surface of f sq cm is $f\kappa$, under similar circumstances.

If the plane is placed askew to the direction of motion B, so that the normal N of the plane makes an angle α with B the direction of motion, the resistance depends in some manner on α.

The resistance as a function of α is according to Newton $= \kappa f \cos^2\alpha$; according to F. v. Lössl $= \kappa f \cos\alpha$; according to G. Kirchhoff and Lord Rayleigh

$$\kappa f \frac{(4+\pi)\cos\alpha}{4+\pi\cos\alpha};$$

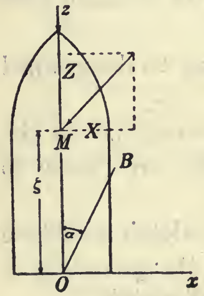
and according to Duchemin

$$\kappa f \frac{2\cos^2\alpha}{1+\cos^2\alpha}.$$

As to the direction of the resistance to a surface placed askew, it is always assumed that the thrust which the slanting surface experiences is at right angles to the surface.

In the sequel it will also be assumed that when the normal to a surface of f sq cm makes an angle α with the wind direction B, the resistance is at right angles to the surface and has the magnitude $\kappa f \cos^m\alpha$ where κ denotes the resistance to 1 sq cm in perpendicular movement at the same velocity.

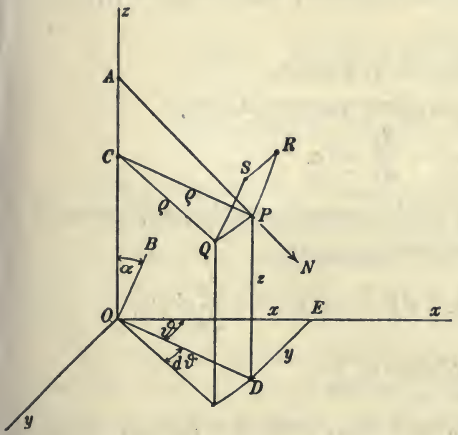
Further it will be assumed that this law is equally true for an infinitesimal element of surface, and that the resistance against a finite part of the surface can be calculated by integration over the surface.

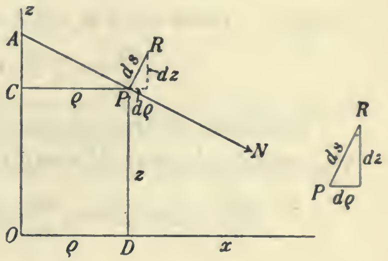


In the treatment of a shell, let a rectangular system of coordinates in space be taken as a basis. Let the shell be a body of rotation with the axis of figure on the longitudinal axis of the shell along the z axis. Let the base of the shell be the xy plane. Let the direction of motion of the centre of gravity on the tangent to the trajectory be parallel to the xz plane, and make the given angle α with the axis of the shell, on the z axis.

Everything is then symmetrical with respect to the xz plane; and we have to determine the components X and Z of the air-resistance in the x and z directions, as well as the position of the point of application M of the resultant $\sqrt{(X^2+Z^2)}$ on the axis of the shell.

Let P be a point on the surface of the shell with rectangular coordinates $OE=x$, $ED=y$, $DP=z$; or with cylindrical coordinates, $\angle EOD=\vartheta$, radius vector $OD=CP=\rho$, $DP=z$.





Take a meridian section at P through the surface of the shell, along the z axis or the axis of the shell; also a section through P at right angles to the axis of the shell.

In the first section let $ds=PR$ be an infinitesimal element of the meridian curve of the surface of the shell. In the last section, which is circular, let $PQ=\rho d\vartheta$ be an infinitesimal element of the circle of the cross-section.

In this manner an infinitesimal element of surface $PQSR$ is taken at P, with surface $df=\rho d\vartheta ds$. The resistance on this surface element (according to the first assumption) is directed along the normal APN and (according to the second and third assumptions) has the magnitude $\kappa df \cos^m \omega$, where ω denotes the angle between the normal to the surface and the direction OB of the tangent to the trajectory.

Let the normal to the surface AN make angles β_1, β_2, β_3 respectively with the x, y, z axes. Now the cosine of the angle between AN and the direction CP or OD is equal to $\frac{dz}{ds}$, and so

$$\cos\beta_1 = \frac{dz}{ds}\cos\vartheta,$$

and
$$\cos\beta_2 = \frac{dz}{ds}\sin\vartheta.$$

Further
$$\cos\beta_3 = -\frac{d\rho}{ds},$$

(as in the figure where the meridian section through P is indicated). If the direction of the tangent to the trajectory makes angles $\gamma_1, \gamma_2, \gamma_3$ with the three axes, then

$$\cos\gamma_1 = \sin\alpha,\quad \cos\gamma_2 = 0,\quad \cos\gamma_3 = \cos\alpha,$$

thence
$$\cos\omega = \cos\beta_1\cos\gamma_1 + \cos\beta_2\cos\gamma_2 + \cos\beta_3\cos\gamma_3$$
$$= \frac{dz}{ds}\cos\vartheta\sin\alpha + 0 - \frac{d\rho}{ds}\cos\alpha.$$

Furthermore the components of the normal reaction $\kappa df\cos^m\omega$ of the surface element df along the axes x, y, z are

$$dX = \kappa df\cos^m\omega\frac{dz}{ds}\cos\vartheta,\quad dY = \kappa df\cos^m\omega\frac{dz}{ds}\sin\vartheta,$$
$$dZ = -\kappa df\cos^m\omega\frac{d\rho}{ds},$$

where $m = 2$ according to Newton's law, and $df = \rho\, d\vartheta\, ds$. These expressions for dX, dY, dZ are to be integrated over the part of the surface of the shell exposed to the stream of air; and in them Y is zero by reason of the symmetry with respect to the plane xz.

To obtain the distance $\zeta = OM$ of the point of application M of the resultant of the air-resistance from the base of the shell, the equation of the moment of the resistance components about O must be written down. Herein only the x component comes into consideration, because the z component has no moment and the y component is nothing. The normal resistance on the surface element df has the x component

$$\kappa df\cos^m\omega\frac{dz}{ds}\cos\vartheta;$$

and this component cuts the axis of the shell, in A. The moment arm is

$$OA = OC + CA = z + \rho\frac{d\rho}{dz},$$

so that the moment is

$$\kappa\, df \cos^m \omega \left(z + \rho \frac{d\rho}{dz}\right) \frac{dz}{ds} \cos \vartheta.$$

The integration of all these moments gives the moment ζX of the resultant. If X has been calculated, then ζ is known.

The complete result is then laid down in the following formulae:

$$X = \quad \kappa \iint \cos^m \omega \rho\, dz \cos \vartheta\, d\vartheta \quad \text{......................(1)}$$

$$Z = -\kappa \iint \cos^m \omega \rho\, d\rho\, d\vartheta \quad \text{............................(2)}$$

$$X\zeta = \quad \kappa \iint \left(z + \rho \frac{d\rho}{dz}\right) \cos^m \omega \rho\, dz \cos \vartheta\, d\vartheta \quad \text{.........(3)}$$

$$\cos \omega = \sin \alpha \frac{dz}{ds} \cos \vartheta - \cos \alpha \frac{d\rho}{ds}. \quad \text{......................(4)}$$

Here X denotes the component of the air-resistance at right angles to the long axis of the shell, Z that along the axis.

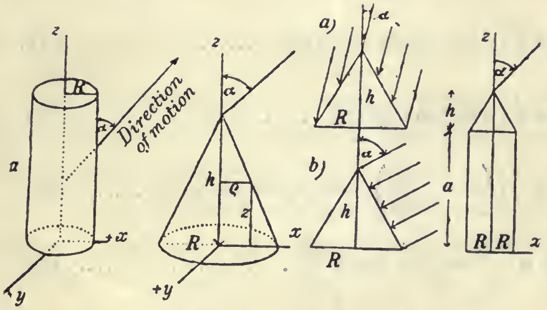
The resultant air-resistance $\sqrt{(X^2 + Z^2)}$ cuts the axis of the shell in a point M, which is at a distance ζ from the base of the shell.

The angle β between the resultant and the axis is in general not identical with the angle α between shell axis and the tangent of the trajectory, but is given by $\tan \beta = X : Z$.

The factor κ denotes the air-resistance against unit surface in a direction at right angles to it, with the corresponding velocity v of the centre of gravity of the shell, for which it is considered. On Newton's law $m = 2$, and $m = 1$ when Lössl's law is assumed as the basis of the calculation. The equation $\rho = f(z)$ of the meridian curve of the shell is given by the shape. In the operation of the integration, this is carried out over the part of the shell struck directly by the air-resistance, or else over some part under consideration; so that for the whole shell the calculation can be carried out at once; with respect to z from the base of the shell to the point; with respect to ρ from the inside to the outside of the surface of the shell, and finally with respect to ϑ from one to the other limit of the stream of air tangential to the surface; so that this is only from 0 to 2π when the whole curved outer surface is struck by the stream of air; in other cases the limits are to be determined with regard to the form of the shell and the angle α.

Kummer and St Robert have carried out calculations of this kind with the assumption $m=2$ (Newton) for many shapes. So also W. Gross with the assumption $m=1$ (Lössl); mention however must be made of the fact that the calculations of Gross are only approximate.

Similar calculations of this sort have been made by de Sparre, v. Wuich, Mayevski, Siacci, Charbonnier.



Examples.

1. Resistance of the outer surface of a circular cylinder, open at the top, of radius R and height a. Components and point of resultant action to be calculated; on Newton's assumption $m=2$.

The equation of the meridian curve is $\rho=R$; so that

$$d\rho=0,\quad ds=dz,\quad \cos\omega=\sin a\cos\delta;$$

thence

$$X=\kappa R\sin^2 a\iint\cos^3\vartheta\, d\vartheta\, dz$$

$$Z=0,\ \text{(since } d\rho=0\text{, and the cylinder is open above)}$$

$$X\zeta=\kappa R\sin^2 a\iint\cos^3\vartheta\, d\vartheta\, z\,dz.$$

Since half of the curved surface of the cylinder is exposed directly to the air-resistance, the integration is made only from $\vartheta=-\frac{1}{2}\pi$ to $\vartheta=+\frac{1}{2}\pi$, and besides from $z=0$ to $z=a$; and so

$$X=\tfrac{4}{3}\kappa Ra\sin^2 a,\quad X\zeta=\tfrac{2}{3}\kappa Ra^2\sin^2 a,\ \text{and thence } \zeta=\tfrac{1}{2}a,$$

that is the point of application lies at the middle of the cylinder height.

If the cylinder is closed at the top by the circular area $r^2\pi$ at right angles, then $Z=\kappa R^2\pi\cos^2 a$ (on Newton's assumption) and

$$\tan\beta=\frac{X}{Z}=\frac{4a}{3\pi R}\tan^2 a.$$

2. Cone, radius R, height h (see figure). Same assumption; $m=2$.

The equation of the meridian curve, that is of the straight generating line, is

$$\rho=\frac{R}{h}(h-z)$$

and thence

$$\frac{d\rho}{dz}=-\frac{R}{h},\quad \frac{ds}{dz}=\frac{\sqrt{(h^2+R^2)}}{h},\quad \frac{d\rho}{ds}=\frac{-R}{\sqrt{(h^2+R^2)}},$$

$$\cos\omega=\frac{h\sin a\cos\vartheta+R\cos a}{\sqrt{(h^2+R^2)}};$$

and so

$$X = \frac{\kappa Rh}{h^2+R^2}\iint\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2 (h-z)\,dz\cos\vartheta\,d\vartheta$$

$$Z = \frac{\kappa R^2}{h^2+R^2}\iint\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2 (h-z)\,dz\,d\vartheta$$

$$X\zeta = \frac{\kappa Rh}{h^2+R^2}\iint\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2\left[z - \frac{R^2}{h^2}(h-z)\right](h-z)\,dz\cos\vartheta\,d\vartheta.$$

Integrated with respect to z from 0 to h,

$$X = \frac{\kappa Rh^3}{2(h^2+R^2)}\int\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2\cos\vartheta\,d\vartheta$$

$$Z = \frac{\kappa R^2h^2}{2(h^2+R^2)}\int\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2 d\vartheta$$

$$X\zeta = \frac{\kappa Rh^2(h^2-2R^2)}{6(h^2+R^2)}\int\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2\cos\vartheta\,d\vartheta.$$

Thence it follows, for every angle a, that

$$\zeta = \frac{h^2-2R^2}{3h}.$$

Two cases are to be distinguished in regard to the integration with respect to ϑ (compare figs. a and b); firstly the case where the whole curved surface of the cone is exposed to the air-resistance, and this occurs when the angle a is smaller than the angle which the axis of the cone makes with the side of it, i.e., when $\tan a < \frac{R}{h}$; and secondly the case where only a part of the surface of the cone is exposed, that is when $\tan a > \frac{R}{h}$.

In the first case $\left(\tan a < \frac{R}{h}\right)$, the two limits of integration with respect to ϑ are $\vartheta = -\pi$ and $\vartheta = +\pi$; and since

$$\int_{-\pi}^{\pi}\cos^3\vartheta\,d\vartheta = 0,\quad \int\cos^2\vartheta\,d\vartheta = \pi,\quad \int\cos\vartheta\,d\vartheta = 0,$$

we have

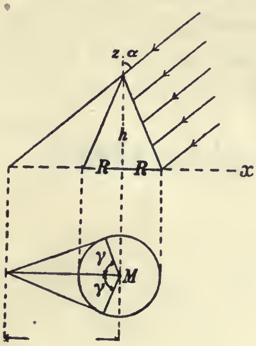
$$X = \frac{\kappa h^2R^2\pi\sin a\cos a}{h^2+R^2},$$

$$Z = \frac{\kappa h^2R^2\pi\left(\frac{1}{2}\sin^2 a + \frac{R^2}{h^2}\cos^2 a\right)}{h^2+R^2}.$$

In the second case $\left(\tan a > \frac{R}{h}\right)$ only that part of the surface of the cone is exposed, for which $\cos\omega$ is positive (ω = angle between normal and direction of motion of the air); the integration with respect to ϑ has then the limits where a parallel to the direction of the air touches the surface of the cone; that is for $\cos\omega = 0$, or

$$h\sin a\cos\vartheta + R\cos a = 0,\quad \cos\vartheta = -\frac{R}{h}\cot a.$$

If γ denotes the angle given by



$$\cot\gamma = \frac{R}{h}\cot a, \text{ or } \gamma = \text{arc cos}\left(\frac{R}{h}\cot a\right),$$

the limits of integration are

$$\vartheta = -\pi + \gamma \text{ and } \vartheta = +\pi - \gamma;$$

and we require for the integration with respect to these limits the three integrals

$$\int_{-\pi+\gamma}^{+\pi-\gamma} \cos^3\vartheta\, d\vartheta = \tfrac{2}{3}\sin\gamma\,(2+\cos^2\gamma),$$

$$\int \cos^2\vartheta\, d\vartheta \qquad = \pi - \gamma - \sin\gamma\cos\gamma,$$

$$\int \cos\vartheta\, d\vartheta \qquad = 2\sin\gamma.$$

If we take

$$\int_{-\pi+\gamma}^{+\pi-\gamma}\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2 \cos\vartheta\, d\vartheta = P,$$

$$\int\left(\sin a\cos\vartheta + \frac{R}{h}\cos a\right)^2 d\vartheta = Q,$$

then

$$P = \tfrac{2}{3}\sin^2 a\sin\gamma\,(2+\cos^2\gamma) + \frac{2R}{h}\sin a\cos a\,(\pi - \gamma - \sin\gamma\cos\gamma) + \frac{R^2}{h^2}\cos^2 a\sin\gamma,$$

$$Q = \sin^2 a\,(\pi - \gamma - \sin\gamma\cos\gamma) + \frac{4R}{h}\sin a\cos a\sin\gamma + \frac{2R^2}{h^2}(\pi - \gamma)\cos^2 a;$$

and when the angle γ is expressed by the angle a,

$$P = \tfrac{2}{3}\left(\frac{R^2}{h^2}\cos^2 a + 2\sin^2 a\right)\sqrt{\left(1 - \frac{R^2}{h^2}\cot^2 a\right)} + \frac{2R}{h}\sin a\cos a\left[\pi - \text{arc cos}\left(\frac{R}{h}\cot a\right)\right];$$

$$Q = \left(\frac{2R^2}{h^2}\cos^2 a + \sin^2 a\right)\left[\pi - \text{arc cos}\left(\frac{R}{h}\cot a\right)\right] + \frac{3R}{h}\sin a\cos a\sqrt{\left(1 - \frac{R^2}{h^2}\cot^2 a\right)},$$

and thence for the case when $\tan a > \frac{R}{h}$

$$X = \frac{\kappa h^3 R P}{2(h^2+R^2)}, \quad Z = \frac{\kappa h^2 R^2 Q}{2(h^2+R^2)}.$$

3. Combination of cylinder and cone (same assumption; $m = 2$).

(*a*) For the case where $\tan a < \frac{R}{h}$; by simple addition we have

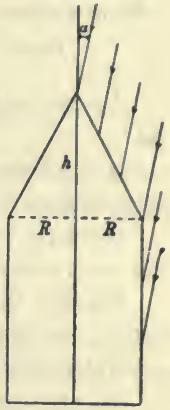
$$X = \tfrac{4}{3}\kappa R a\sin^2 a + \frac{\kappa h^2 R^2 \pi\sin a\cos a}{h^2+R^2},$$

$$Z = \frac{\tfrac{1}{2}\kappa h^2 R^2 \pi\left(\sin^2 a + \frac{2R^2}{h^2}\cos^2 a\right)}{h^2+R^2},$$

$$X\zeta = \tfrac{2}{3}\kappa R a^2\sin^2 a + \frac{\kappa h^2 R^2\pi\sin a\cos a}{h^2+R^2}\left(a + \frac{h^2-2R^2}{3h}\right),$$

and so

$$\zeta = \frac{\frac{2}{3} a^2 \sin\alpha + \frac{h^2 R\pi}{h^2+R^2}\left(a + \frac{h^2-2R^2}{3h}\right)\cos\alpha}{\frac{4}{3} a \sin\alpha + \frac{h^2 R\pi \cos\alpha}{h^2+R^2}}.$$



(*b*) For the case where $\tan\alpha > \frac{R}{h}$

$$X = \tfrac{4}{3}\kappa R a \sin^2\alpha + \frac{\kappa h^3 R P}{2(h^2+R^2)},$$

$$Z = \frac{\kappa h^2 R^2 Q}{2(h^2+R^2)},$$

$$X\zeta = \tfrac{2}{3}\kappa R a^2 \sin^2\alpha + \frac{\kappa h^3 R}{2(h^2+R^2)}\left(a + \frac{h^2-2R^2}{3h}\right)P,$$

and so

$$\zeta = \frac{\frac{2}{3} a^2 \sin^2\alpha + \frac{h^3\left(a + \frac{h^2-2R^2}{3h}\right)P}{2(h^2+R^2)}}{\frac{4}{3} a \sin^2\alpha + \frac{h^3 P}{2(h^2+R^2)}}.$$

4. By approximate calculation on the basis of Lössl's Law ($m=1$) W. Gross finds the following values for the resultant resistance W to a shell with ogival head:

rounding radius, 2 calibres: $W = \kappa R^2\pi\,(0{\cdot}3655 + 1{\cdot}3606 \sin^2\alpha)$ } for angle α up to
" " 2·5 " : $W = \kappa R^2\pi\,(0{\cdot}3312 + 1{\cdot}6344 \sin^2\alpha)$ } $\sin\alpha = 0{\cdot}3$.

Here again κ denotes the resistance of unit of surface moving at right angles with the same velocity; and so for a 2·5 calibre rounded head $\kappa R^2\pi \times 0{\cdot}3312$ is the resistance of this shell in the case where the axis of the shell lies in the tangent of the trajectory.

As for the distance ζ of the point of application from the base of the shell, the following values are obtained by W. Gross for a shell 3·5 calibre long overall, and 1·5 calibre length of head of shell, $2R$ denoting the calibre:

$\sin\alpha = 0{\cdot}1$	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9	1·0
$\zeta = 5{\cdot}3$	4·8	4·4	4·1	3·9	3·7	3·5	3·4	3·3	3·0 R.

The centre of gravity was distant 2·97 R from the base of the shell.

Thus even for a complete crosswise position ($\alpha = 90°$) the point of application lies ahead of the centre of gravity. For very small angle α, the point of application would lie somewhere about the middle of the head of the shell.

Under the assumption of Newton's value, $m=2$, calculation shows that here also the point of application lies in general ahead of the centre of gravity, and near the point for small values of the angle α; so that ζ depends on α. It is only for the right circular cylinder cut straight across, as well as for the combination of cylinder and cone, for which the height of head $h = 0{\cdot}41\, R$, that ζ is seen to be independent of α.

In this theory the velocity v of the shell is involved only in the factor κ.

Remark on the uncertainty of all such calculations as these, and on the necessity for experiment.

Against calculations of the previous kind the following must be said:

In the first place nothing certain is known of the three assumptions employed at the outset.

Secondly, even when these assumptions hold good, no one knows the most suitable value for m.

Thirdly, the flow of the air away from the shell, with waves and eddies, has not been considered, and cannot at present be taken into account mathematically in a satisfactory manner.

On the other hand no possibility exists at the present time of proceeding in a manner free from objection, though this would be very desirable, because the influences of the obliquity of the axis of the shell make themselves felt in deviation of the shell and diminution of the range.

On the dependence of the position of the point of application on the angle α, Kummer (1875) has made numerous and accurate experiments with bodies of shell-like shape, but with small velocities only, and with no rotation. He investigated the relation between ζ and α, for a body of revolution.

For this he chose several different values of ζ, and investigated the corresponding value of α, in the following manner:

The model of the shell (of cardboard, so as to increase the sensibility of the method) was suspended freely on a horizontal axis, so that the shell was moved in still air at a velocity of about 8 m/sec (by means of a whirling apparatus); the model of the shell being on an arm over 2 m long, which was revolved about a vertical axis.

Kummer's method was as follows: he determined the position of equilibrium for a large number of positions of the transverse axis, assumed by the body under the action of the air-resistance alone. The distance of the transverse axis from the base of the shell was thus ζ; then the corresponding angle α was observed, at which the long axis stood.

All other forces of rotation must naturally be eliminated: above all, the force of gravity must be removed by making the centre of gravity come into the transverse axis by means of an internal mechanism.

Kummer carried out his observations for the plane, cylinder, combination of cylinder and cone, hemisphere, and semi-ellipsoid, and finally for a model of the Mauser bullet and the 4-pounder Prussian shell. Some details of the arrangement of the research were improved by Kummer in a second series of experiments.

The results of Kummer's research may be given here for the model of the shell with a height $a = 112{\cdot}5$ mm of the cylindrical body, radius $R = 37{\cdot}5$ mm, and height $h = 47{\cdot}5$ mm of the ogival head. He found

$\zeta=$	68	70	72	74	76	78	80	82	84	86	88	90
$\alpha=$	86	83	82	79	73	70	69	68	64	55	48	43

92	94	96	98	100	102	104	106	108	110 mm
39	36	34	33	32	30	25	23	21	18°.

As the stream of air is directed at a continually decreasing angle with respect to the axis, so the point of application approaches more and more to the upper end of the cylindrical part ($\zeta = 112{\cdot}5$).

For an angle smaller than $\alpha = 18°$ the research gave no longer a definite result. The Newtonian assumption gives for this case

$$\zeta = \frac{\frac{3}{8} aR\pi + a^2 \tan \frac{1}{2}\alpha}{\frac{3}{8} R\pi + 2a \tan \frac{1}{2}\alpha} \left(\begin{array}{l} a \text{ the height of the cylindrical part,} \\ 2R \text{ the calibre} \end{array}\right).$$

Putting $\alpha = 0$, we get $\zeta = a$, in agreement with experiment: on the contrary, the rest of the calculated and observed values of ζ for α are remarkably discordant.

Such experiments with a velocity of 8 m/sec cannot give the results required in ballistics; because at velocities up to 1000 m/sec much more is required, and this would naturally involve very serious experimental difficulties.

For if it is attempted to make such a stream of air act for a long time on a stationary model of a shell, the stream lines of the air must possess a uniform parallel motion in front of the model, and in addition the velocity of the air must be the same against each square centimetre of the cross-section. Considerable expenditure of power is then required, and special preparations for the measurement of the directions and velocity of the air at every point are required, without interfering with the flow of the air.

Probably the employment of Neesen's method, described above, will lead to results. In any case a wide field is open for such work.

III. Calculations relating to the shape of the shell head.

§ 13. When α is $=0$ in the formulae for X, Z, $X\zeta$ of § 12, it is assumed that the axis of the shell lies in the tangent to the path. Integration with respect to ϑ must then be carried out from 0 to 2π, and then $\int \cos\vartheta d\vartheta$ is zero, and so too X and $X\zeta$, (but the value of ζ approaches a finite limit, which is obtained by calculating the value of ζ at first for a small finite angle α, and putting $\alpha=0$).

Then the resistance Z in the direction of the z axis must be considered. But since

$$\cos\omega = -\frac{d\rho}{ds},$$

then writing x for ρ, in the case where the axis lies in the tangent to the path, we have

$$W = 2\pi\kappa \int \left(\frac{dx}{ds}\right)^m x dx, \text{ where } ds = \sqrt{(dx^2+dz^2)}.$$

Here $\kappa(v)$ is the resistance to unit surface for motion at right angles, and for the velocity v corresponding to the motion of the centre of gravity of the shell.

Examples.

1. A shell, consisting of a circular cylinder of calibre $2R$, combined with a truncated cone of height h, and radius a of the uppermost section. Lössl's and Gross's assumptions, $m=1$.

The resistance W_1 of the curved surface in the direction of the shell-axis is

$$W_1 = 2\pi\kappa \int_a^R \frac{x dx}{\sqrt{\left\{1+\left(\frac{dz}{dx}\right)^2\right\}}},$$

where $\quad x-a=(h-z)\cot\beta, \quad \cot\beta=\dfrac{R-a}{h};$

$$dx = -\cot\beta dz, \quad \sqrt{\left\{1+\left(\frac{dz}{dx}\right)^2\right\}} = \frac{1}{\cos\beta};$$

so that $\quad W_1 = 2\pi\kappa\cos\beta \displaystyle\int_a^R x dx = \kappa\pi(R^2-a^2)\cos\beta.$

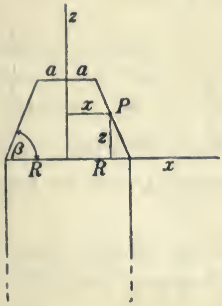
To this must be added the resistance of the flat head

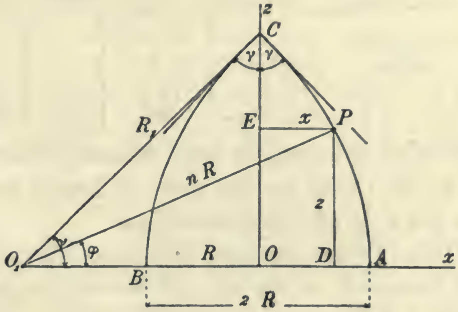
$$W_2 = \kappa\pi a^2.$$

The total resistance $W = W_1 + W_2$; compared with the resistance $\kappa R^2\pi$ of the direct cross-section of the cylinder of calibre $2R$, this is in the ratio of

$$\cos\beta\left(1 - \frac{a^2}{R^2}\right) + \frac{a^2}{R^2} \text{ to } 1.$$

2. Ogival shell with a rounding radius $= n$ half calibres. Lössl's assumption, $m = 1$.





Let AC be the generating circular arc of the ogive, with centre O_1, and P any point (xz) of the circle. It is convenient to take the central angle $AO_1P = \phi$ as the independent variable instead of x. Then

$$O_1P\cos\phi = O_1D = O_1A - AD,$$

or $$nR\cos\phi = nR - (R - x),\quad x = nR\left(\cos\phi - \frac{n-1}{n}\right),$$

$$dx = -nR\sin\phi\,d\phi,\quad ds = nR\,d\phi,$$

so that

$$W = 2\pi\kappa\int\frac{dx}{ds}x\,dx$$

$$= 2\pi\kappa\int\frac{nR\sin\phi\,d\phi}{nR\,d\phi}\,nR\left(\cos\phi - \frac{n-1}{n}\right)nR\sin\phi\,d\phi$$

$$= 2\pi\kappa R^2n^2\int_0^\gamma \sin^2\phi\left(\cos\phi - \frac{n-1}{n}\right)d\phi.$$

$$W = \kappa R^2\pi n^2\left(\sin\gamma - \tfrac{1}{3}\sin^3\gamma - \gamma\cos\gamma\right),$$

in which γ denotes the angle AO_1C, which is given by

$$\cos\gamma = \frac{nR - R}{nR} = \frac{n-1}{n}.$$

It may be mentioned in this connexion that in such ogival shells the semi-ogival angle γ, the height of the head $OC = h$, and the radius of rounding $R_1 = O_1C = nR$, are connected in the following manner:

$$\cos\gamma = \frac{n-1}{n}, \quad \sin\gamma = \frac{h}{R_1}, \quad \left(\frac{h}{2R}\right)^2 = \frac{R_1}{2R} - \tfrac{1}{4};$$

as for example

Rounding radius in calibres:	$\frac{R_1}{2R} =$	0·5	1·0	1·5	2	3
Height of head in calibres:	$\frac{h}{2R} =$	0·5	0·866	1·118	1·323	1·658
Semi-ogival angle:	$\cos\gamma =$	0	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{5}{6}$
	$\gamma =$	90°	60°	48° 11′	41° 25′	33° 34′

Similar calculations for different forms of head have been carried out by W. Gross on the base of Lössl's assumption, and by Ingalls with the help of Duchemin's Law.

Hélie (France) assumed that the coefficient i of ogival shell increased and decreased as the sine of the semi-ogival angle γ; and this was established by numerous experiments.

A. Hamilton (North America) stated that it was proved that the value of i for a shell should be proportional to the mean value of the sine of the angle that the tangent to the ogive at the various points made with the axis of the shell.

This would make the i values of two shells inversely proportional to the surface of the heads of the shells.

If we put $i = 1$ for an ogive of 2 calibre radius of rounding, then we should have

$i = 1·00,\ 0·82,\ 0·71,\ 0·64,\ 0·58,\ 0·54,$

for $n = 2,\ 3,\ 4,\ 5,\ 6,\ 7$ calibres.

But it will be shown later on, that these assumptions cannot be considered to be proved.

The following table gives the specific resistance of shells of equal calibre, as calculated on the basis of the laws of v. Lössl, Duchemin, and Newton.

As is seen, these values are very discrepant.

Laboratory experiments at small velocity have been made in great numbers. For example, Borda, Hutton and Vince have obtained the following results:

The resistance to a hemisphere is to that on the diametral cross-section as 0·407 : 1 (Borda 0·405 : 1; Hutton 0·413 : 1; Vince 0·403 : 1). Further the resistance on a circular cone with angles 90°, 60°, 51° 24′ bears to that on the plane base the ratio 0·691, 0·543, 0·433 to 1, respectively.

By experiments with cylinders 10 cm high, moved axially, fitted with conical heads 1, 1·5, 2, 3, 4 half-calibres high, and with a hemisphere, Didion arrived at the result that the resistances in these six cases were to each other respectively as 73·26, 53·99, 47·74, 44·29, 40·69, and (for the hemisphere) 43·03.

With spheres moving with velocity of 9 m/sec he found

$$W(\text{kg}) = 0.0275\,\delta R^2 \pi v^2;$$

δ the weight of air in kg/m³, $R^2\pi$ cross-section in m², and v the velocity in m/sec.

Height of head in calibres	Radius of rounding in calibres	Ogival form of head: Law of F. v. Lössl (W. Gross)	Ogival form of head: Law of Duchemin (Ingalls)	Ogival form of head: Law of Newton (Kummer, &c.)	Cone-shaped head: Law of F. v. Lössl (W. Gross)	Cone-shaped head: Law of Duchemin (Ingalls)	Cone-shaped head: Law of Newton (Kummer, &c.)
0·5	0·5	0·666	0·858	0·500	0·707	0·943	0·500
0·866	1	0·504	0·752	0·292	0·500	0·800	0·250
1·118	1·5	0·419	0·675	0·204	0·409	0·663	0·167
1·323	2	0·366	0·617	0·156	0·353	0·628	0·125
1·5	2·5	0·331	0·571	0·127	0·317	0·575	0·100

A. Frank averaged the numerical values of the air-resistance at low velocity for bodies of very different forms on the basis of experiments. Finally F. v. Lössl was led by his research to the view that the resistance of a cone moving axially, with semi-vertical angle of opening a, bears to the resistance of the plane base area the ratio of $0.83 \sin a$ to 1.

Further, according to his experiments the resistance of a sphere was one-third of the resistance to the plane diametral section (according to Lössl's law it would be $\frac{2}{3}$).

All this shows clearly that this law cannot hold exactly.

Based on the German experiments at the target, the relative values for ogival shell according to W. Heydenreich would be

0·5	0·7	1	1·5	2	3	4	6	8	calibres of rounding radius,
1350	1200	1100	1000	950	850	800	700	650	

and these form values, other things being equal, can be "immediately applied to all shells, independently of calibre."

After further developments W. Heydenreich himself appears to infer the inapplicability of the above numbers as fixed form values, and to introduce a doubt as to the universal transference of the form value from one shell to another.

It will be shown in the sequel how the form values were obtained, and how a figure obtained in this way must at best be a makeshift.

Critical remarks on experiments to determine the effect of the shape of the shell head.

The initial velocity v_0, range X, angle of departure ϕ and air density are observed. By means of approximations from the shell-calibre $2R$ and shell-weight P the product $i\beta$ is obtained. Here β is a factor of adjustment to compensate for the errors caused in the integration of the differential equation of the problem. (See Chapter V.)

Division by β thus gives the i-value in comparison with a normal value $i=1$, which must be defined in an agreed but arbitrary manner.

(*a*) When the i-value for any given shell on the basis of the same value of v_0, ϕ, X, $2R$, P, δ, is calculated by means of two different systems of solution, based on the same laws of air-resistance, the same value is not always obtained; discrepancies up to 13 °/₀ can be met with.

The reason of this is that the compensation for the errors of integration in the different systems of solution has been more or less successful (compare § 33).

Even in the same systems of solution (see for example Siacci II) the errors, for different departure angles ϕ and ranges X, of the corresponding β values are not the same.

Suppose for example a calculation is made grounded on Siacci II, and β is taken out of the β table, and i is thence determined; a part of the error in β is then transferred to the value of i.

(*b*) Moreover as stated the air-resistance is not exactly proportional to the cross-section of the shell. But in the calculation of i this proportionality has been assumed; and so again an error ensues. This error too appears in the i-value in the calculation.

Now the more two shells of the same calibre differ from each other in shape of head, the more will the fact that the resistance and cross-section are not proportional to each other make itself evident in the calculated value of i.

(*c*) The air-density δ is in fact variable, because it depends on the height of the flight of the shell. But in the calculation the air-density will have been assumed constant, and either equal to that on the ground, or to some mean value of air-density; here again an error arises, that also affects the coefficient i.

(*d*) As stated above, the resistance is not exactly proportional to a single coefficient; the relationship is really a very complicated one, depending on the shape.

But in the calculation of its value this proportionality is assumed.

(*e*) If the angle of departure ϕ is measured corresponding to the same shell, the same initial velocity v_0 and the same air-density δ, for many ranges X, the i values of the shell can be calculated from each separate range.

But it appears frequently that the series of the values thus obtained is not constant.

If the solution of the ballistic problem was exact and complete, and if the long axis of the shell remained steadily in the tangent of the trajectory, in accordance with one of the assumptions of Sections 3—5, the i values would necessarily be equal.

As a matter of fact the calculated i values vary, and in some of the new infantry bullets to a very marked extent from one trajectory to another.

The fundamental cause of this alteration in the form-coefficients arises firstly from the fact that the reduction factor β is different for the different trajectories of the same shell; and secondly in that the change of i with the shape and velocity has not been considered, or insufficiently; thirdly in the fact that the shell is to some extent performing violent oscillations in the air.

In this last case the calculation is inexact because the calculation of the trajectory should take into account the oscillation of the shell. As this is not the case, and as moreover approximate methods are employed here, the errors arising out of it must give a variation of the form-coefficient i.

Conversely however this variation may not be employed for the quantitative measure of the extent of the oscillation.

On these grounds no certainty can be expected that the values of i obtained in this manner can settle the true value of form; and still less that the results are generally valid.

§ 14. Calculations concerning the most effective form of head of a shell. The August-head.

The problem is to determine that profile of the head of a shell which at given velocity shall give a minimum resistance. In the figure the long axis of the shell is taken as the axis of x, and the y-axis stands at right angles to it. The half-calibre $R = BB_1 = CC_1$ is given and the height $h = AB = x_1 - x_0$, and the front surface is at a given distance x_0 from the origin of coordinates O. The question is to determine the meridian curve A_1B_1, such that by its rotation about the axis of x the surface generated shall have the least resistance, when the shell moves with given velocity v in the direction CA in still air, or when the air streams past with the same relative velocity in the direction AC against the head of the shell. Here $\kappa(v)$ may denote the resistance normal to the surface per unit area. The resistance to the complete head $B_1A_1A_2B_2$ is to be calculated and made a minimum.

This leads to a problem in the Calculus of Variations. The following result is stated without proof. When it is required to determine y as a function of x (curve A_1B_1) for which the given definite integral

$$\int_a^b F(x, y, y', y'', \ldots)\, dx$$

is a maximum or minimum, the differential equation

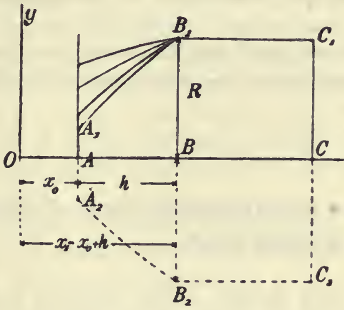
$$0 = \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right) - \ldots \qquad \ldots\ldots\ldots\ldots(1)$$

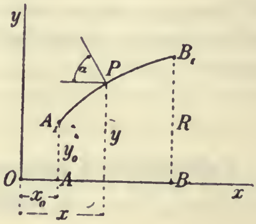
requires to be integrated. The constants of integration will then be calculated as follows.

Suppose first the ends $(x_0 y_0)$, $(x_1 y_1)$ of the corresponding branch of the curve are given; and that then

$$x = x_0,\ y = y_0, \text{ and } x = x_1,\ y = y_1.$$

Let the point $(x_1 y_1)$ be fixed at B_1; and on the other hand let the other point $(x_0 y_0)$ be capable of sliding along a parallel to the y-axis;





that is, let the desired curve run from the fixed point $(x_1 y_1)$ to a line $x = x_0$ parallel to the y-axis, and end there. Then if $x = x_1$, $y = y_1$; and if $x = x_0$, $\dfrac{\partial F}{\partial y'} = 0$, from which the constant can be calculated.

This last case is the one before us; for B_1 is fixed, and A_1 lies on the vertical A_1A_2 produced as required, since the height h of the head is given.

Let A_1B_1 be part of the curve in question, P any point on it; and ds the element of the curve at P. By rotation of the element ds about the axis of x an infinitesimal zone of surface $2\pi y\,ds$ is made, which is the element df of the outer surface of the head of the shell.

If α is the angle between the direction of motion (the x-axis) and the normal to the surface element, on the assumption of Newton's law the resistance on the element of surface is

$$\kappa\, df \cos^2\alpha = \kappa \,.\, 2\pi y\, ds \left(\frac{dy}{ds}\right)^2$$

directed along the normal to df.

The component resistance along the x-axis is

$$\kappa \,.\, 2\pi y\, ds \left(\frac{dy}{ds}\right)^3 = 2\pi\kappa y\, dy \frac{1}{1+\left(\frac{dx}{dy}\right)^2}.$$

Denote $\frac{dx}{dy}$ or x' by q; $\frac{dy}{dx}$ or y', or $\frac{1}{q}$ by p. Then the sum of the x components of the resistances against the curved surface of the head of the shell

$$W = 2\pi\kappa \int_{y_0}^{R} \frac{y\,dy}{1+x'^2} = 2\pi\kappa \int \frac{y\,dy}{1+q^2}. \quad \text{............(2)}$$

Here y is the independent variable, and the function under the integral is $\psi(y, x') = \frac{y}{1+x'^2} = \frac{y}{1+q^2}$. If the rule (1) of the Calculus of Variations is to be employed, it must be noticed that here x and y have exchanged their rôle; that is, the differential equation to be integrated is

$$0 = \frac{\partial\psi}{\partial x} - \frac{d}{dy}\left(\frac{\partial\psi}{\partial x'}\right) + \ldots .$$

Since only y and x' occur in ψ, but not x, then $\frac{\partial\psi}{\partial x} = 0$, and so

$$0 = \frac{d}{dy}\left(\frac{\partial\psi}{\partial x'}\right); \quad \frac{\partial\psi}{\partial x'} = \text{constant}.$$

Now
$$\frac{\partial\psi}{\partial x'} = \frac{\partial\psi}{\partial q} = \frac{-2qy}{(1+q^2)^2};$$

so that
$$\frac{-2qy}{(1+q^2)^2} = \text{constant} = -2C, \quad y = C\frac{(1+q^2)^2}{q}.$$

Moreover
$$dx = q\,dy = qC\frac{4q^2(1+q^2) - (1+q^2)^2}{q^2}\,dq,$$

$$\frac{dx}{C} = \left(2q + 3q^3 - \frac{1}{q}\right) dq.$$

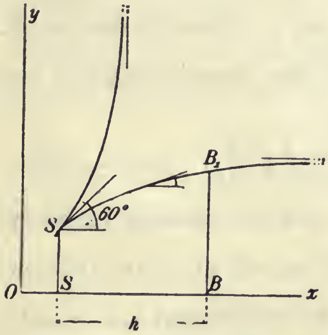
The solution of the problem is thus given by the simultaneous equations

$$x = C(q^2 + \tfrac{3}{4}q^4 - \log q + C_1), \quad y = \frac{C}{q}(1+q^2)^2. \quad \text{......(3)}$$

Here is the equation of a curve in the form $x=f_1(q)$, $y=f_2(q)$, with parameter q.

When C and C_1 are determined, the curve can be discussed and traced by points. It appears that in the general case the curve has a cusp S_1 and two asymptotes; the first parallel to the x-axis, the second parallel to the y-axis. The first and lower branch of the curve extends from $p=\sqrt{3}$ to $p=0$: the second and upper branch from $p=\sqrt{3}$, that is from the cusp S_1, to $p=\infty$.

It will be shown later that only the first branch S_1B_1 comes under consideration here.



As to the determination of the constants C and C_1 it was worked out in the following manner by N. v. Wuich (1882) and later by August (1882).

The first condition requires in any case that when $x=x_1=x_0+h$, $y=R$, since the head of the shell is to make a direct prolongation of the cylindrical part. Also the upper head surface, A_1A_2, should be as small as possible, and so the ordinate SS_1 of the cusp S_1 should be the radius AA_1 of the head surface at the forward flat end of the shell.

The condition $\frac{dy}{dq}=0$ shows that the curve ordinate y has its minimum value for $q=\frac{1}{\sqrt{3}}$, or $p=\sqrt{3}$, at the cusp S_1, and the angle between the tangent and x-axis $=60°$.

Thence the two conditions for the calculation of C and C_1 are the following: for $x=x_0+h$, y must $=R$; for $x=x_0$, $q=\frac{1}{\sqrt{3}}$.

August has examined the corresponding solution in detail; Armanini and Lampe however have shown that his solution is incorrect. The latter proved numerically that with the same calibre $2R$ of the cylindrical part of the shell, and with the same height h of the head, a hyperboloidal rotation surface can be found, with plane front surface, giving a resistance somewhat smaller than the surface of August.

The error of August's calculation lies in the fact that the part of the resistance due to the plane head surface A_1A_2 has not been taken into account in the correct way.

The total resistance against the curved surface of the head of the shell and against the plane head surface is to be made a minimum.

The variation of the end point A_1 on the parallel to the y-axis influences not only the meridian curve A_1B_1, but also the head surface $AA_1^2.\pi$ or $y_0^2\pi$.

The total resistance against the head of the shell is

$$W = 2\pi\kappa \int_0^{y_0} \frac{y\,dy}{1+q^2} + 2\pi\kappa \int_{y_0}^{R} \frac{y\,dy}{1+q^2}.$$

Here the first part denotes the resistance to the plane head surface, along which $q = 0$, since the head surface is at right angles to the x-axis; the second part is the resistance to the curved surface.

Let the first integral be divided into two parts,

$$\int_0^{y_0} = \int_0^R + \int_R^{y_0} = \int_0^R - \int_{y_0}^R.$$

This is equal to

$$\int_0^R \frac{y\,dy}{1+0} - \int_{y_0}^R \frac{y\,dy}{1+0} = \frac{R^2}{2} - \int_{y_0}^R y\,dy.$$

The following is thus to be made a minimum

$$\begin{aligned} W &= 2\pi\kappa\left(\frac{R^2}{2} - \int_{y_0}^R y\,dy\right) + 2\pi\kappa \int_{y_0}^R \frac{y\,dy}{1+q^2} \\ &= \kappa R^2\pi - 2\pi\kappa \int_{y_0}^R \left(y - \frac{y}{1+q^2}\right) dy \\ &= \kappa R^2\pi - 2\pi\kappa \int_{y_0}^R \frac{yq^2\,dy}{1+q^2}. \end{aligned}$$

To make W in this expression a minimum, the integral

$$\int_{y_0}^R \frac{yq^2\,dy}{1+q^2}$$

must be a maximum, since $\kappa R^2\pi$ is constant.

The function under the integral is now

$$\phi = \frac{yq^2}{1+q^2};$$

and the solution of the differential equation

$$0 = \frac{\partial\phi}{\partial x} - \frac{d}{dy}\left(\frac{\partial\phi}{\partial q}\right) + \ldots$$

gives $$\frac{\partial\phi}{\partial q} = \text{constant}, \quad \text{or} \quad y\,\frac{2q(1+q^2) - 2q^3}{(1+q^2)^2} = 2C$$

$$y = \frac{C}{q}(1+q^2)^2;$$

and in addition, with $dx = q\,dy$, we have, as above in (3)

$$x = C\left(q^2 + \tfrac{3}{4}q^4 - \log q + C_1\right), \quad y = \frac{C}{q}(1+q^2)^2.$$

The integration constants C and C_1 are to be determined from the conditions: $y = R$ for $x = x_1$; $\dfrac{\partial F}{\partial y'} = 0$ for $x = x_0$ (see the explanatory remark above).

Since the integral to be treated is

$$\int \frac{yq^2\,dy}{1+q^2} \quad \text{or} \quad \int \frac{yy'\,dx}{1+y'^2},$$

therefore

$$F = \frac{yy'}{1+y'^2}, \quad \frac{\partial F}{\partial y'} = y\,\frac{1+y'^2-2y'^2}{(1+y'^2)^2} = y\,\frac{1-y'^2}{(1+y'^2)^2},$$

and this $=0$; so that for $x = x_0$, $y' = \pm 1$; as is easily seen, only the upper sign comes into consideration, when a real curve is to be employed.

Thus the slope of the tangent of the curve with the x-axis must not be 60°, but 45°, at the end point A_1 of the part of the curve A_1B_1.

That a maximum of the integral arises in fact under the assumptions of the mathematical data, and with it a minimum of W, is seen from the second variation, $\dfrac{\partial^2 F}{\partial y'^2}$; this will be

$$= + \frac{yy'(y'^2-3)}{(1+y'^2)^3}.$$

Since that branch of the curve under consideration is drawn, for which the asymptote is parallel to the x-axis, and since the curve stretches from the point B_1 to the point A_1 where the slope of the tangent is 45°, y and y' are positive, and $y'^2 < 3$, and so $\dfrac{\partial^2 F}{\partial y'^2}$ is negative, and the integral is a maximum. (Kneser has examined the conditions more fully.)

Remarks on the preceding solution of the problem.

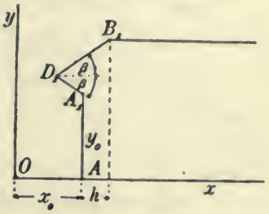
(*a*) It is unlikely that the Newtonian law can be applied to the high velocities which are here to be considered.

(*b*) The normal resistance on a surface element df is not merely dependent on $\kappa\,df$ and α, and so not exactly equal to $\kappa\,df\cos^2\alpha$, but is probably also a function of the distance y of the surface element from the long axis of the shell.

(c) The shape of the shell is involved in the air-resistance function $\kappa(v)$.

The influence of friction is parallel to the axis, and perpendicular also to the profile line of the shell, since the shell rotates; wave and eddy making are neglected completely.

The following will show where the above theory leads, when the outward flow of the air from the shell, and the wave and eddy making are not taken into account.



It might be tacitly assumed that the curve in question $y=f(x)$ has always a finite differential coefficient, and yet with an arbitrary ordinate y_0 of the initial point A_1, a curve might be imagined between A_1 and B_1 as profile of the curved surface of the shell, made up of a broken line $A_1D_1B_1$.

Suppose this to consist of the two straight lines A_1D_1 and D_1B_1, making equal angles with the x-axis at any arbitrary angle β (the point D_1 can then easily be constructed geometrically from the assumed angle β).

Then along $A_1D_1B_1$ the value of p is equal to $\pm\tan\beta$,

$$p^2=+\tan^2\beta, \text{ or } q^2=+\cot^2\beta.$$

Then will

$$W=\kappa R^2\pi-2\pi\kappa\int_{y_0}^{R}\frac{yq^2dy}{1+q^2}=\kappa R^2\pi-2\pi\kappa\frac{\cot^2\beta}{1+\cot^2\beta}\int y\,dy;$$

(since $\cot^2\beta$ is constant) we have

$$W=\kappa R^2\pi-\kappa\pi\cos^2\beta\,(R^2-y_0^2).$$

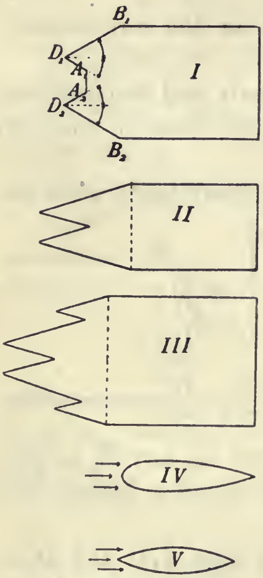
The angle β can be chosen as small as desired; and then in the limit

$$W=\kappa R^2\pi-\kappa\pi\,.\,1\,.\,(R^2-y_0^2)=\kappa\pi y_0^2.$$

And in particular if A_1 is placed on the long axis ($y_0=0$), the limit of air-resistance W against it will be zero (Legendre and Weierstrass have remarked already on this solution).

No one however will suppose that the resistance to shells of the form I, or II, or III will be very small.

The reason for this apparent contradiction lies in the fact that the outflow of air from the shell has not been taken into account.



On these grounds such theoretical calculations can have no practical importance. Experiment can alone decide the most suitable form of the head of a shell.

Moreover in relation to this subject it may be mentioned that many things indicate the essential importance of the rear end of the shell.

The shapes in IV and V were put forward by d'Alembert in 1744 and by Piobert in 1831; the egg shape by Robins; a truncated cone on the rear end of the shell by Dreyse in 1840 and Whitworth in 1860; as well as a sharp-pointed head by the latter and by Hebler.

The torpedo form is doubtless useful on purely hydrodynamical principles; but the need for the stability of the shell in the bore and in flight in the air, and other practical reasons are opposed to this form of shell.

IV. Calculation of the density of the air δ.

§ **15.** The air-resistance, according to the preceding, depends above all on the density of the air surrounding the shell.

It is required then, from the temperature of the air t° C., the barometer height H_0 mm, and the percentage of moisture in the air, to calculate the weight δ, of 1 m³ of air.

The weight of 1 cubic metre of perfectly dry air is 1·29303 kg, at sea level in latitude 45°; or 1·29388 kg in Berlin, latitude 52° 30′ and 40 m above the sea. At the same time the weight P of 1 m³ of dry air at t° C. temperature, and at H_0 mm barometer height, in accordance with the laws of Mariotte and Gay-Lussac is given at Berlin by

$$P = 1{\cdot}2939 \cdot \frac{H_0}{760} \cdot \frac{1}{1 + 0{\cdot}00367t}. \qquad \text{.............. (1)}$$

But air contains moisture, and therefore $\delta < P$; because the vapour of water, which the air holds, has only $\frac{5}{8}$ the weight of an equal volume of dry air. The height of the barometer refers to the pressure of the moist air.

We have to imagine then that water vapour of tension e has penetrated into one cubic metre of dry air, and that in consequence a certain quantity of dry air has disappeared, so that the pressure is the same as before.

Let the dry air which remains behind in the cubic metre weigh G_1 kg; the pressure due to it is given by H_1 mm. Let the water vapour which has flowed in weigh G_2 kg, with a pressure e mm.

Then according to Dalton's Law that the pressure of the mixture is the sum of the partial pressures, which each gas would have if it filled the space alone,

$$H_0 = H_1 + e. \quad \text{.........................(2)}$$

Introducing this assumption, we have to consider that at first the cube was filled with G_1 kg of dry air, at a pressure H_1. Then according to the Boyle-Mariotte law we have

$$\frac{G_1}{P} = \frac{H_1}{H_0} = \frac{H_0 - e}{H_0} \quad \text{.........................(3)}$$

Now, if only the G_2 kg of water vapour was present in the m³, at a pressure e (weight per m³ $\frac{5}{8} P$), then practically

$$\frac{G_2}{\frac{5}{8}P} = \frac{e}{H_0}. \quad \text{.............................(4)}$$

These values

$$G_1 = P\,\frac{H_0 - e}{H_0}, \text{ and } G_2 = \tfrac{5}{8} P\,\frac{e}{H_0},$$

introduced into the equation $\delta = G_1 + G_2$, give

$$\delta = \frac{P}{H_0}\,(H_0 - \tfrac{3}{8} e), \quad \text{.........................(5)}$$

or, from (1)

$$\delta\,(\text{kg/m}^3) = \frac{1{\cdot}2939}{760\,(1 + 0{\cdot}00367t)}\,(H_0 - \tfrac{3}{8} e). \quad \text{.........(I)}$$

When the air is saturated with vapour then $e = E$ (tension of water vapour at $t°$ C.), and E can be taken from the tables.

If however this is not the case, then e is a fraction of E; let $e = sE$; and 100 s will be given as the hygrometric percentage in the table; then

$$\delta = \frac{1{\cdot}2939\,H_0}{760} \cdot \frac{273}{273 + t} - 0{\cdot}174 \cdot \frac{sE}{273 + t}. \quad \text{.........(II)}$$

Thus t is the temperature of the air in degrees Centigrade (Celsius).

s is the relative moisture, that is the ratio of the tension e of the water vapour actually in the air to the tension E of water vapour in the case of saturation (for E consult Table no. 4). $100s$ is given directly by Koppe's or Lamprecht's hygrometer.

H_0 the barometric height in mm, reduced to 0° C.

The height H in mm will be read off on the mercury barometer. The readings of the barometer are reduced for purposes of comparison to one and the same temperature, viz., 0° C.

The coefficient of expansion of mercury being 1 : 5550, the height read off on the barometer

$$H = H_0\left(1 + \frac{t}{5550}\right), \text{ so that practically } H_0 = H\left(1 - \frac{t}{5550}\right);$$

and the correction amounts to

$$\frac{Ht}{5550} = 0{\cdot}000181 \,.\, Ht,$$

which is to be subtracted from H.

On the other hand the scale on the barometer changes by expansion. If the scale is made of brass (coefficient of expansion 0·000019), we have to subtract again from the last number, 0·000019 Ht.

The whole correction then to be subtracted from the reading H of the barometer is thus only 0·000162 Ht (compare Table no. 5).

Frequently the humidity of the air is not taken into account. In this case

$$\delta = \frac{0{\cdot}465 \,.\, H_0}{273 + t} \quad (H_0 \text{ in mm}),$$

and this is sufficient in most cases.

In this way the density of the air is obtained near the ground level.

Air-density δ_y at height y.

When the alteration of air-density with the elevation y m above the place of observation is taken into account, it can be allowed for by the formula

$$\delta_y = \delta\,(1 - 0{\cdot}00011\, y),$$

where δ_y denotes the density at height y (the number 0·00011, according to the calculations of Charbonnier, is more accurate than that of 0·00008 of St Robert).

It will often be sufficient, after the corresponding problem of the trajectory has been calculated with δ equal to the air-density on the ground, and in this way the height y of the vertex has been determined, to perform the operation again with an air-density δ_Y, which refers to the height Y at which the shell would be found in its flight on the average.

This average height Y, in the case where it refers to a trajectory which comes down again to the horizontal plane through the muzzle, is given by $Y = \frac{2}{3} y_s$.

The density δ_Y of the air at the average height Y is thus

$$\delta_Y = \delta (1 - 0{\cdot}00011 \,.\, \tfrac{2}{3} y_s),$$

where δ denotes the density at the ground, and y_s the height of the vertex.

The calculation is then repeated with this value of the air-density.

In all such calculations it is nevertheless essential in the calculation of the air-density δ on the ground, that the air-temperature is to be taken not at the moment of firing, but as a mean temperature based on a series of observations, because the air-temperature alters in general more slowly at a great height than near the ground. The quickly moving periodic oscillations in the temperature of the lower air-strata do not travel much upwards, and so must be ignored.

It is preferable then to measure the temperature of the air on the ground at 6 in the morning, 2 in the afternoon, and 10 in the evening, and to take the mean of these three readings.

To obtain the mean diurnal temperature of the air, more simply and almost as exactly, a single measurement will serve, if taken either in the morning between 8 and 10 o'clock (in Winter a little before 10, in Summer a little after 8), or else in the evening at 8.

In the reduction of ranges to a normal air-density, as will be explained in § 43 and § 45, it is a question of the percentage change with average height, or of $\Delta\delta_y/\delta_y$. This relation is usually replaced by the equivalent at the ground level

$$\frac{\Delta\delta_y}{\delta_y} = \frac{\Delta\delta}{\delta}.$$

But this is only correct when the relation between the air-density δ_y at height y and the air-density on the ground is a definite function $f(y)$ of the height y (for example the linear function above), i.e., $\delta_y = \delta \,.\, f(y)$, and when moreover this function $f(y)$ does not alter when the air-density changes from δ_y to $\delta_y + \Delta\delta_y$.

It is only then that $\Delta\delta_y = \Delta\delta \,.\, f(y)$.

The last relation includes the other, because the rapid variations of temperature, which can be observed near the ground and at a limited height, have been eliminated, when δ has been assumed as the mean of the last 24 hours.

When the air-density δ_y at a height y m is to be calculated more accurately than in the approximate formulae of St Robert or Charbonnier, the procedure is as follows:

The temperature $t°$ C. is calculated by assuming (see note 15) the air-temperature to diminish on the average $0^\circ{\cdot}57$ for every 100 m of height.

Thence, and from the barometer reading H_0 mm on the ground, the barometer reading H_y at that height (t_m the mean temperature between the upper and the lower height) is given by

$$\log H_y = \log H_0 - \frac{y}{18400\,(1+0{\cdot}004\,t_m)},$$

and thus
$$\delta_y = \frac{1{\cdot}294\,.\,H_y}{760}\cdot\frac{273}{273+t} - \frac{0{\cdot}174\,.\,\frac{1}{2}E}{273+t},$$

in which expression in most cases the second term may be neglected; and s is taken $=\frac{1}{2}$ at that height; E is given in terms of t in Volume IV, Table 4. Better still, the direct measurement of δ_y by means of registering kites and pilot balloons might be carried out.

An empirical table obtained in this way is given in Vol. III, § 111, from which δ_y for any height is to be taken; thence the calculation is made according to the observed air-density on the ground.

Example on § 15.

1. Reading of barometer on the ground 751·8 mm, air-temperature 15°, hygrometric state 50 °/₀.

Then $H_0 = 751{\cdot}8 - 1{\cdot}8 = 750$ mm; and according to Table 4, $E = 12{\cdot}8$ mm, $s=\frac{1}{2}$, $t = 15^\circ$; on the ground $\delta = 1{\cdot}206$ kg/m³.

2. Height of barometer on the ground $H_0 = 750$ mm, mean temperature of the air 15°. How great is the air-density δ_y at a height $y = 2000$ m?

Temperature at this height is $t = 3^\circ{\cdot}6$, so that

$$\log H_y = \log 750 - \frac{2000}{18400\left(1+0{\cdot}004\,\frac{15+3{\cdot}6}{2}\right)},\quad H_y = 578;$$

thence
$$\delta_y = 0{\cdot}93 \text{ kg/m}^3.$$

Charbonnier's expression gives

$$\delta_y = 1{\cdot}206\,(1-0{\cdot}00011\,.\,2000) = 0{\cdot}94;$$

St Robert's gives

$$\delta_y = 1{\cdot}206\,(1-0{\cdot}00008\,.\,2000) = 1{\cdot}01.$$

§ 16. Critical remarks concerning air-resistance.

We thus see that the experimental results are not entirely in agreement with those obtained by a theoretical examination of the question, and it is consequently evident that the matter has hitherto been insufficiently investigated in its theoretical aspect.

The attempts to arrive, through purely theoretical considerations, at a law of the resistance of the air to an elongated shell, moving axially and rotating, have so far led to no satisfactory results, because the phenomena cannot be properly examined.

The shell loses energy in its flight through the air by reason of the fact that the particles of the surrounding air are accelerated. These accelerations are associated with wave making, and eddy motion in consequence of friction. These complicated results of the air motion may be treated from one point of view as the results of impact, and from another as thermodynamic effects: in either case with only partial success.

Other laws, such as those of Lorenz and Vieille, giving at least the most important facts of the movement of the air round the shell in a mathematical form, are such that it is not yet known whether they are directly applicable to practical purposes.

It has been shown that the various quantities required in measurement of the air-resistance, namely the cross-section $R^2\pi$, the form-coefficient i, the velocity v, etc., do not occur in the simple manner assumed formerly; that is, as separate factors of a product, in the true function of air-resistance; and that, strictly speaking, a single form-coefficient i does not exist, as characteristic of the influence of the shape of the shell.

In the case where the elongated shell does not move axially through the air, but where the long axis makes a finite angle with the tangent to the path of the centre of gravity, the components of the air-resistance, parallel and perpendicular to the long axis, and also the point of application of the resultant air-resistance on the axis may be calculated by help of some elementary law (Newton, Lössl, etc.), but these calculations are very uncertain; because nothing is definitely known as to what elementary law is to be adopted in calculations for the high velocity of the shell; and above all whether any law can be applied with sufficient accuracy to give any practical result.

The so-called August head-form cannot possibly be the final solution of the Newtonian problem as to the most suitable shape of surface.

Not only is there a fundamental discrepancy in the mere mathematical statement of the problem, but the assumptions are in contradiction with the actual facts of the air movement round the shell.

We shall in future speak of the retardation of the shell as being $= cf(v)$, where c is proportional to the cross-section $R^2\pi$, the air-density δ, a form-coefficient i, and inversely proportional to the weight P of the shell. But this hypothesis is only adopted because there is nothing better to replace it.

CHAPTER III

Problems relating to the trajectory

§ 17. The general equations. The Principal Equation and its integrability.

The parabolic, as well as the elliptic path of a shell, considered in Chapter I for a vacuum, is altered in general by the air resistance so that the range is shortened, the vertex height and final velocity are diminished, and the angle of descent is increased.

On the other hand it is not fundamentally impossible that the air resistance may not increase the range, as recorded by v. Minarelli from the observation of such cases: this is possible with elongated shell in cases where the front part of the axis of the shell lies always, or at least for the greater part of the trajectory, above the tangent of the path; and so the action of the air against the slanting shell is of the same kind as that on a sailing ship with the sails set on the slant.

(The same thing may happen with a spherical shell, when the shell has a rotation about a horizontal axis; compare § 51 and § 58.)

These cases rarely arise, and they are excluded here, on the assumption that the axis of the elongated shell lies continuously in the tangent of the path (or that no rotation occurs in a spherical shell).

Moreover disturbing influences will be neglected, such as the rotation of the Earth, and the wind, and for the present the air density will be assumed as of its constant mean value.

Strictly speaking the ballistic coefficient c, occurring in the retardation $cf(v)$ of a shell from air resistance, is a given function of the height y of flight, because the air density δ is involved in c, and this, as in § 15, varies with the height y.

Further there might be special shells to be considered in which the weight included in c is a function of the time t (smoke-producing shells, or star shells come under this head), or others in which the cross-section varies with the time.

In such cases $cf(v)$ would be a function of the velocity v, height y, and time t, so that

$$cf(v) = \phi(v, y, t).$$

Leaving out of sight all these considerations for the present, $cf(v)$ will be considered merely as a function of v alone.

The problem then is to calculate the elements of the trajectory, under these limiting assumptions.

The trajectory is sometimes very different from that in a vacuum, as is shown in the following examples.

Examples. (*a*) French infantry bullet: $v_0 = 701$ m/sec; $2R = 8$ mm; $P = 12{\cdot}8$ g. At $\phi = 4° 59'$, the range $= 2400$ m $= 28$ °/ₒ of that in a vacuum.

(*b*) French 22 cm mortar shell M. 87; $P = 118$ kg. With $\phi = 45°$, and $v_0 = 78$, 146, 193, 228 m/s, the range is respectively 97, 92, 87, 83 °/ₒ of that in a vacuum.

The System of Equations.

Denoting the retardation due to the air resistance by $cf(v)$, and employing

$$x, x_s, X, y, y_s, \theta, \phi, \omega, v, v_0, v_e, v_s, t, T, t_s, s$$

with the same meaning as in Chapter I, the problem is formulated by the two equations

$$d(v \cos\theta) = -cf(v)\cos\theta\, dt, \qquad \text{(1)}$$

$$d(v \sin\theta) = -cf(v)\sin\theta\, dt - g\, dt, \qquad \text{(2)}$$

because the forces which act on the shell, of mass $m = \dfrac{P}{g}$, are the weight mg, in the direction of the $-y$ axis, and the resistance of the air $mcf(v)$ in the direction of the tangent.

These two equations (1) and (2) can be transformed into the five following equations, (3) to (7)

$$gd(v\cos\theta) = vcf(v)\, d\theta \qquad \text{(3)}$$

or

$$\frac{dv}{v} = \frac{d\theta}{\cos\theta}\left(\frac{cf(v)}{g} + \sin\theta\right) = \frac{dq}{1-q^2}\left(\frac{cf(v)}{g} + q\right) \qquad \text{(3a)}$$

where $q = \sin\theta$; or

$$\frac{du}{d\zeta} = \operatorname{th}\zeta + F(u) \qquad \text{(3b)}$$

where $u = \log v$, and $\sin\theta = \operatorname{th}\zeta$; also $F(u)$ is equal to $\dfrac{cf(v)}{g}$; or

$$\frac{d\tau}{dw} = -\frac{1-\tau^2}{F(w)-\tau}, \qquad \text{(3c)}$$

where $v = e^w$, $\frac{cf(v)}{g} = F(w)$, $\sin\theta = -\tau$.

$$gdx = -v^2\,d\theta, \quad\text{.............................}(4)$$

$$gdt = -\frac{v\,d\theta}{\cos\theta}, \quad\text{.............................}(5)$$

$$gdy = -v^2\tan\theta\,d\theta, \quad\text{.........................}(6)$$

$$gds = -\frac{v^2\,d\theta}{\cos\theta}, \quad\text{...........................}(7)$$

where ds is the element of arc. Also

$$\frac{dx}{dt}\frac{dp}{dt} = -g, \quad\text{...........................}(8)$$

in which $\frac{dy}{dx}$ or $\tan\theta$ is replaced by p.

Proof of the equations (3) *to* (8).

The component acceleration in the direction of the normal on the one hand is $g\cos\theta$ at the point P of the trajectory considered, and on the other hand is $\frac{v^2}{\rho}$, where ρ is the radius of curvature at point $P(x, y)$.

Now $\rho = \frac{ds}{d\theta}$; therefore (7) becomes

$$g\cos\theta = -v^2\frac{d\theta}{ds}$$

(the minus sign is required, because as s increases the angle θ is diminishing, and $d\theta$ is negative).

The elimination of $\cos\theta$ between (7) and (1) gives

$$d(v\cos\theta) = +\frac{cf(v)\,v^2 d\theta\,dt}{g\,ds},$$

or, since $\frac{ds}{dt} = v$, the equation (3) follows, which can be written in the form (3a).

The horizontal component of the velocity $\frac{dx}{dt} = v\cos\theta$; substituting in this equation the value of $\cos\theta$ from (7), we have equation (4), and then (6), since $dy = \tan\theta\,dx$ and $ds = v\,dt$.

Equation (5) is merely another form of (7), since $ds = v\,dt$.

Finally, to obtain (8),

we have from (5) $$\frac{v^2 d\theta}{dt}\frac{d\theta}{\cos^2\theta\,dt} = g^2;$$

and since $$v^2\frac{d\theta}{dt} = -g\frac{dx}{dt}, \text{ and } \frac{d\theta}{\cos^2\theta} = d\tan\theta,$$

therefore $$-\frac{dx\,d\tan\theta}{dt^2} = g.$$

The proof of these equations can be given without the aid of the expression for the centripetal acceleration $\frac{v^2}{\rho}$; because in equations (1) and (2), when the left-hand side is expanded, and the retardation $cf(v)$ of the air resistance is eliminated by multiplying the equations respectively by $\sin\theta$ and $\cos\theta$, and subtracting, then equation (5) follows immediately; and also (7).

Among these equations (1) to (8), the one which involves only two elements of the trajectory is (3*a*) or (3*b*).

We have next to integrate the differential equation (3), and to determine the constant of integration so as to make $\theta = \phi$, $v = v_0$.

If v has been obtained as a function of θ, so that $v = F(\theta)$, then

$$dx = -\frac{1}{g}(F(\theta))^2\, d\theta, \qquad dy = -\frac{1}{g}(F(\theta))^2 \tan\theta\, d\theta,$$

$$dt = -\frac{1}{g}F(\theta)\sec\theta\, d\theta, \qquad ds = -\frac{1}{g}(F(\theta))^2 \sec\theta\, d\theta;$$

and we have only to integrate with respect to θ, or sum up dx, dy, dt, ds.

As for the different methods which can be used to carry out this plan, they are considered in Chapters IV and V.

Integrability of the Chief Equation.

It is only on a definite assumption of the form of the function $cf(v)$, the retardation due to air resistance, that a first integral can exist in a finite form of this equation,

$$gd(v\cos\theta) = vcf(v)\, d\theta,$$

or, of the equation

$$\frac{dv}{v} = \frac{d\theta}{\cos\theta}\left(\frac{cf(v)}{g} + \sin\theta\right).$$

The integration was worked out in 1719 by John Bernoulli, on the assumption $cf(v) = cv^n$. Thence it is possible, as we have already seen for the quadratic law $cf(v) = cv^2$, as well as for the cubic law cv^3, the biquadratic law cv^4, and so on, to solve the problem.

And when as in § 10, the empirical value of air resistance is given by tables in a series of zones, the trajectory in this more general case can be calculated with accuracy, by dividing it up into a number of successive parts. This is discussed in the later articles, §§ 20 to 22, 32 to 34, and § 37.

Afterwards, in 1744, d'Alembert showed how the integration is possible for the more general law $cf(v) = cv^n + b$, which includes the Bernoulli law as a special case.

He examined also the functions $a\log v + b$, $av^n + R + bv^{-n}$, $a(\log v)^2 + R\log v + b$. But these three additional forms are not of any real importance for our present purpose.

In 1901, Siacci resumed the work of investigating other forms of an integrable function, and published fourteen other forms, such as

$$Av\sqrt{(2c+v^2)}+B(c+v^2), \text{ where } A, B, c \text{ are constants.}$$

Putting $\sqrt{(2c+v^2)}=vz(Bc-\sin\theta)$, equation (3) becomes

$$\frac{zdz}{z^2(B^2c^2-1)+2Acz+1}+\frac{d\theta}{\cos\theta(Bc-\sin\theta)}=0,$$

in which the variables are separated.

The form employed by Legendre, in 1782, is included in this, with $A=0$. Other functions of this kind have been examined by P. Appell, M. E. Ouivet, and T. Hayashi (Tokyo).

It need hardly be stated that there is an infinite number of such integrable functions; for it is only necessary to assume any relation between $\cos\theta$ and v, say, $\cos\theta=\psi(v)$, and to insert this, with $d\theta=\frac{-d\psi}{\sqrt{1-\psi^2}}$, into the equation (3), and then solve it for $cf(v)$; and such a function is then obtained.

Again, as in § 18, other integrable functions are obtained on the assumption of $y=\psi(x)$ for the equation of the trajectory.

The equation, (3) or (3a), is the dynamical expression of the hodograph curve of the trajectory.

Consider the line drawn through the origin O of coordinates parallel to the tangent at any point P of the path; on this parallel through O let the magnitude v of the corresponding velocity of the moving point be measured to scale so as to form a radius vector from O. When this construction is carried out for all points of the path, the ends of these vectors trace out a curve, which is called the hodograph of the corresponding path, and the variables v and θ are the polar coordinates of the hodograph.

The hodograph in general is a curved line; but in the special case of a vacuum it is the perpendicular to the axis of x at a distance $v_0\cos\phi$ from O: for in this case $cf(v)=0$, so that the equation (3) becomes

$$d(v\cos\theta)=0, \quad v\cos\theta=\text{constant}=v_0\cos\phi.$$

Recently C. Cranz and R. Rothe have shown that the equation (3) in the problem can be integrated graphically with satisfactory accuracy, in the case of an entirely arbitrary law of air resistance, without employing zonal laws.

The equation

$$\frac{dv}{v}=\frac{d\theta}{\cos\theta}\left\{\sin\theta+\frac{cf(v)}{g}\right\}$$

is first simplified by the introduction of two new variables u and ζ (instead of the former variables v and θ). We take $u=\log v$, so that $du=\frac{dv}{v}$, and $\sin\theta=\text{th}\,\zeta$, and then $d\zeta=\frac{d\theta}{\cos\theta}$.

Writing $F(u)$ for $\frac{cf(v)}{g}$, the equation becomes $\frac{du}{d\zeta}=\text{th}\,\zeta+F(u)$. Here the value of $\frac{cf(v)}{g}$ or $F(u)$ is given for v, and so for u, in the table of air resistance; and $\text{th}\,\zeta$ is given in terms of ζ in a convenient form in a table of hyperbolic functions (those for instance of W. Ligowski, Berlin 1890, published by Ernst and Korn, or in the Table of Functions of E. Jahnke and F. Emde, Leipzig 1909, published by Teubner).

Runge's graphical method of integration may be employed for the solution of a differential equation of the first order; and he has shown that his method converges, that is, the solution always becomes more accurate with a repetition of the process.

In a system of u, ζ coordinates, a number of isoclinals $\text{th}\,\zeta+F(u)=C$ are first drawn, and then starting from the origin (u_0, ζ_0) a polygon is constructed, of which the successive sides cut the isoclinals at the angles given by the corresponding value of C. A first approximation is thus obtained of $u=\phi(\zeta)$. Substitute this value of u in the differential equation, and we obtain

$$du=\{\text{th}\,\zeta+F[\phi(\zeta)]\}\,d\zeta.$$

As the variable ζ above is contained in the right-hand side, we can integrate with respect to ζ, from ζ_0 to ζ; and so a closer solution is obtained.

Moreover, Runge shows how to follow graphically the substitution and integration, as well as the other details.

We can imagine then the construction of a special integraph contrived for the integration of the equation, so as to carry out mechanically the remaining integrations.

An apparatus is described by E. Pascal in his book on Integrators which is specially devised to carry out the integration of the equation.

Filloux has proposed to use Prytz's planimeter for the same purpose.

§ 18. An Inversion Problem.

If the actual path of the shell is known (either by the equation $y=\psi(x)$ between the coordinates x and y of any point on it, or graphically in any other manner) then for any such point (x, y) of the trajectory the corresponding velocity of the C.G. of the shell, the inclination θ to the horizon of the tangent of the path, the time of flight t, and the air resistance $W=mcf(v)$ are obtained in the following manner.

We find the first three derivatives of y with respect to x; denote them by y', y'', y'''; then

$$v=\sqrt{\left(g\frac{1+y'^2}{-y''}\right)}, \quad v\cos\theta=\sqrt{\frac{g}{-y''}},$$

$$cf(v)=-g\frac{y'''\sqrt{(1+y'^2)}}{2y''^2}, \quad dt=dx\sqrt{\frac{-y''}{g}};$$

so that the time t is given by an integration of this last equation.

For if we differentiate $\tan\theta=\frac{dy}{dx}=y'$ with respect to θ, then

$$\frac{1}{\cos^2\theta}=\frac{dy'}{dx}\frac{dx}{d\theta}=y''\frac{dx}{d\theta},$$

$$\frac{dx}{d\theta}=-\frac{v^2}{g}, \quad \frac{1}{\cos^2\theta}=-\frac{v^2}{g}y'', \quad v\cos\theta=\sqrt{\frac{g}{-y''}}.$$

And further,

$$\cos\theta=\frac{1}{\sqrt{(1+y'^2)}},$$

so that

$$v=\sqrt{\left(g\frac{1+y'^2}{-y''}\right)},$$

and along the trajectory y'' is negative, so that $\sqrt{(-y'')}$ is real.

The relation for t follows from

$$v\cos\theta=\frac{dx}{dt}, \quad dt=\frac{dx}{v\cos\theta}=dx\sqrt{\frac{-y''}{g}}.$$

Finally, in the equation (3)

$$cf(v)=\frac{gd(v\cos\theta)}{vd\theta};$$

and here, since

$$v\cos\theta=\sqrt{\frac{g}{-y''}},$$

$$\frac{d}{d\theta}(v\cos\theta)=\frac{d(v\cos\theta)}{dx}\frac{dx}{d\theta}=-\frac{v^2}{g}\sqrt{(-g)}(-\tfrac{1}{2})(y'')^{-\frac{3}{2}}y'''=+\frac{v^2y'''}{2\sqrt{(-gy''^3)}}$$

so that

$$cf(v)=\frac{gvy'''}{2\sqrt{(-gy''^3)}}=-\frac{gy'''\sqrt{(1+y'^2)}}{2y''^2}.$$

Examples.

1. The trajectory in many cases may be replaced conveniently by an hyperbola with a vertical asymptote; this has been stated by Newton, Indra, Ökinghaus, and Stauber.

E. Ökinghaus formerly stated that the trajectory was actually such an hyperbola; later he assumed the two asymptotes to be slanting, and discussed the hyperbolic solution of the problem as merely an approximate solution; see § 19, theorem 6.

On the assumption that the trajectory should be an hyperbola

$$y=\frac{ax-x^2}{b-x}\cdot\frac{b}{a}\tan\phi,$$

then
$$cf(v)\cos\theta = \frac{3ga(b-x)^2}{4b^2(b-a)\tan\phi},$$
$$(v\cos\theta)^2 = \frac{ga(b-x)^3}{2b^2(b-a)\tan\phi},$$
$$\tan\theta = \tan\phi \cdot \frac{b}{a}\left\{1 - \frac{b(b-a)}{(b-x)^2}\right\},$$
$$t = 2\left\{\frac{1}{\sqrt{(b-x)}} - \frac{1}{\sqrt{b}}\right\}\sqrt{\left\{\frac{2b^2}{ga}(b-a)\tan\phi\right\}}.$$

2. Piton-Bressant assumes the trajectory to be a parabola of the 3rd order,
$$y = x\tan\phi - \frac{gx^2}{2v_0^2\cos^2\phi}(1+mx),$$
where m is an empirical constant to be determined (compare also § 25).

Then in this case, since
$$y''' = -\frac{3gm}{v_0^2\cos^2\phi},$$
the law for the retardation due to air resistance will be as follows:
$$cf(v) = -\frac{3mv^4\cos^3\theta}{2v_0^2\cos^2\phi},$$
also
$$\tan\theta = \tan\phi - \frac{gx}{v_0^2\cos^2\phi}(1+\tfrac{3}{2}mx),$$
$$v\cos\theta = v_0\cos\phi\,(1+3mx)^{-\frac{1}{2}},$$
$$t = \frac{2\{(1+3mx)^{\frac{3}{2}}-1\}}{9mv_0\cos\phi}.$$

3. The following proposition was made by C. F. Close (see above, § 11). Assuming the principle of the tilting of the trajectory as satisfactory, and that a range table has been constructed, the trajectory of the gun may be constructed for the extreme range by a tilting of the lesser trajectories. The points on the trajectory are then given in polar coordinates.

The relation between radius vector and polar angle is given by an equation, and thence on the above principle the values of v, $v\cos\theta$, t, $cf(v)$ for any point of the longest trajectory are found.

It is possible then with the aid of the range table to obtain the air resistance; and corresponding calculations have been made by G. Greenhill and C. E. Wolff. Consult § 11 and §§ 38 to 40 on the probability of these assumptions.

§ 19. General properties of every trajectory.

A knowledge of the differential equations established in § 17 is sufficient to deduce a series of general values, independently of any assumption of a special law of air resistance, for any trajectory.

It is assumed, however, that the resultant air resistance acts along the tangent of the trajectory, and the retardation $cf(v)$, due to air resistance, is a continuous function of the velocity alone.

Those general properties of the trajectory are now to be discussed, which differ between the paths in air and in a vacuum.

1. The horizontal component $v \cos\theta$ of the velocity v in the flight of the shell diminishes along the trajectory.

Proof. In the equation

$$gd(v\cos\theta) = cf(v)\,v\,d\theta,$$

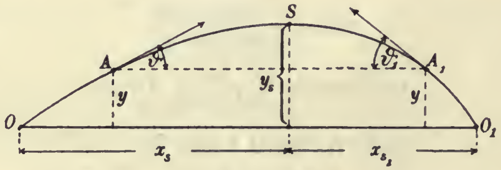
where $cf(v)$ is positive, $d\theta$ is always negative; because the angle of slope θ with the horizontal decreases from its original value ϕ; thus the right-hand side of the equation is negative, and so $d(v\cos\theta)$ is negative; that is, $v\cos\theta$ diminishes always.

Numerical example. A shell from a field gun, $v_0 = 442$ m/sec, $\phi = 15\frac{14}{16}$ degrees, calibre 8·8 cm, weight of shell $P = 7{\cdot}5$ kg.

For the horizontal distances

$$x = 0,\ 3000,\ 5000 \text{ m},\quad v\cos\theta = 425,\ 223,\ 168 \text{ m/sec}.$$

2. The angle of descent ω is greater than the angle of departure ϕ.



In general, at two points A and A_1 with equal ordinates y (A on the ascending branch, A_1 on the descending) the angle of slope θ_1 at A_1 is greater than θ at A.

Proof. The equation

$$gdy = -v^2\tan\theta\,d\theta, \quad \text{or} \quad -\frac{\tan\theta\,d\theta}{\cos^2\theta} = +\frac{gdy}{(v\cos\theta)^2}$$

is to be integrated, first from the origin O up to the vertex S, or from $\theta = \phi$ to $\theta = 0$, or also, from $y = 0$ to $y = y_s$; on the other hand, back from the point of descent O_1 to the vertex S; so that

$$\text{on the one side, } +\tfrac{1}{2}\tan^2\phi = \int_0^{y_s} \frac{gdy}{(v\cos\theta)^2},$$

$$\text{on the other side, } +\tfrac{1}{2}\tan^2\omega = \int_0^{y_s} \frac{gdy}{(v\cos\theta)^2}.$$

Since $v\cos\theta$ is always diminishing, the denominator in the second integral is always less than that in the first integral; or the fraction under the integral sign in the second integral is always greater than that in the first; the second integral is thus greater than the first; so that

$$\tan\omega > \tan\phi, \quad \omega > \phi.$$

The same holds when the integration is taken from A or A_1.

Numerical example, as in No. 1.

We have $\omega = 24°\ 53'$, $\phi = 15\frac{14}{16}$ degrees.

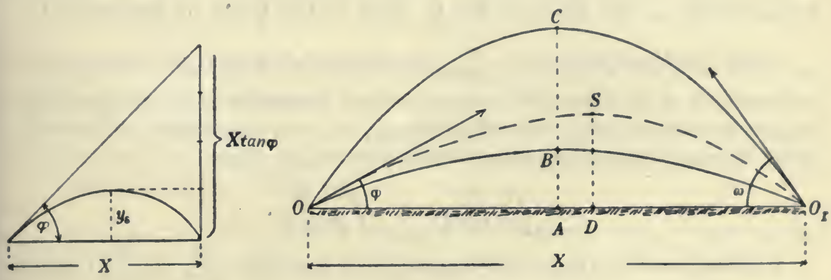
3. The vertex height y_s of the trajectory always lies between $\frac{1}{4}X\tan\phi$ and $\frac{1}{4}X\tan\omega$, where X is the range.

Captain Anér, of the Swedish infantry, has worked out a rigorous proof of this law. In the parabolic flight in a vacuum

$$y_s = \frac{v_0^2\sin^2\phi}{2g}, \quad X = \frac{v_0^2}{g}\sin 2\phi,$$

so that
$$y_s = \tfrac{1}{4}X\tan\phi.$$

Next, $OO_1 = X$, and consider first the parabolic path OBO_1, with angle ϕ of departure and descent, for which the vertex height $AB = \frac{1}{4}X\tan\phi$; next, the parabolic path OCO_1, with departure and descent angle ω, for which the vertex height $AC = \frac{1}{4}X\tan\omega$; thirdly, the actual trajectory OSO_1, with angle of departure ϕ, and of descent ω, for which the vertex height is DS.



Then the last, DS, is in every case smaller than AC and greater than AB, because the actual trajectory OSO_1 must lie between the two parabolic paths OBO_1 and OCO_1; thus DS lies between $\frac{1}{4}X\tan\phi$ and $\frac{1}{4}X\tan\omega$.

Numerical example, as in No. 1; $X = 4501$ m, vertex height lies between 320 m and 520 m. The most probable value is then the mean, 420 m.

A calculation by Siacci (§ 28) gave 416 m, and Table 12 of Volume IV gave 425 m.

Either the arithmetic or geometric mean may be taken in the calculation of the vertex height, or

$$y_s = \tfrac{1}{8}X(\tan\phi + \tan\omega), \text{ or } y_s = \tfrac{1}{4}X\sqrt{(\tan\phi\tan\omega)}.$$

4. If A and A_1 are two points on the trajectory at the same height y above the horizontal through the muzzle, the velocity v at the point A in the ascending branch of the trajectory is greater than the velocity v_1 at the point A_1 of the descending branch.

Proof. The equation of motion of the shell resolved along the tangent, is

$$\frac{dv}{dt} = -cf(v) - g\sin\theta,$$

or if s denotes the arc of the trajectory to the point considered

$$\tfrac{1}{2}d(v^2) = -cf(v)\,ds - g\sin\theta\,ds = -cf(v)\,ds - g\,dy.$$

If this equation is integrated from A to A_1, then $\int dy = 0$, and there remains

$$\tfrac{1}{2}(v_1^2 - v^2) = -c\int_{s_1}^{s_2} f(v)\,ds;$$

the right-hand side is negative, and so $v_1 < v$.

This equation can be obtained at once by means of the fact that the alteration $\frac{1}{2}m(v_1^2 - v^2)$ of the kinetic energy of the shell, in passing from A to A_1, is the sum of the work of air resistance and gravity; and as the last part is zero when A and A_1 stand at the same height,

$$\tfrac{1}{2}m(v_1^2 - v^2) = -\int mcf(v)\,ds.$$

Numerical example, as in No. 1: $y=0$, $v_0 = 442$ m/sec, $v_e = 197$ m/sec.

5. The vertex point S of the trajectory is nearer, measured horizontally, to the point of fall O_1 than to the point of departure O.

Proof. Let the equation $dx = \frac{dy}{\tan\theta}$ be integrated, first from the origin O up to the vertex S, or from $y=0$ to $y=y_s$, and let θ denote the angle of slope of the tangent to the horizontal. Secondly let the equation be integrated from the point of fall O_1 back to the vertex, with θ_1 the angle of slope. Then

$$x_s = \int_0^{y_s}\frac{dy}{\tan\theta}, \quad x_{s_1} = \int_0^{y_s}\frac{dy}{\tan\theta_1}.$$

But here, according to 2, for the same y, $\theta_1 > \theta$, so that $\frac{dy}{\tan\theta_1} < \frac{dy}{\tan\theta}$; thence $x_{s_1} < x_s$.

Numerical example, as in No. 1. A calculation according to Siacci gave

$$x_s = OD = 2500 \text{ m}, \quad x_{s_1} = O_1D = 2001 \text{ m}.$$

6. The descending branch of the trajectory has a vertical asymptote, at a distance $\frac{1}{g}\int_{-\frac{1}{2}\pi}^{\phi} v^2\,d\theta$ from the origin; the velocity in the path increases there and approaches a limiting value v_1, to be calculated from the equation $cf(v_1) = g$.

Proof. We have $dt = -\frac{v}{g}\frac{d\theta}{\cos\theta} = -\frac{v\cos\theta}{g}\frac{d\theta}{\cos^2\theta}$. When this equation is integrated from $t=0$ to $t=t$, it is allowable to replace $v\cos\theta$ by a mean value μ, since $v\cos\theta$ is always finite and continuous, and $\frac{1}{\cos^2\theta}$ does not alter in sign.

According to this

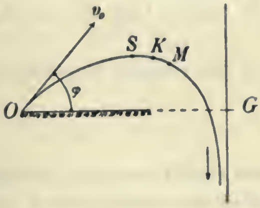
$$t = -\frac{\mu}{g}\int_{\phi}^{\theta}\frac{d\theta}{\cos^2\theta} = -\frac{\mu}{g}(\tan\theta - \tan\phi).$$

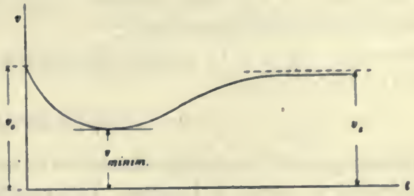
The prolongation of the path of the shell past the horizontal through the muzzle, will converge to a vertical direction because when $t=\infty$, the left-hand side of the equation becomes infinite, and consequently the right-hand side also: since μ and $\tan\phi$ are finite, $\tan\theta$ must $= -\infty$, and $\theta = -\frac{\pi}{2}$.

It is evident that this vertical, to which the descending branch approaches, is a line at a finite distance, because of the relation

$$dx = -\frac{v^2}{g}\,d\theta, \quad x = -\frac{1}{g}\int_{\phi}^{\theta} v^2\,d\theta.$$

Here v^2 is always finite, because, starting from the initial value v_0, the velocity diminishes at first, provided ϕ is different from zero and is positive, in consequence of the effect of gravity and air resistance.





After v has reached a minimum, it increases again under the effect of gravity, till finally the air resistance becomes equal to the weight of the shell. Then when this limiting value v_1 is reached, which is theoretically only after an infinite time, the forces $mcf(v)$ and mg balance one another, and the shell moves on with the constant velocity v_1.

Thus the integral $\int_{\phi}^{\theta} v^2\,d\theta$ is always finite, whatever value θ may assume between ϕ and $-\frac{1}{2}\pi$; and the limiting value of x is thus

$$OG = -\frac{1}{g}\int_{\phi}^{-\frac{1}{2}\pi} v^2\,d\theta = +\frac{1}{g}\int_{-\frac{1}{2}\pi}^{\phi} v^2\,d\theta.$$

Numerical example. A shell was fired at Meppen on April 28, 1892, with the following initial conditions: calibre 24 cm, weight of shell 215 kg, radius of ogival head 2 calibres, initial velocity 640 m/sec, angle of departure 44°, air density taken at 1·22 kg/m³.

Calculation gave therefore the following results: horizontal range 19,066 m, time of flight 68·8 seconds, final velocity 380·4 m/sec, angle of descent 58° 21′·5, vertex abscissa 10,840 m, vertex ordinate 6,150 m. Moreover the limiting value v_1 to which the velocity was tending continually, was about 580 m/sec, and the distance of the vertical asymptote from the point of departure = 29,300 m.

7. The minimum value v_m is given by the equation

$$cf(v_m) = -g\sin\theta.$$

The point of the trajectory, where this value is reached, lies beyond the vertex, in the descending branch.

Proof. To obtain the slope of the tangent of the path where v is a minimum, the derivative of v with respect to θ must be made zero; but as, in general,

$$\frac{dv}{d\theta} = \frac{v}{\cos\theta}\left\{\frac{cf(v)}{g} + \sin\theta\right\},$$

the first part of the theorem follows immediately.

Further, the velocity v in the path at any point can be resolved into a horizontal component $v \cos\theta$, and a vertical component $v \sin\theta$.

From the starting-point up to the vertex, both these components diminish, as also their resultant v.

At the vertex, the horizontal component is still decreasing; the vertical component on the other hand has reached its minimum; or, in other words, is constant for a moment; consequently the rate of change of the resultant v depends on $v \cos\theta$, and as this is diminishing, v is decreasing at the vertex.

But since v in any case increases again later, it follows that the minimum must lie on this side of the vertex.

The exact place is to be determined by means of the relation between v and θ, and by the equation

$$cf(v) + g \sin\theta = 0.$$

Example, as in No. 6. By calculation it was found that if $\theta = -15°$, v became a minimum, at about 251 m/sec. The coordinates of the corresponding point were $x = 12{,}570$ m, $y = 5880$ m.

8. Curvature of the trajectory. The point K of maximum curvature is given by means of the relation $cf(v) = -\frac{3}{2} g \sin\theta$; it lies always on the descending branch, and between the vertex S, and the point M of least velocity.

Proof. The acceleration of the shell in the direction of the normal to the curve is on the one hand $g \cos\theta$, and on the other $\frac{v^2}{\rho}$, where ρ is the radius of curvature; and so

$$|\rho| = \frac{v^2}{g \cos\theta};$$

and this expression will reach a minimum, or the curvature a maximum when $\frac{d\rho}{d\theta} = 0$. But

$$\frac{d\rho}{d\theta} = \frac{2v \frac{dv}{d\theta} \cos\theta + v^2 \sin\theta}{g \cos^2\theta};$$

and (§ 17, equation 3a)

$$\frac{dv}{d\theta} = v \tan\theta + \frac{v\,cf(v)}{g \cos\theta}.$$

Therefore the condition for an extreme value is given by

$$0 = 2v \cos\theta \left\{ v \tan\theta + \frac{v\,cf(v)}{g \cos\theta} \right\} + v^2 \sin\theta,$$

or $3g \sin\theta + 2cf(v) = 0$, as above; and from this condition the point K of maximum curvature of the path can be obtained.

As to the position of this point and the nature of the extreme value, the change in $\rho = \frac{v^2}{g \cos\theta}$ must be considered, first from the origin O to the vertex S, and then secondly from the point M of minimum velocity and beyond.

The velocity v diminishes up to the vertex S; θ diminishes at the same time or $\cos\theta$ increases, and $\frac{1}{\cos\theta}$ diminishes; on both grounds the radius of curvature ρ diminishes from O to S, that is the trajectory becomes more and more curved from the origin to the vertex.

On the other hand, at the point M of smallest velocity v, the numerator v in the expression $\frac{v^2}{g\cos\theta}$ is constant for a moment, θ has become negative, and $\cos\theta$ is diminishing, i.e., $\frac{1}{\cos\theta}$ is increasing. Thus the alteration of ρ in the neighbourhood of the point M depends on the alteration of $\frac{1}{\cos\theta}$; that is, ρ is also diminishing there. But the curvature is continuous, so that a minimum value of ρ must lie between S and M.

It may be remarked further that the expression $\rho=\frac{v^2}{g\cos\theta}$ is completely independent of any special law of air resistance, since $f(v)$ is not involved in it; and so it follows that all trajectories with the same v and θ have three consecutive points in common with the parabolic path in a vacuum.

It follows also in consequence that the actual trajectory may often be replaced with advantage for a short arc by the corresponding parabolic path in a vacuum; for instance in the neighbourhood of the origin O (see figure on p. 97) by the parabola OBO_1 with the same v_0 and ϕ, or near the point O_1 by the parabola OCO_1 with the same v_e and ω.

This is convenient in the ballistics of small arms, in the measurement of the error of the angle of departure, for the determination of the drop y of the bullet, or in the determination of the range.

Numerical example, as in No. 6. Calculation gave the coordinates of the point K of greatest curvature as $x=12{,}000$ m, $y=6000$ m, and thence $\theta=-10°$.

9. The vertical component of the velocity increases throughout the whole descending branch. At two points A and A_1 with equal ordinate y (figure, page 96) the vertical component is greater in the ascending branch at A than at A_1 in the descending branch.

Proof. Substitute in equation (2) on p. 89, that is, in

$$d(v\sin\theta)=-\{g+cf(v)\sin\theta\}\,dt,$$

the value of dt from $\frac{dy}{dt}=v\sin\theta$; then

$$\tfrac{1}{2}d(v\sin\theta)^2=-\{g+cf(v)\sin\theta\}\,dy.$$

This equation is to be integrated first in the ascending branch from A to S, from $v\sin\theta$ to $v_s\sin 0$, on the left-hand side, and on the right from y to y_s; secondly on the descending branch from the vertex S down to A_1, that is on the left from $v_s \sin 0$ to $v \sin\theta$, on the right from y_s to y; we have then

$$(a)\quad 0-\tfrac{1}{2}(v\sin\theta)^2=-\int_y^{y_s}\{g+cf(v)\sin\theta\}\,dy,$$

$$(b)\quad \tfrac{1}{2}(v\sin\theta)^2-0=-\int_{y_s}^{y}\{g+cf(v)\sin\theta\}\,dy$$

$$=+\int_y^{y_s}\{g+cf(v)\sin\theta\}\,dy.$$

In this last equation (b), as in (a), we take θ as the acute angle between the horizontal and the tangent of the path; then in (b), $\sin\theta$ is negative, and so

$$(a)\quad \tfrac{1}{2}(v\sin\theta)^2=+\int_y^{y_s}\{g+cf(v)\sin\theta\}\,dy,\ \text{ascending branch;}$$

$$(b)\quad \tfrac{1}{2}(v\sin\theta)^2=+\int_y^{y_s}\{g-cf(v)\sin\theta\}\,dy,\ \text{descending branch.}$$

In the integrals on the right-hand side, the values of (b) are less than those of (a), and so the integral in (b) is less than that in (a), and thus $v\sin\theta$ in (b) is less than $v\sin\theta$ in (a).

10. For a range on the horizontal through the muzzle, the time of flight in the descending branch is greater than the time in the ascending branch.

Proof. Let the time of flight in the ascending branch from O to S (figure on p. 96) be denoted by t_1, and the time of flight in the descending branch from S to O_1 by t_2.

From equation (5) of the system in § 17,

$$dt=-\frac{v\,d\theta}{g\cos\theta}.$$

This equation is to be integrated from O to S, i.e., on the left from $t=0$ to $t=t_1$, and on the right from $\theta=\phi$ to $\theta=0$; and next from O_1 to S, i.e., on the left from $t=0$ to $t=t_2$, and on the right, where θ must denote again the acute angle, from $\theta=\omega$ to $\theta=0$.

This gives

$$(a)\quad t_1=-\int_{\theta=\phi}^{\theta=0}\frac{v\,d\theta}{g\cos\theta}=+\int_{\theta=0}^{\theta=\phi}\frac{v\,d\theta}{g\cos\theta},$$

$$(b)\quad t_2=-\int_{\theta=\omega}^{\theta=0}\frac{v\,d\theta}{g\cos\theta}=+\int_{\theta=0}^{\theta=\omega}\frac{v\,d\theta}{g\cos\theta}.$$

Both integrals are finite, since v and $\cos\theta$ are finite, and therefore t_1 and t_2 are also finite.

We can also employ the equation

$$dt=\frac{dy}{v\sin\theta},$$

and integrate it, in spite of the zero value in the denominator at the vertex; first in the ascending branch from O to S, that is, from $y=0$ to $y=y_s$, and secondly in the descending branch, backwards from O_1 to S, that is, from $y=0$ to $y=y_s$ (with θ again the acute angle).

Then (a) $t_1 = \int_{y=0}^{y=y_s} \frac{dy}{v \sin \theta}$, ascending branch,

(b) $t_2 = \int_{y=0}^{y=y_s} \frac{dy}{v \sin \theta}$, descending branch.

Here, as was shown before in Law 9, $v \sin \theta$ in (b) for the same y is less than $v \sin \theta$ in (a), so that $\frac{1}{v \sin \theta}$ in (b) is greater than $\frac{1}{v \sin \theta}$ in (a); and so it follows that the integral in (b) is greater than the integral in (a) (term by term), or $t_2 > t_1$.

11. The arc s_1 of the ascending branch from the origin O to the vertex S is longer than the descending arc s_2, from the vertex S to the point of fall O_1 on the horizontal through the muzzle.

Proof. Take the result $ds = \frac{dy}{\sin \theta}$, and first integrate it from O to S, and so from $s=0$ to $s=s_1$ on the left; and on the right from $y=0$ to $y=y_s$; secondly from the point of fall O_1 backwards to S, where θ again denotes the upward acute angle. We have thus

(a) $s_1 = \int_0^{y_s} \frac{dy}{\sin \theta}$, ascending branch;

(b) $s_2 = \int_0^{y_s} \frac{dy}{\sin \theta}$, descending branch.

According to Law 2, for the same y the angle θ in (b) is always greater than in (a), so that $\frac{1}{\sin \theta}$ in (b) is always less than $\frac{1}{\sin \theta}$ in (a), and so $s_2 < s_1$.

Consult the remarks in the notes in the appendix concerning the question of the angle of departure ϕ, with given initial velocity v_0, which corresponds to the greatest range in air.

CHAPTER IV

First group of calculations in the approximate solution of the ballistic problem. Approximate solution of the exact differential equations

§ **20.** It was shown in § 17 that the procedure in a numerical solution must be such that the equation,

$$gd\,(v\cos\theta) = cf(v)\,v\,d\theta,$$

must first be integrated, where $cf(v)$ is the retardation due to the air resistance; and then its integral equation, in the form $v = F(\theta)$, must be employed to carry out the integration or summation in

$$x = -\int \frac{v^2}{g}\,d\theta, \quad y = -\int \frac{v^2}{g}\tan\theta\,d\theta, \quad t = -\int \frac{v}{g}\sec\theta\,d\theta.$$

Strictly speaking, the problem cannot be solved analytically in a finite form except for a law of air resistance $cf(v) = cv$.

In other cases, as we shall see, only approximations are possible.

As a first group of approximate solutions, let those be examined for which the equation itself has been solved exactly; and let approximations be used for the summation of dx, dy, dt. This first group is the one with which this chapter is concerned.

Later a second group will be taken in which an approximate method is employed, for the equation itself, in that the equation is replaced by another approximation, for which all further integrations are made possible.

1. Solution of the equation on the assumption of a retardation = a + cv^n.

The equation now becomes

$$gd\,(v\cos\theta) = v\,(a + cv^n)\,d\theta.$$

When the left-hand side of this equation is expanded, and the equation is divided by v^{n+1}, and $av^{-n}d\theta$ brought to the left-hand side,

$$g\cos\theta\,.\,v^{-n-1}\,dv - (a + g\sin\theta)\,v^{-n}\,d\theta = cd\theta. \quad \text{......(1)}$$

Put $v^{-n} = u$; then $-nv^{-n-1}\,dv = du$, so that we have an equation in Euler's form

$$\frac{du}{d\theta} + uR(\theta) = Q(\theta),$$

where $$R = +\frac{a + g\sin\theta}{\cos\theta}\cdot\frac{n}{g}, \quad Q = -\frac{nc}{g\cos\theta}.$$

The integral of this differential equation of the first order is

$$u = e^{-\int R d\theta}\left[\int (Qe^{+\int R d\theta})\,d\theta + \text{the constant of integration}\right].$$

Consequently, in the preceding case, the required relation between v and θ is the following:

$$\frac{1}{(v\cos\theta)^n} = e^{-\frac{n}{g}\int\frac{a\,d\theta}{\cos\theta}}\left[-\frac{n}{g}\int\frac{ce^{+\frac{n}{g}\int\frac{a\,d\theta}{\cos\theta}}}{(\cos\theta)^{n+1}}\,d\theta + \text{the constant of integration}\right]. \quad \ldots(2)$$

2. In the special case $a = 0$, the law becomes $cf(v) = cv^n$ for the retardation of the air resistance.

Then for any value of the angle θ, the velocity v of the shell is obtained from

$$\frac{1}{(v\cos\theta)^n} = -\frac{nc}{g}\int\frac{d\theta}{(\cos\theta)^{n+1}} + \text{a constant.} \quad \ldots\ldots\ldots(3)$$

This constant can be obtained, in the foregoing special case of $cf(v) = cv^n$, from

$$gd(v\cos\theta) = cf(v)\,vd\theta.$$

Thus

$$gd(v\cos\theta) = cv^n \,.\, vd\theta = c(v\cos\theta)^{n+1}\frac{d\theta}{(\cos\theta)^{n+1}},$$

$$\frac{d(v\cos\theta)}{(v\cos\theta)^{n+1}} = \frac{c}{g}\frac{d\theta}{(\cos\theta)^{n+1}};$$

and integrating

$$\frac{1}{(v\cos\theta)^n} = -\frac{nc}{g}\int\frac{d\theta}{(\cos\theta)^{n+1}} + \text{a constant}$$

For the calculation of this integral,

$$\int\frac{d\theta}{(\cos\theta)^{n+1}},$$

we know that

$$\int \frac{dx}{(\cos x)^n} = \frac{\sin x}{(n-1)(\cos x)^{n-1}} + \frac{n-2}{n-1}\int \frac{dx}{(\cos x)^{n-2}}.$$

Thus, for example,

$$\int \frac{dx}{(\cos x)^3} = \frac{\sin x}{2(\cos x)^2} + \tfrac{1}{2}\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) \quad\text{.......... (4)}$$

$$= \tfrac{1}{2}\sin x(1+\tan^2 x) + \tfrac{1}{2}\log\frac{1+\sin x}{\cos x}$$

$$= \tfrac{1}{2}p\sqrt{(1+p^2)} + \tfrac{1}{2}\log\{\sqrt{(1+p^2)}+p\},$$

where $p = \tan x$.

This function, which may be denoted for short by ξ, is given in Vol. IV, Table 10*b*, for values of

$$\xi(x) = \frac{1}{2}\left(\frac{\sin x}{\cos^2 x} + \log\frac{1+\sin x}{\cos x}\right) = \frac{1}{2}\frac{\sin x}{\cos^2 x} + \tfrac{1}{2}\log\tan(45^\circ + \tfrac{1}{2}x),$$

for angles x from 0° to 87°.

A more extended Table is found in Otto's Tables for bomb throwing, Berlin 1842.

$$\int \frac{dx}{(\cos x)^4} = \tan x + \tfrac{1}{3}(\tan x)^3 \quad\text{.......... (5)}$$

$$\int \frac{dx}{(\cos x)^5} = \frac{\sin x}{4\cos^4 x} + \frac{3\sin x}{8\cos^2 x} + \tfrac{3}{8}\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) \quad\text{...... (6)}$$

$$\int \frac{dx}{(\cos x)^6} = \tan x + \tfrac{2}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x, \text{ etc.} \quad\text{.......... (7)}$$

With $n = 2$, for example,

$$\frac{1}{(v\cos\theta)^2} = -\frac{c}{g}\left(\frac{\sin\theta}{\cos^2\theta} + \log\frac{1+\sin\theta}{\cos\theta}\right) + \text{constant};$$

or, writing $\tan\theta = p$,

$$\frac{1+p^2}{v^2} = -\frac{c}{g}\left\{p\sqrt{(1+p^2)} + \log[\sqrt{(1+p^2)}+p]\right\} + \text{const.}$$

When this expression for v, or rather for v^2, is substituted in the general equations

$$g\,dx = -v^2 d\theta, \qquad g\,dy = -v^2\tan\theta\, d\theta,$$

$$g\,dt = -v\sec\theta\, d\theta, \qquad g\,ds = -v^2\sec\theta\, d\theta,$$

then dx, dy, dt, ds are expressed entirely in θ, or in p, with $\tan\theta = p$, or in z with $\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) = z$; so that it only remains to work out these integrations, that is, the problem is reduced to quadratures.

3. In the special case of $a=0$ and $n=1$, i.e., on the assumption $cf(v)=cv$, the expressions for all the elements of the path can be obtained in a finite form, as stated already.

As functions of t, we have then

$$x = v_0 \cos\phi \frac{1-e^{-ct}}{c}, \quad \text{.........(8)}$$

$$y = -\frac{gt}{c} + \frac{g + cv_0 \sin\phi}{c^2}(1-e^{-ct}), \quad \text{.........(9)}$$

$$v\cos\theta = v_0 \cos\phi\, e^{-ct}, \quad \text{.........(10)}$$

$$v\sin\theta = -\frac{g}{c} + \frac{g + cv_0\sin\phi}{c} e^{-ct}. \quad \text{.........(11)}$$

But this law of air resistance, $cf(v) = cv$, does not generally come into consideration.

4. In the special case of $a=0$ and $n=2$, and the assumption $cf(v)=cv^2$, the quadratic law of air resistance is obtained. As above

$$\frac{1}{(v\cos\theta)^2} = -\frac{2c}{g}\xi(\theta) + \text{a constant of integration.}$$

The integration constant is determined from the condition that $v=v_0$ and $\theta=\phi$ at the origin; and so

$$v^2 = \frac{g}{2c}\frac{1}{\cos^2\theta\{C-\xi(\theta)\}}, \quad \text{.........(12)}$$

where
$$C - \xi(\phi) = \frac{g}{2c(v_0\cos\phi)^2}, \quad \text{.........(13)}$$

and
$$\xi(\phi) = \tfrac{1}{2}\left\{\frac{\sin\phi}{\cos^2\phi} + \log\tan(45^\circ + \tfrac{1}{2}\phi)\right\}.$$

This constant is connected with the velocity v_s of the shell at the vertex of the path by a simple relation. Since $\theta=0$ at the vertex, and there $\xi(\theta)=0$, and $v=v_s$, therefore

$$C = \frac{g}{2cv_s^2}. \quad \text{.........(14)}$$

Introducing this value of v^2 or v in the system of equations

$$g\,dx = -v^2 d\theta, \quad g\,dy = -v^2\tan\theta\, d\theta,$$

$$g\,dt = -\frac{v\,d\theta}{\cos\theta}, \quad g\,ds = -\frac{v^2 d\theta}{\cos\theta},$$

then
$$2c\,dx = -\frac{d\theta}{\cos^2\theta\{C-\xi(\theta)\}}, \quad \text{.........(15)}$$

$$2c\,dy = -\frac{\tan\theta\,d\theta}{\cos^2\theta\,\{C-\xi(\theta)\}}, \qquad (16)$$

$$\sqrt{(2gc)}\,dt = -\frac{d\theta}{\cos^2\theta\,\sqrt{\{C-\xi(\theta)\}}}, \qquad (17)$$

$$2c\,ds = -\frac{d\theta}{\cos^3\theta\,\{C-\xi(\theta)\}}. \qquad (18)$$

This last equation (18) can be expressed in a finite form by another integration.

Since, as above, $\frac{d\xi}{d\theta} = \frac{1}{(\cos\theta)^3}$, we can write

$$2c\,ds = -\frac{d\xi}{C-\xi} = +\frac{d(C-\xi)}{C-\xi},$$

and thence by integration

$$2cs = \log\{C-\xi(\theta)\} + \text{a constant}. \qquad (19)$$

(*a*) If the arc s of the trajectory is measured from the origin O, and $s=0$ when $\theta=\phi$,

$$s = \frac{1}{2c}\log\frac{C-\xi(\theta)}{C-\xi(\phi)}, \quad \xi(\theta) = C - \frac{g}{2c\,(v_0\cos\phi)^2}e^{2cs}.$$

(*b*) If on the other hand, as below, the arc s is measured from the vertex, and $s=0$ for $\theta=0$, $\xi(\theta)=0$, then

$$s = \frac{1}{2c}\log\frac{C-\xi(\theta)}{C}, \quad \xi(\theta) = C(1-e^{2cs}). \qquad (20)$$

The trajectory possesses two asymptotes.

The prolongation of the descending branch approaches a vertical line more and more, at a distance from the origin, according to § 19, 6,

$$\frac{1}{g}\int_{-\frac{1}{2}\pi}^{\phi} v^2\,d\theta.$$

At the vertex the variable of integration θ changes sign; $\xi(\theta)$ becomes zero and then negative; so that this distance is

$$\frac{1}{2c}\int_{\theta=\phi}^{\theta=0}\frac{d\theta}{\cos^2\theta\,\{\xi(\theta)-C\}} + \frac{1}{2c}\int_{\theta=0}^{\theta=\frac{1}{2}\pi}\frac{d\theta}{\cos^2\theta\,\{\xi(\theta)+C\}}.$$

The prolongation backward of the ascending branch (to $t=-\infty$, $x=-\infty$, $y=-\infty$) approximates to a slanting line, inclined to the horizon at an angle β given by $C-\xi(\beta)=0$; and the expression for its distance from the origin is easily determined.

This distance is given here, without proof, as

$$\frac{1}{2c\cos\beta}\int_{\theta=\phi}^{\theta=\beta}\frac{(\tan\beta-\tan\theta)\,d\theta}{\cos^2\theta\,\{\xi(\beta)-\xi(\theta)\}},$$

where β is supposed to be calculated from the relation $\xi(\beta)=C$.

A similar result holds in general for the law of air resistance $cf(v)=cv^n$.

Statement of Results.

1. Retardation due to air resistance $= cv^n + a$:

$$\begin{cases} g\,dx = -v^2 d\theta = -\dfrac{2v^2 dz}{1+z^2}, \\ g\,dy = -v^2 \tan\theta\, d\theta = -v^2 \dfrac{z^2-1}{z^2+1}\dfrac{dz}{z}, \\ g\,dt = -v\sec\theta\, d\theta = -\dfrac{v dz}{z}, \\ g\,ds = -v^2 \sec\theta\, d\theta = -\dfrac{v^2 dz}{z}: \end{cases}$$

and here v is expressed in terms of θ or z through

$$\frac{1}{(v\cos\theta)^n} = e^{-\frac{n}{g}\int\frac{a\,d\theta}{\cos\theta}}\left\{-\frac{n}{g}\int\frac{ce^{+\frac{n}{g}\int\frac{a\,d\theta}{\cos\theta}}}{(\cos\theta)^{n+1}}d\theta + \text{constant}\right\},$$

or,

$$\frac{1}{v^n} = -\frac{n}{g}(1+z^2)^{-n} z^{-n\left(\frac{a}{g}-1\right)}\left\{\int c(1+z^2)^n z^{\frac{na}{g}-n-1}dz + C\right\},$$

in which z denotes $\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$, and the constant of integration C is to be calculated from the initial conditions

$$v = v_0 \text{ for } \theta = \phi, \quad \text{or} \quad z_0 = \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi).$$

2. Retardation due to air resistance of the form $cf(v) = cv^n$:

$$g\,dx = -v^2 d\theta, \qquad g\,dy = -v^2\tan\theta\, d\theta,$$
$$g\,dt = -v\sec\theta\, d\theta, \quad g\,ds = -v^2\sec\theta\, d\theta;$$

and then

$$\frac{1}{(v\cos\theta)^n} = -\frac{nc}{g}\int\frac{d\theta}{(\cos\theta)^{n+1}} + \text{constant}.$$

When the constant of integration is determined from the condition $v = v_0$ for $\theta = \phi$, then

$$\frac{1}{(v\cos\theta)^n} - \frac{1}{(v_0\cos\phi)^n} = -\frac{nc}{g}\{F(\theta) - F(\phi)\},$$

or

$$v = \frac{\sec\theta}{\left\{C - \dfrac{nc}{g}F(\theta)\right\}^{\frac{1}{n}}},$$

where we write for brevity

$$F(\theta)=\int_0^\theta \frac{d\theta}{(\cos\theta)^{n+1}},\ F(\phi)=\int_0^\phi \frac{d\theta}{(\cos\theta)^{n+1}},\text{ and } C=\frac{1}{(v_0\cos\phi)^n}+\frac{nc}{g}F(\phi):$$

and then

$$dx=-\frac{1}{g}\frac{\sec^2\theta\, d\theta}{\left[C-\frac{nc}{g}F(\theta)\right]^{\frac{2}{n}}},$$

$$dy=-\frac{1}{g}\frac{\sec^2\theta\tan\theta\, d\theta}{\left[C-\frac{nc}{g}F(\theta)\right]^{\frac{2}{n}}},$$

$$dt=-\frac{1}{g}\frac{\sec^2\theta\, d\theta}{\left[C-\frac{nc}{g}F(\theta)\right]^{\frac{1}{n}}},$$

$$ds=-\frac{1}{g}\frac{\sec^3\theta\, d\theta}{\left[C-\frac{nc}{g}F(\theta)\right]^{\frac{2}{n}}}.$$

(*a*) $n=2$, $cf(v)=cv^2$ (quadratic law):

$$v^2=\frac{g}{2c}\frac{1}{\cos^2\theta[C-\xi(\theta)]}$$

with the notation

$$\xi(\theta)=\frac{1}{2}\left[\frac{\sin\theta}{\cos^2\theta}+\log\tan(45^\circ+\tfrac{1}{2}\theta)\right],$$

$$\xi(\phi)=\frac{1}{2}\left[\frac{\sin\phi}{\cos^2\phi}+\log\tan(45^\circ+\tfrac{1}{2}\phi)\right]$$

(for which consult Table 10, Vol. IV), and for brevity we put

$$C=\frac{g}{2c(v_0\cos\phi)^2}+\xi(\phi),\text{ or } C=\frac{g}{2cv_s^2};$$

and then we have

$$2c\,dx=-\frac{d\theta}{\cos^2\theta[C-\xi(\theta)]},$$

$$2c\,dy=-\frac{\tan\theta\, d\theta}{\cos^2\theta[C-\xi(\theta)]},$$

$$\sqrt{2gc}\,dt=-\frac{d\theta}{\cos^2\theta\sqrt{[C-\xi(\theta)]}},$$

$$2c\,ds=-\frac{d\theta}{\cos^3\theta[C-\xi(\theta)]},$$

$$s=\frac{1}{2c}\log\frac{C-\xi(\theta)}{C-\xi(\phi)},$$

when s is measured from the origin;

and $$s=\frac{1}{2c}\log\frac{C-\xi(\theta)}{C},$$

when s is measured from the vertex.

(*b*) $n=3$; retardation $cf(v)=cv^3$ (cubic law):

$$\left.\begin{aligned} dx&=-\frac{v^2}{g}d\theta\\ dy&=-\frac{v^2}{g}\tan\theta\,d\theta\\ dt&=-\frac{v}{g}\frac{d\theta}{\cos\theta}\\ ds&=-\frac{v^2}{g}\frac{d\theta}{\cos\theta}\end{aligned}\right\}$$

where

$$v=\frac{1}{\cos\theta\left[C-\frac{3c}{g}(\tan\theta+\frac{1}{3}\tan^3\theta)\right]^{\frac{1}{3}}},$$

and

$$C=\frac{1}{(v_0\cos\phi)^3}+\frac{3c}{g}(\tan\phi+\tfrac{1}{3}\tan^3\phi),$$

or $C=\frac{1}{v_s^3}$.

The integrations for x, y, t, s, lead to elliptic integrals, when $a=0$, and $n=3$ or 4.

Corresponding tables based on the tables of Legendre have been constructed by Greenhill for $n=3$, and Sabudski for $n=4$.

Remarks on similar trajectories. Rules for comparison.

On the assumption of the law $cf(v)=cv^n$, and integrating with respect to θ from ϕ to θ, we had

$$\frac{1}{(v\cos\theta)^n}-\frac{1}{(v_0\cos\phi)^n}=-\frac{nc}{g}\int_\phi^\theta\frac{d\theta}{(\cos\theta)^{n+1}},$$

whence $$v\cos\theta=\frac{v_0\cos\phi}{\left[1-\frac{nc}{g}(v_0\cos\phi)^n\int_\phi^\theta\frac{d\theta}{(\cos\theta)^{n+1}}\right]^{\frac{1}{n}}}.\qquad\text{(1)}$$

In the general case

$$g\,ds=-v^2\cos^2\theta\frac{d\theta}{\cos^3\theta}$$

and $$g\,dt=-v\cos\theta\frac{d\theta}{\cos^2\theta},$$

so that, between limits θ_1 and θ,

$$\frac{gs}{(v_0\cos\phi)^2}=-\int_{\theta_1}^\theta\frac{d\theta}{\left[1-\frac{nc}{g}(v_0\cos\phi)^n\int_\phi^\theta\frac{d\theta}{(\cos\theta)^{n+1}}\right]^{\frac{2}{n}}\cos^3\theta},\qquad\text{(2)}$$

$$\frac{gt}{v_0\cos\phi}=-\int_1^\theta\frac{d\theta}{\left[1-\frac{nc}{g}(v_0\cos\phi)^n\int_\phi^\theta\frac{d\theta}{(\cos\theta)^{n+1}}\right]^{\frac{1}{n}}\cos^2\theta}.\qquad\text{(3)}$$

Next, we consider two trajectories A and A' of different projectiles, fired at the same angle of departure ϕ, but with different initial velocities, v_0 and v_0', and with corresponding coefficients, c and c'. Let s denote the arc of the A trajectory between an initial inclination θ_1 and final inclination θ; and s' the corresponding arc of A' between the same inclinations.

The trajectories A and A' are called similar when the ratio $s : s'$ of the arcs of the trajectories is constant between the same initial and final inclinations.

It is assumed that the same zone of air resistance is under consideration, and also the same power law; and so, with constant ϕ, θ, θ_1 and n, that

$$\frac{gs}{(v_0 \cos \phi)^2} = \frac{gs'}{(v_0' \cos \phi)^2},$$

that is, $s : s'$ is constant and equal to $v_0^2 : v_0'^2$, provided that

$$c\,(v_0 \cos \phi)^n = c'\,(v_0' \cos \phi)^n, \quad \text{or} \quad cv_0^n = c'v_0'^n.$$

In this case then,

$$\frac{gt}{v_0 \cos \phi} = \frac{gt'}{v_0' \cos \phi}, \text{ or } t : t' \text{ is in the constant ratio of } v_0 : v_0'.$$

On the two trajectories, and at the points where the final slope θ of the tangent is the same, let the velocities of the two shells be denoted by v and v'.

It is obvious, as $cv_0^n = c'v_0'^n$, that from (1) we have the relation $v : v' = v_0 : v_0'$.

Thence also $cv^n = c'v'^n$, that is the retardation due to the air resistance is of the same magnitude at homologous points of the two paths, with equal θ.

To sum up, we have then the following proposition. If the same power law holds for the two shells, and further, if they are fired at the same departure angle ϕ, with velocities such that the initial retardation due to the air resistance is the same for the two shells, then the arcs s and s' of the trajectories of equal curvature, contained between equal tangent inclinations, are in a constant ratio, and this ratio is that of the squares of the initial velocities, $s : s' = v_0^2 : v_0'^2$.

So also the corresponding times of flight are in a constant ratio, viz., that of the initial velocities; $t : t' = v_0 : v_0'$.

Finally, the retardation due to air resistance at homologous points of the two paths, that is between equal tangent inclinations, is the same for both the shells.

These laws of similar trajectories are due to St Robert and F. Siacci.

An application of these laws of similar trajectories has been made lately by E. Röggla. By means of the range table of a known gun, he obtains the elements of the trajectory of a howitzer or a mortar.

With the same air density, and the same shape of the shell, the ballistic coefficient c is inversely proportional to $\frac{P}{R^2\pi}$ where P is the weight of the shell, $R^2\pi$ the cross-section.

Thence the statement can be enunciated in the following manner:

Denote by v_0 the initial velocity of a gun A, x the horizontal distance, and v the velocity after t seconds; x_s is the vertex abscissa, v_s the vertex velocity, t_s the time of flight to the vertex; X is the maximum range, at about 45° elevation, T the corresponding total time of flight, v_e the final velocity, P the weight of the shell, $R^2\pi$ the cross-section, $q = P : R^2\pi$ the sectional density of load.

Denote the corresponding expressions for another gun B by v_0', x', v', t', x_s', v_s', t_s', X', T', v_e', q'. The values x', t', v' for the gun B are to refer to a point on the trajectory at the same tangent slope as x, t, v for the gun A.

In both cases the angle of departure, the air density and shape of the shell are taken as being the same.

It follows then that, if $v_0^2 : v_0'^2 = q : q'$,

then $$v_0^2 : v_0'^2 = q : q' = x : x' = x_s : x_s' = X : X', \text{ nearly,}$$

and $$v_0^2 : v_0'^2 = v^2 : v'^2 = v_s^2 : v_s'^2 = v_e^2 : v_e'^2, \text{ nearly;}$$

and further $$t : t' = t_s : t_s' = v_0 : v_0' = v : v' = v_e : v_e', \text{ nearly,}$$

where t, t' denote the times of flight of the two shells over the arcs s, s'.

Example. In an American coast mortar, $2R = 25{\cdot}4$ cm., $P = 274$ kg, sectional density of load $q = 0{\cdot}54$ kg/cm², $v_0 = 352$ m/sec, maximum range $X = 10{,}500$ m, when $v_e = 298$ m/sec.

These numbers are to be applied to another gun with $q' = 0{\cdot}343$ kg/cm² (with equal δ and i). How great should be the initial velocity v_0'; what will be the maximum range X'; and what the final velocity v_e'?

$$X' : 10{,}500 = 0{\cdot}343 : 0{\cdot}54, \quad v_0'^2 : 352^2 = 0{\cdot}343 : 0{\cdot}54, \quad v_e' : 298 = v_0' : v_0.$$

Thence $$X' = 6670 \text{ m}, \quad v_0' = 280 \text{ m/sec}, \quad v_e' = 237 \text{ m/sec}.$$

This rule stands in intimate connexion with the one given by Newton and Froude, and in its most general form by von Helmholtz (see notes to § 13).

An extract from these rules states :—If experiments with a model are to be carried out, so as to give results for another body moving in the same medium, such as air, and if the linear dimensions are to be altered in the ratio of n, then the squares of the velocities and of the times must be altered in the same ratio n.

In the preceding case, with similar shape of shell and same material, the sectional density is nearly as the length of the shell, and this is proportional to the other corresponding lengths.

§ 21. Method of Euler-Otto.

In the year 1753, the well known mathematician L. Euler gave an approximate solution of the ballistic problem, which is still of importance for an initial velocity $v_0 < 240$ m/sec, and at the same time a guide to the calculation of corresponding tables.

His method depends on the summation of dx, dy, dt, ds. He treats the trajectory as a polygon of an infinite number of straight arcs Δs, and thence makes up the finite expression for the corresponding projections, Δx and Δy, as well as for the corresponding time Δt; he then sums up the Δx, Δy, Δt to x, y, t. He assumes the quadratic law of air resistance.

The expressions which arise in the calculation are deduced as in § 20, and it need only be shown here how Euler carried out the remaining summations.

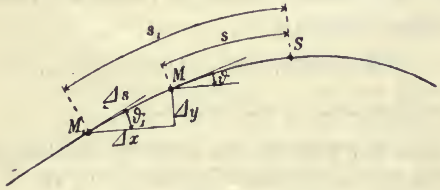
Consider two adjacent points, M and M_1, of the same branch of the trajectory, and put $SM = s$, $SM_1 = s_1$ for the arcs measured from the vertex S.

According to the above,

$$SM = s = \frac{1}{2c} \log \frac{C - \xi(\theta)}{C}, \quad SM_1 = s_1 = \frac{1}{2c} \log \frac{C - \xi(\theta_1)}{C},$$

and so the small element of arc MM_1, or

$$\Delta s = \frac{1}{2c} \log \frac{C - \xi(\theta_1)}{C - \xi(\theta)}.$$



The inclination to the horizontal of the trajectory is θ at the point M, and θ_1 at the point M_1, and Euler now treats the element of arc as nearly straight, and having a mean inclination $\frac{1}{2}(\theta + \theta_1)$; and therefore

$$\Delta x = \frac{1}{2c} \log \frac{C - \xi(\theta_1)}{C - \xi(\theta)} \cos \frac{\theta + \theta_1}{2},$$

$$\Delta y = \frac{1}{2c} \log \frac{C - \xi(\theta_1)}{C - \xi(\theta)} \sin \frac{\theta + \theta_1}{2}.$$

These projections, Δx and Δy, of the arc Δs are then to be summed; thus $\Sigma \Delta x = x$, $\Sigma \Delta y = y$.

In an example Euler took the difference of the inclinations θ and θ_1 of the trajectory as being 5 degrees. The corresponding time of flight was given then from the relation

$$2c (v \cos \theta)^2 = \frac{g}{C - \xi(\theta)},$$

and so

$$\Delta t = \frac{\Delta x}{v \cos \theta} = \frac{1}{2c} \log \frac{C - \xi(\theta_1)}{C - \xi(\theta)} \cos \frac{\theta + \theta_1}{2} \frac{\sqrt{2c}\sqrt{C - \xi(\theta)}}{\sqrt{g}},$$

whence the time of flight follows by summation $t = \Sigma \Delta t$.

As it would be exceedingly wearisome to carry out the calculations in every case, together with the summations $\Sigma \Delta x$, $\Sigma \Delta y$, $\Sigma \Delta t$, we must set to work to construct Tables which allow us, for any trajectory whatever, given by the values of c, ϕ, v_0, to obtain the elements of the trajectory, viz., range X, angle of descent ω, final velocity v_e, vertex abscissa x_s, vertex ordinate y_s, and so forth.

Euler set out on the following principle, so as to reduce to a minimum the trouble of constructing the Tables.

Let the formulae above, for $\Sigma\Delta x$, $\Sigma\Delta y$, $\Sigma\Delta t$, be written in the form

$$2cx = 2c\Sigma\Delta x = \Sigma \log \frac{C-\xi(\theta_1)}{C-\xi(\theta)} \cos \frac{\theta+\theta_1}{2},$$

$$2cy = 2c\Sigma\Delta y = \Sigma \log \frac{C-\xi(\theta_1)}{C-\xi(\theta)} \sin \frac{\theta+\theta_1}{2},$$

$$\sqrt{2c}t = \sqrt{2c}\Sigma\Delta t = \Sigma \log \frac{C-\xi(\theta_1)}{C-\xi(\theta)} \cos \frac{\theta+\theta_1}{2} \frac{\sqrt{[C-\xi(\theta)]}}{\sqrt{g}}.$$

The right-hand sides of these equations involve only C (or the angle β of inclination of the asymptote) and θ, which vary along any trajectory, and from one trajectory to another.

We assume then $c=1$ for the present, and consider first for any assumed value of C the series of Δx, Δy, Δt, calculated from degree to degree of θ, by the formulae above, for the ascending branch; and after this with $C+\xi$ for the descending branch, the summation being made starting from the vertex where $\theta=0$.

The different trajectories can then differ only in the different values of the angle ϕ of departure.

But it is evident that all these trajectories are congruent to each other; for they are merely greater or smaller parts cut off the same curve, reckoned from the vertex.

We have obtained then, for any and every C, the elements x, y, t of a complete series of trajectories, with different values of ϕ.

Expressed otherwise, when c is not restricted to the value unity, we have, for any given value of C or β, the elements $2cx$, $2cy$, $\sqrt{(2c)}\,t$ of the various trajectories, which differ in the angle of departure ϕ.

We suppose this to be carried out for another value of C or β, and so the elements $2cx$, $2cy$, $\sqrt{(2c)}\,t$ are known for a second series of trajectories, for various values of ϕ; and so forth.

The Tables should then be calculated, grouped for the different values of C which can occur in practice; and they will show in each group, and for every single ϕ, the values of cx, cy, $\sqrt{(2c)}\,t$, as well as of $\frac{x}{t^2}$, which does not involve c.

8—2

Tables were calculated on these fundamental laws by H. Fr. von Jacobi (these tables are lost), by Fr. P. von Grävenitz, 1764, but notably by J. C. F. Otto, 1842.

They have been extended later by Mola, Scheve, Siacci, Lardillon, Braccialini, and others, and arranged differently for convenience of use.

Otto calculated his Tables, in 1842, for the various values of β between 35° and 87°, mostly rising by 2 degrees at a time; and then between 30° and 75° rising by one degree at a time.

Since the value of C (or of β) is given by cv_0^2 and ϕ, the Tables are arranged for practical use in groups for different values of ϕ, and in each group according to cv_0^2.

Also the elements are given of cX, ω, v_e, $T : \sqrt{X}$ of the point of fall on the muzzle horizon, and also $y_s : X$.

This method can be illustrated by the following extract, where for instance $\phi = 60°$.

$2cX$	$\frac{cv_0^2}{g}$	$\frac{v_0^2}{2gX}$	ω	$\frac{v_e}{v_0}$	$T\sqrt{\frac{g}{X}}$	$\frac{y_s}{X}$
1·30	1·633	1·256	72° 19′	0·584	2·135	0·573
1·35	1·759	1·303	72° 44′	0·570	2·146	0·580
1·40	1·894	1·353	73° 9′	0·556	2·157	0·586

Use of Otto's Tables for the solution of a particular trajectory.

(See also Vol. IV.)

The range in metres on the muzzle horizon is denoted by X, the angle of departure by ϕ, the initial velocity in m/sec by v_0, the acute angle of descent by ω, the final velocity by v_e in m/sec, the time of flight by T seconds, height of vertex by y_s m.

The retardation due to air resistance is cv^2, and here

$$c = \frac{R^2\pi \, . \, i \, . \, \lambda \, . \, \delta \, . \, g}{P \times 1{\cdot}206},$$

where R is the half-calibre in metres, δ is the air density in kg/m³, $g = 9{\cdot}81$ m/sec², i is the coefficient of shape of head ($=1$ for Krupp's normal shell of 2 calibres as radius of the ogival head, or 1·3 calibre as height of ogive, or 41°·5 half ogival angle at the point), P the weight of the shell in kg; $\lambda = 0{\cdot}014$ for a velocity less than 240 m/sec (but if less accuracy is necessary, the tables can be taken with $\lambda = 0{\cdot}014$ for all velocities less than normal sound velocity, and with $\lambda = 0{\cdot}039$ for velocities greater than sound velocity, up to about 1000 m/sec).

1. Given c, v_0, ϕ.

Let us start with $\frac{cv_0^2}{g}$, which can be calculated from the given values of v_0, δ, i, R, P.

Look out in the group of given angles of departure ϕ and for the calculated $\frac{cv_0^2}{g}$ along the horizontal line the corresponding value of $2cX$, whence X is found; as also the value of ω, and of $\frac{v_e}{v_0}$ whence v_e is found. Interpolate where required.

2. Given c, X, ϕ.

Starting with $2cX$, look out in the ϕ Tables the values, on the horizontal line corresponding to $2cX$, of $\frac{v_e}{v_0}$, $\frac{cv_0^2}{g}$, $\frac{v_0^2}{2gX}$, ..., and so obtain, with interpolation where required, the trajectory elements v_e, v_0, X, ...

3. Given c, ϕ, ω.

A start is made from ω.

4. Given v_0, X, ϕ.

The start is made from $\frac{v_0^2}{2gX}$.

5. Given c, v_0, X.

Start from $2cX$ and $\frac{cv_0^2}{g}$, and interpolate.

If the alteration of air density with the height is to be taken into account, a calculation is made with a first approximation of δ, and c, to find the vertex height y_s; and the calculation is repeated with a closer value of c.

Example. Given the angle of departure $\phi = 60°$, time of flight $T = 40{\cdot}65$ seconds, range 3,520 m. To find v_0, v_e, ω, y_s.

Here $T\sqrt{\frac{g}{X}} = 40{\cdot}65\sqrt{\frac{9{\cdot}81}{3520}} = 2{\cdot}1416$; and thence from Otto's Tables,

$$\frac{v_0^2}{2gX} = \frac{v_0^2}{2 \times 9{\cdot}81 \times 3520} = 1{\cdot}303; \text{ thence } v_0 = 300 \text{ m/sec.}$$

Further, $\frac{y_s}{X} = \frac{y_s}{3520} = 0{\cdot}580$, $y_s = 2042$ m; $\frac{v_e}{v_0} = \frac{v_e}{300} = 0{\cdot}570$, $v_e = 171$ m/sec.

Lastly $\omega = 72° 44'$.

Notes. (*a*) A. M. Legendre pointed out in 1782 that an error is contained in Euler's procedure, where the Δx and Δy are calculated as if they were the projections of the ends of the arc elements Δs, treated as straight lines, as the projections Δx and Δy are too great. Therefore he takes a circular element for Δs, instead of a straight line; and finds (consult Didion for the proof)

$$\Delta x = (\text{Euler } \Delta x)\frac{\sin\frac{\theta_1-\theta}{2}}{\frac{\theta_1-\theta}{2}} \quad \Delta y = (\text{Euler } \Delta y)\frac{\sin\frac{\theta_1-\theta}{2}}{\frac{\theta_1-\theta}{2}}.$$

Didion showed later, in 1848, that this procedure of Legendre leads to no closer result than that of Euler.

(*b*) A method corresponding to that of Euler, for the quadratic law of air resistance cv^2, was brought forward by Bashforth in 1873, based on the cubic law cv^3.

This theory is considered below.

(*c*) A. Bassani has carried out the integration on the assumption of the quadratic law of air resistance, on a method where the function

$$\tfrac{1}{2}p\sqrt{(1+p^2)}+\tfrac{1}{2}\log[\sqrt{(1+p^2)}+p]$$

which occurs in Euler's solution, is replaced by the approximate value

$$\frac{p(1+0{\cdot}2523p^2)}{1+0{\cdot}091p^2}.$$

§ 22. Method of F. Bashforth.

As mentioned briefly already, F. Bashforth assumed the cubic law of air resistance, and retardation $cf(v)=cv^3$, where c varied in different zones of the velocity, as the basis of a method of solution; and corresponding Tables were constructed on the same principles as were employed by Euler and Otto for the quadratic law cv^2 and by Sabudski for the biquadratic law.

The relation between the velocity v of the shell in its path, and the corresponding angle θ of slope of the tangent of the path to the horizon, is given, as proved in § 20, in the form

$$\frac{1}{(v\cos\theta)^3} = -\frac{3c}{g}(\tan\theta+\tfrac{1}{3}\tan^3\theta) + \text{the integration constant } A.$$

Let us put

$$v\cos\theta = v_x, \quad \sqrt[3]{\frac{g}{c}} = \kappa, \quad 3\tan\theta+\tan^3\theta = B(\theta),$$

and determine the integration constant A from the relation at the vertex, when $\theta=0$, $v=v_s$; then the equation can be written

$$v_x = \frac{v_s}{\sqrt[3]{\left[1-\frac{v_s^3}{\kappa^3}B(\theta)\right]}}.$$

The general expressions for dt, dx, dy, namely

$$dt = -\frac{v_x\,d\theta}{g\cos^2\theta},\quad dx = -\frac{v_x^2\,d\theta}{g\cos^2\theta},\quad dy = -\frac{v_x^2\tan\theta\,d\theta}{g\cos^2\theta},$$

lead then to

$$t = -\frac{v_s}{g}\int_\phi^\theta \frac{d\theta}{\cos^2\theta\sqrt[3]{\left[1-\frac{v_s^3}{\kappa^3}B(\theta)\right]}},$$

$$x = -\frac{v_s^2}{g}\int_\phi^\theta \frac{d\theta}{\cos^2\theta\sqrt[3]{\left[1-\frac{v_s^3}{\kappa^3}B(\theta)\right]^2}},$$

$$y = -\frac{v_s^2}{g}\int_\phi^\theta \frac{\tan\theta\,d\theta}{\cos^2\theta\sqrt[3]{\left[1-\frac{v_s^3}{\kappa^3}B(\theta)\right]^2}}.$$

We see that these integrals depend only on θ and the value of $\frac{v_s}{\kappa}$, because $B(\theta)$ involves θ only.

These integrals are denoted for brevity by T, X, Y; and F. Bashforth has calculated Tables for them, corresponding to values of $\frac{v_s}{\kappa}$ and θ.

Values of the integrals are required between the limits ϕ and θ; for instance,

$$T_\theta{}^\phi = T_\theta{}^0 + T_0{}^\phi = T_0{}^\phi - T_0{}^\theta.$$

It is sufficient then to have the integrals reckoned from the vertex ($\theta = 0$).

The formulae are therefore

$$x = +\frac{v_s^2}{g}X_\theta{}^\phi = +\frac{v_s^2}{g}(X_0{}^\phi - X_0{}^\theta) \quad\text{..................(1)}$$

$$y = +\frac{v_s^2}{g}Y_\theta{}^\phi = +\frac{v_s^2}{g}(Y_0{}^\phi - Y_0{}^\theta) \quad\text{..................(2)}$$

$$t = +\frac{v_s^2}{g}T_\theta{}^\phi = +\frac{v_s}{g}(T_0{}^\phi - T_0{}^\theta) \quad\text{..................(3)}$$

$$v\cos\theta = \frac{v_s}{\sqrt[3]{\left[1-\frac{v_s^3}{\kappa^3}B(\theta)\right]}} \quad\text{..............................(4)}$$

$$v_s = \frac{v_0 \cos\phi}{\sqrt[3]{\left[1 + \frac{v_0^3}{\kappa^3} B(\phi)\cos^3\phi\right]}} \quad \text{.....................(5)}$$

$$B(\theta) = 3\tan\theta + \tan^3\theta \quad \text{..............................(6)}$$

$$\kappa = \sqrt[3]{\frac{g}{c}} \quad \text{..(7)}$$

The calculation proceeds then with the aid of the Tables for values of $\left(\frac{v_s}{\kappa}\right)^3$ and θ, for X, Y, T, and a Table of $B(\theta)$ for values of θ.

For instance, if the calibre is $2R$, weight of shell P, air density δ, initial velocity v_0, and angle of departure ϕ, we calculate from (7) the value of κ, take out from the Table of $B(\theta)$ the value of $B(\phi)$, and calculate from (5) the vertex velocity v_s.

For any selected value of θ, and the value of the fraction $\frac{v_s^3}{\kappa^3}$, we have the tabular values of X, Y, T; and from (1), (2), (3), the trajectory elements x, y, t, which correspond to θ, the angle of slope chosen.

In the special case of $\theta = 0$, we have the vertex coordinates x_s, y_s, and also the time t_s of reaching the vertex.

Numerical example. Given $2R = 0{\cdot}2286$ m, $P = 110{\cdot}9$ kg, $v_0 = 315{\cdot}5$ m/sec, $\phi = 43^\circ{\cdot}5$, $\delta = 1{\cdot}206$ kg/m³, $i = 1$, to determine the height y_s of the vertex.

$$\kappa^3 = \frac{g}{c} = \frac{P}{cv^3} = \frac{110{\cdot}9}{0{\cdot}000060 \times (0{\cdot}1143)^2 \times 3{\cdot}1416} = 45{,}033{,}000,$$

$$B(\phi) = 3\tan 43^\circ{\cdot}5 + \tan^3 43^\circ{\cdot}5 = 3{\cdot}7015,$$

$$v_s = \frac{315{\cdot}5\cos 43^\circ{\cdot}5}{\sqrt[3]{\left[1 + \frac{(315{\cdot}5\cos 43^\circ{\cdot}5)^3 \times 3{\cdot}7015}{45{,}033{,}000}\right]}} = 182{\cdot}1,$$

$$\frac{v_s^3}{\kappa^3} = \frac{(182{\cdot}1)^3}{45{,}033{,}000} = 0{\cdot}1341.$$

The Y table gives, for $\left(\frac{v_s}{\kappa}\right)^3 = 0{\cdot}1341$, and $\theta = 0$, the value $Y_0^{43\cdot5} = 0{\cdot}58195$; so that the vertex ordinate

$$y_s = \frac{v_s^2}{g} Y_0^{\phi} = \frac{(182{\cdot}1)^2}{9{\cdot}81} \times 0{\cdot}58195 = 1966 \text{ m}.$$

In the approximation methods of the First Group, the "Method of Velocities" can be included, which E. Vallier employed for the integration of the exact differential equations.

An extension of the Simpson Rule is used which can be employed for the calculation of any long trajectory, in which the air density is variable to a marked extent.

See also the notes on this chapter.

The methods of series-expansion can also be included in the First Group, of which an account is given in § 22 *a*.

§ 22 a. Method of Development in Series.

Where a function $F(x)$, in the interval 0 to x, is finite and continuous, together with its first $n+1$ derivatives, assumed also to exist, $F'(x), F''(x), \ldots, F^{(n+1)}(x)$, then, according to Maclaurin,

$$F(x) = F(0) + xF'(0) + \frac{x^2}{2!}F''(0) + \ldots + \frac{x^n}{n!}F^{(n)}(0) + \text{a remainder term } R,$$

where, according to Lagrange,

$$R = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\epsilon x),$$

ϵ being an unknown fraction between 0 and 1. Or otherwise, in an integral form,

$$R = \frac{1}{n!}\int_{t=0}^{t=x} (x-t)^n F^{(n+1)}(t)\, dt.$$

So that if $F(x)$ is calculated by this expansion in powers of x, and if in the calculation the series breaks off at the term $\frac{x^n}{n!}F^{(n)}(0)$, an error R will exist, the limits of which can be determined, but this can only be employed when it has been proved that for $n=\infty$, $R=0$; in other words, that the series converges.

In the preceding case we may calculate the trajectory ordinate y in terms of the corresponding trajectory abscissa x, by an expansion in a series. Then

$$F(x) = y, \quad F(0) = y_{x=0},$$

$$F'(x) = \frac{dy}{dx} = \tan\theta, \quad F'(0) = \tan\phi,$$

$$F''(x) = \frac{d\tan\theta}{dx} = -\frac{g}{(v^2\cos^2\theta)} \text{ (see § 17)}, \quad F''(0) = -\frac{g}{(v_0\cos\phi)^2},$$

$$F'''(x) = -g\frac{d}{dx}(v\cos\theta)^{-2} = +g\,\frac{2v\cos\theta}{(v\cos\theta)^4}\cdot\frac{d(v\cos\theta)}{dt}\,\frac{dt}{dx}$$

or, since

$$\frac{d(v\cos\theta)}{dt} = -cf(v)\cos\theta, \text{ and } \frac{dx}{dt} = v\cos\theta,$$

it follows that

$$F'''(x) = -\frac{2gcf(v)\cos\theta}{(v\cos\theta)^4}, \quad F'''(0) = -\frac{2gcf(v_0)}{v_0^4\cos^3\phi}.$$

Proceeding in the same manner, we find

$$F^{(iv)}(0) = \frac{2gcf(v_0)}{v_0^6\cos^4\phi}\left[g\left\{\frac{v_0(cf(v_0))'}{cf(v_0)} - 1\right\}\sin\phi + v_0(cf(v_0))' - 4cf(v_0)\right],$$

and so on.

So long as the shell is above the muzzle horizon, y and its derivatives with respect to x remain finite and continuous. Therefore we obtain the expansion of y in terms of x by substitution of the calculated values $F(0)$, $F'(0)$,... in the series above.

The slope θ of the tangent will be obtained thence from $\tan\theta = \frac{dy}{dx}$, through a single differentiation, and $v\cos\theta$ the horizontal component of the velocity from $v\cos\theta = \sqrt{\frac{g}{-y''}}$ (see §18) through a second differentiation.

Finally the time of flight t is calculated from $dt = \frac{dx}{v\cos\theta}$, by means of an integration, while $t=0$ for $x=0$.

In this way we obtain

$$y = x\tan\phi - \frac{gx^2}{2(v_0\cos\phi)^2} - \frac{g\,cf(v_0)}{3v_0}\left(\frac{x}{v_0\cos\phi}\right)^3$$
$$-\frac{g}{4}\left[\left\{\frac{cf(v_0)}{v_0}\right\}^2 - \frac{1}{3}\left\{\frac{cf(v_0)}{v_0}\right\}'\{cf(v_0) + g\sin\phi\}\right]\left(\frac{x}{v_0\cos\phi}\right)^4$$
$$+ \dots + \text{a remainder } R;$$

or $$y = x\tan\phi - \frac{gx^2}{2(v_0\cos\phi)^2}\left[1 + \frac{2}{3}\frac{cf(v_0)}{v_0^2\cos\phi}x + \dots\right] + \text{a remainder } R \quad \dots\dots\dots\dots(1)$$

$$\tan\theta = \tan\phi - \frac{gx}{(v_0\cos\phi)^2}\left[1 + \frac{cf(v_0)}{v_0^2\cos\phi}x + \dots\right] + \text{a remainder } R \quad \dots\dots\dots\dots(2)$$

$$t = \frac{x}{v_0\cos\phi}\left[1 + \frac{cf(v_0)}{2v_0^2\cos\phi}x + \dots\right] + \text{a remainder } R \quad \dots\dots\dots(3)$$

$$v\cos\theta = v_0\cos\phi\left[1 - \frac{cf(v_0)}{v_0^2\cos\phi}x + \dots\right] + \text{a remainder } R. \quad \dots\dots(4)$$

Here, for instance, if the biquadratic law of air resistance, $cf(v) = cv^4$, is taken as the basis,

$$cf(v_0) = cv_0^4, \quad (cf(v_0))' = (cv^4)'_{x=0} = 4cv_0^3, \quad \left[\frac{cf(v_0)}{v_0}\right]' = 3cv_0^2, \text{ and so on.}$$

The expansion in a series can be constructed in this way for any law of air resistance.

Expansions of this kind have been worked out in different ways since the end of the 18th century, with either x, or t, or θ, or s, as the independent variable (Lambert, Borda, Tempelhof, Otto, Heim, Français, Pfister, Denecke, Ligowski, Neumann).

The convergence of the series was either assumed as evident, or briefly discussed.

Even the later work of P. Haupt (see Note) is not rigorous. C. Veithen was the first to give a rigorous proof that the ballistic expansion in a series of powers of x and y, as a function of t, converges for all finite values of t, in all cases where a certain assumption is made with regard to the air resistance function.

The following abstract may be given of the method of C. Veithen (see Note). A system of real ordinary differential equations of the First Order may be considered:

$$\frac{d\xi}{dt} = \phi(t, \xi, \eta), \quad \frac{d\eta}{dt} = \psi(t, \xi, \eta),$$

and the functions ϕ and ψ may be supposed to be expanded in a power series of $t - t_0$, $\xi - \xi_0$, $\eta - \eta_0$, which converges for all values of these differences.

In this case, ξ and η represent definite uniform functions of t, which satisfy the differential equations, and assume the values $\xi = \xi_0$, $\eta = \eta_0$, for $t = t_0$.

These functions can be expanded in power series of $t - t_0$,

$$\xi = \xi_0 + \xi_1(t - t_0) + \ldots\ldots\ldots$$
$$\eta = \eta_0 + \eta_1(t - t_0) + \ldots\ldots\ldots$$

which converge for all values of $t - t_0$.

Now the fundamental equations can be written in the form (compare § 17, equations (1) and (2)),

$$\frac{d\xi}{dt} = F(v)\,\xi, \qquad \frac{d\eta}{dt} = F(v)\,\eta - g,$$

wherein we find

$$\xi = \frac{dx}{dt} = v\cos\theta, \qquad \eta = \frac{dy}{dt} = v\sin\theta,$$

$$v = \sqrt{(\xi^2 + \eta^2)}, \quad F(v) = -\frac{cf(v)}{v}.$$

For the finite region of v which comes into consideration, we can approximate with sufficient accuracy to the empirical function $F(v)$, so long as it is everywhere continuous, by means of a finite polynomial in v^2,

$$F(v) = a_1 v^2 + a_2 v^4 + a_3 v^6 + \ldots\ldots\ldots + a_n v^{2n};$$

that is, we can express, with the required accuracy, the right-hand side of the differential equation by polynomials in ξ, η.

On these assumptions of the air resistance, the above laws hold; and we see that ξ and η are capable of being expressed in a power series of t,

$$\xi = \xi_0 + \xi_1 t + \xi_2 t^2 + \ldots\ldots\ldots$$

$$\eta = \eta_0 + \eta_1 t + \eta_2 t^2 + \ldots\ldots\ldots$$

which converge for all values of t, and $\xi_0 = v_0 \cos\phi$, $\eta_0 = v_0 \sin\phi$.

It follows then, that the expansion of x and y, in terms of t, converges also for all values of t, where

$$x = \int_0^t \xi\, dt = \xi_0 t + a_2 t^2 + \ldots\ldots$$

$$y = \int_0^t \eta\, dt = \eta_0 t + b_2 t^2 + \ldots\ldots$$

Supposing that only the first 3 or 4 terms of the series in (1) are employed, we know then that the trajectory can be regarded as a parabola of the 3rd or 4th order.

This is the procedure, for example, of B. Prehn, Dolliak, Piton-Bressant, Hélie, Mieg (the last calculating with arithmetic series of the 3rd or 4th order).

SOME APPLICATIONS.

1. Method of Piton-Bressant, and of Hélie; formulae of the Gâvre Committee.

Series (1) was taken, breaking off at the 3rd term. The factor $\frac{2}{3}\,\frac{cf(v_0)}{v_0^2 \cos\phi}$ was denoted by K, and determined from the range.

Thus we have (compare also § 18)

$$y = x \tan\phi - \frac{gx^2}{2v_0^2 \cos^2\phi}(1 + Kx),$$

and thence $$\tan\theta = \tan\phi - \frac{gx}{2v_0^2 \cos^2\phi}(2 + 3Kx).$$

When $x = X$, $y = 0$, $\theta = -\omega$; so that if we put $1 + KX = Z$, then

$$\tan\phi = \frac{gXZ}{2(v_0 \cos\phi)^2}, \quad Z = \frac{v_0^2 \sin 2\phi}{gX}:$$

and Z denotes the ratio between the range in a vacuum, and the actual range X in air, for the same ϕ and v_0.

At the end of the path,

$$\tan(-\omega) = \tan\phi - \frac{gX}{2v_0^2 \cos^2\phi}(2 + 3KX) = \tan\phi - \frac{\tan\phi}{Z}[2 + 3(Z - 1)]$$
$$= \tan\phi\left(\frac{1}{Z} - 2\right).$$

And further, since $-y'' = \frac{g}{v_0^2 \cos^2\phi}(1 + 3Kx)$,

we have
$$v\cos\theta = \frac{v_0\cos\phi}{\sqrt{(1+3Kx)}},$$
and, at the end point,
$$\frac{v_e\cos\omega}{v_0\cos\phi} = \frac{1}{\sqrt{(1+3KX)}} = \frac{1}{\sqrt{(3Z-2)}}.$$

Finally, the time of flight is given by integration of
$$dt = dx\sqrt{\frac{-y''}{g}} = \frac{1}{v_0\cos\phi}\sqrt{(1+3Kx)}\,dx,$$
$$t = \frac{2}{9Kv_0\cos\phi}\left[(1+3Kx)^{\frac{3}{2}} - 1\right];$$
and this, with $K = \frac{Z-1}{X}$, can easily be calculated for the end point of the path.

The abscissa of the vertex x_s follows from the relation $\theta = 0$.

The following system of formulae is thus obtained:

(*a*) For any trajectory,
$$y = x\tan\phi - \frac{gx^2}{2(v_0\cos\phi)^2}(1+Kx) \quad \text{.........................(5)}$$
where K is determined from
$$1 + KX = \frac{v_0^2\sin 2\phi}{gX} \quad \text{.............................(6)}$$
$$\tan\theta = \tan\phi - \frac{gx}{2(v_0\cos\phi)^2}(2+3Kx) \quad \text{.........................(7)}$$
$$\frac{v\cos\theta}{v_0\cos\phi} = \frac{1}{\sqrt{(1+3Kx)}} \quad \text{.................................(8)}$$
$$t = \frac{2}{9v_0\cos\phi}\,\frac{(1+3Kx)^{\frac{3}{2}} - 1}{K}. \quad \text{..............................(9)}$$

(*b*) For the vertex,
$$x_s = \frac{\sqrt{[1+3KX(1+KX)]} - 1}{3K} \quad \text{.........................(10)}$$
$$y_s = X\tan\phi\,\frac{1+2KX}{2+3KX}. \quad \text{.................................(11)}$$

(*c*) For the end point in the horizontal plane through the muzzle,
$$\frac{\tan\omega}{\tan\phi} = 2 - \frac{1}{Z} = f_1(Z) \quad \text{.................................(12)}$$
$$\frac{v_e\cos\omega}{v_0\cos\phi} = \frac{1}{\sqrt{(3Z-2)}} = f_2(Z) \quad \text{...........................(13)}$$
$$\frac{Tv_0\cos\phi}{X} = \frac{2}{9}\,\frac{(3Z-2)^{\frac{3}{2}} - 1}{Z-1} = f_3(Z). \quad \text{.....................(14)}$$

And here
$$Z = 1 + KX = \frac{v_0^2\sin 2\phi}{gX}. \quad \text{.........................(15)}$$

It is often found that, for the same shell and the same initial velocity, K is independent of ϕ, and so can be treated as a constant in the corresponding Range Table. As a matter of fact, K depends on ϕ, as is seen immediately from the expansion in (1). It is somewhat better to work in equations (5) to (9) with K_1 as constant, where $K_1 = K\cos\phi$.

A table is given for $f_1(Z)$, $f_2(Z)$, $f_3(Z)$ on the next page.

Example. Given $X = 3300$ m, with $\phi = 10°$, and $v_0 = 354$ m/sec. To find ϕ, v_e, ω, and T for $X = 4000$ m.

We find K from

$$1 + 3300K = \frac{354^2 \sin(2\times 10°)}{9{\cdot}81\times 3300}, \quad K = \frac{0{\cdot}32}{3300}.$$

For the range $X = 4000$, the corresponding angle of departure ϕ is calculated from

$$\frac{354^2 \times \sin 2\phi}{9{\cdot}81\times 4000} = 1 + \frac{0{\cdot}32\times 4000}{3300} = 1{\cdot}388, \text{ then } \phi = 12°\ 53'.$$

Then for this range, since $Z = 1{\cdot}388$, and thence from the Table, $f_1(Z) = 1{\cdot}279$, $f_2(Z) = 0{\cdot}680$, $f_3(Z) = 1{\cdot}250$,

$$\frac{\tan\omega}{\tan 12°\ 53'} = 1{\cdot}279; \quad \omega = 16°\ 18';$$

$$\frac{v_e \cos 16°55'}{354\cos 12°53'} = 0{\cdot}680; \quad v_e = 244 \text{ m/sec};$$

$$\frac{T\times 354\cos 12°53'}{4000} = 1{\cdot}250; \quad T = 14{\cdot}5.$$

The values of ϕ, ω, v_e, T, calculated in this way, are in fairly good agreement with the truth. But if we should proceed, for instance, from the data for the range of 3300, and the corresponding value of K, to determine the elements of the path for a range of 6000, the error might be serious.

2. Method of Duchêne (French).

The equation of the trajectory is to be assumed to be of the form

$$y = x\tan\phi - \frac{gx^2}{2(v_0\cos\phi)^2}\left(1 + \frac{Ax}{\cos\phi} + \frac{Bx^2}{\cos^2\phi}\right) \quad\text{............(16)}$$

in which A and B are to be considered as dependent on v_0 for the same shell, but as independent of ϕ, that is, as constant in a corresponding Range Table.

To determine A and B, we employ two definite values of X and ϕ.

Substitute then in the equation

$$\frac{v_0^2 \sin 2\phi}{gX} = 1 + \frac{AX}{\cos\phi} + \frac{BX^2}{\cos^2\phi} \quad\text{..........................(17)}$$

and we have two equations for the unknown quantities A and B. Given, for example, for $v_0 = 529$ m/sec, $X_1 = 1800$ m, $\phi_1 = 2°28'$,

$$X_2 = 2200 \text{ m}, \ \phi_2 = 3°12',$$

we find

$$A = 1{\cdot}984\times 10^{-4}, \quad B = 1{\cdot}724\times 10^{-9}.$$

Z	$f_1(Z)$	$f_2(Z)$	$f_3(Z)$
1·00	1·0000	1·0000	1·0000
1·05	1·0476	0·9325	1·0366
1·10	1·0909	0·8771	1·0716
1·15	1·1304	0·8305	1·1052
1·20	1·1667	0·7906	1·1376
1·25	1·2000	0·7559	1·1689
1·30	1·2308	0·7255	1·1992
1·35	1·2593	0·6984	1·2287
1·40	1·2857	0·6742	1·2573
1·45	1·3103	0·6523	1·2852
1·50	1·3333	0·6325	1·3124
1·55	1·3548	0·6143	1·3390
1·60	1·3750	0·5976	1·3650
1·65	1·3939	0·5822	1·3904
1·70	1·4118	0·5680	1·4153
1·75	1·4286	0·5547	1·4397
1·80	1·4444	0·5423	1·4637
1·85	1·4595	0·5307	1·4873
1·90	1·4737	0·5199	1·5104
1·95	1·4872	0·5096	1·5332
2·00	1·5000	0·5000	1·5556
2·05	1·5122	0·4909	1·5776
2·10	1·5238	0·4822	1·5993
2·15	1·5349	0·4740	1·6207
2·20	1·5455	0·4662	1·6418
2·25	1·5556	0·4588	1·6626
2·30	1·5652	0·4517	1·6832
2·35	1·5745	0·4450	1·7035
2·40	1·5833	0·4385	1·7235
2·45	1·5918	0·4323	1·7433
2·50	1·6000	0·4264	1·7628
2·55	1·6078	0·4207	1·7821
2·60	1·6154	0·4153	1·8012
2·65	1·6226	0·4100	1·8201
2·70	1·6296	0·4049	1·8387
2·75	1·6364	0·4000	1·8571
2·80	1·6429	0·3953	1·8754
2·85	1·6491	0·3907	1·8935
2·90	1·6552	0·3863	1·9114
2·95	1·6610	0·3821	1·9291
3·0	1·6667	0·3780	1·9467
3·1	1·6774	0·3701	1·9813
3·2	1·6875	0·3627	2·0153
3·3	1·6970	0·3558	2·0487
3·4	1·7059	0·3492	2·0816
3·5	1·7143	0·3430	2·1139
3·6	1·7222	0·3371	2·1457
3·7	1·7297	0·3315	2·1771
3·8	1·7368	0·3262	2·2079
3·9	1·7436	0·3211	2·2384
4·0	1·7500	0·3162	2·2684
4·5	1·7778	0·2949	2·4126
5·0	1·8000	0·2774	2·5485
5·5	1·8182	0·2626	2·6772
6·0	1·8333	0·2500	2·8000
7·0	1·8571	0·2294	3·0303
8·0	1·8750	0·2132	3·2441
9·0	1·8889	0·2000	3·4444
10·0	1·9000	0·1890	3·6336

$\tan\omega = \tan\phi \cdot f_1(Z)$; $v_e \cos\omega = v_0 \cos\phi \cdot f_2(Z)$; $T = \dfrac{X}{v_0 \cos\phi} \cdot f_3(Z)$; $Z = 1 + KX = \dfrac{v_0^2 \sin 2\phi}{gX}$.

Then for the same shell and the same v_0, but any arbitrary angle of departure ϕ, we calculate the elements from

$$y = x\tan\phi - \frac{gx^2}{2(v_0\cos\phi)^2}(1+p+qp^2) \quad\text{.....................(18)}$$

$$\tan\theta = \tan\phi - \frac{gx}{(v_0\cos\phi)^2}\left(1+\frac{3}{2}p+2qp^2\right) \quad\text{..................(19)}$$

$$v\cos\theta = v_0\cos\phi\,(1+3p+6qp^2)^{-\frac{1}{2}} \quad\text{...................................(20)}$$

$$t = \frac{x}{v_0\cos\phi}\frac{1}{p}\int_0^p \sqrt{(1+3p+6qp^2)}\,dp \quad\text{.........................(21)}$$

where $$\frac{Ax}{\cos\phi} = p, \text{ and } \frac{B}{A^2} = q.$$

This method of Duchêne, with regard to (1), is more exact than the preceding, but less convenient.

3. If it is proposed to represent a given trajectory by a rational integral algebraical function of the 3rd or 4th degree, so as to obtain, for any given distance x, the height y, and thus to determine the trajectory, a great many methods are possible.

A parabola, for instance, of the 3rd Order,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3,$$

is given by the range X, the angle of departure ϕ, and the acute angle of descent ω, through the four conditions that, for

$$x=0,\ y=0,\ y'=\tan\phi\,;\ \text{and for } x=X,\ y=0,\ y'=-\tan\omega.$$

With these conditions

$$y = x\tan\phi - \frac{2\tan\phi-\tan\omega}{X}x^2 - \frac{\tan\omega-\tan\phi}{X^2}x^3 \quad\text{............(22)}$$

A parabola of the 4th Order,

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4,$$

is given for instance by the angle of departure ϕ, the initial velocity v_0, the range X, and the angle of descent (acute) ω; since, for

$$x=0,\ y=0,\ y'=\tan\phi,\ v_0\cos\phi = \sqrt{\frac{g}{-y''}}\,;$$

and for $x=X$, $y=0$, $y'=-\tan\omega$. And so we have the five conditions.

From the first three relations we have at once

$$y = x\tan\phi - \frac{gx^2}{2(v_0\cos\phi)^2}(1+Ax+Bx^2). \quad\text{...................(23)}$$

The two coefficients A and B are then obtained from

$$\left.\begin{aligned} AX+BX^2 &= \frac{v_0^2\sin 2\phi}{gX} - 1 \\ 3AX+4BX^2 &= (\tan\phi+\tan\omega)\frac{2v_0^2\cos^2\phi}{gX} - 2 \end{aligned}\right\} \quad\text{.............(24)}$$

For example, given $v_0=406$ m/sec, $\phi=35°$, $X=8700$ m, $\omega=46° 19'$. Then $A=\frac{31\cdot3}{8700}$, $B=\frac{-14\cdot2}{(8700)^2}$; thence by means of (23) $y_s=1900$ m (but according to Otto's Tables, $y_s=1855$).

Now if the coefficients are known, the height y of the flight for different distances x is calculated on Horner's method, and if a considerable number of equidistant values of x are considered, we make use of arithmetical series. But without any calculation and merely with the use of millimetre squared paper and a set square, y can be obtained graphically for any value of x.

The tracing of such a parabola of a higher order can be carried out in a simple manner by the use of the Abdank-Abakanowitz Integraph.

The principle is as follows: A function of the 4th Order, for instance, $y=a_1x+a_2x^2+a_3x^3+a_4x^4$ is proposed, with known coefficients.

Differentiating three times, we have

$$y'=a_1+2a_2x+3a_3x^2+4a_4x^3,$$
$$y''=2a_2+6a_3x+12a_4x^2,$$
$$y'''=6a_3+24a_4x.$$

The last equation represents a straight line, in the coordinates x and y'''.

This line is drawn, and then treated by the Integraph tracer, with the integration constant determined from the condition $y''=2a$ for $x=0$.

The parabola so obtained, of the 2nd Order, is treated again in the same way, and the integration constant given by the condition $y'=a_1$, for $x=0$. A parabola of the 3rd Order is obtained, and this again is integrated, with $y=0$ for $x=0$; and in this way the parabola of the 4th Order is drawn, representing the trajectory.

The intersection with the x axis gives a real root of the equation

$$0=a_1x+a_2x^2+a_3x^3+a_4x^4,$$

and therefore the range $x=X$ for which $y=0$.

Consult the *Praktische Analysis* of H. von Sanden for details of the numerical and graphical methods, and for the procedure of the mechanical integration consult the treatise of Abdank-Abakanowitz—*Les Intégraphes*, Paris 1889, and compare the notes to this chapter.

4. When an ordinary Range Table is available for a definite system of gun and projectile (for a target on the muzzle horizon) we often have to determine the elements of the trajectory which do not lie on the muzzle horizon, with the help of the Range Table alone.

For this purpose the methods of the French *Aide-Mémoire* may serve. These depend on the assumption that the equation between x and y of the trajectory has the form

$$y=x\tan\phi-\frac{gx^2}{2v_0^2\cos^2\phi}F(x) \quad\ldots\ldots\ldots(a)$$

This assumption is indeed of a somewhat general nature, but as we know from the expansion in series (1), it cannot be exact, even if we are considering one and the same system of gun, shell, air density and initial velocity. For the angle of departure ϕ is a variable in the Range Table, and the function $F(x)$ depends on ϕ.

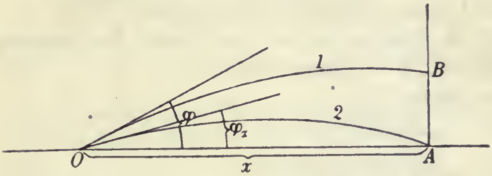
For the same weapon and loading the equation above may be written

$$y = x\tan\phi - \frac{f(x)}{\cos^2\phi}, \quad \text{.........................}(b)$$

whence
$$\tan\theta = y' = \tan\phi - \frac{f'(x)}{\cos^2\phi}, \quad \text{.........................}(c)$$

$$\frac{g}{(v\cos\theta)^2} = -y'' = +\frac{f''(x)}{\cos^2\phi}. \quad \text{.........................}(d)$$

Here x, y denote the coordinates OA, AB of the trajectory OB, or 1, for which the angle of departure is ϕ; θ is the tangent slope at B, and v the velocity, as in the figure.



We next consider the trajectory 2, having the range OA or x, and according to the Range Table having an angle of departure ϕ_x, acute angle of descent ω_x at A, and final velocity v_{ex} and total time of flight T_x.

This trajectory 2 has the equation

$$y = x\tan\phi_x - \frac{f(x)}{\cos^2\phi_x},$$

whence
$$\tan\theta = \tan\phi_x - \frac{f'(x)}{\cos^2\phi_x}, \quad \frac{g}{(v\cos\theta)^2} = \frac{f''(x)}{\cos^2\phi_x}.$$

Then if in this equation x denotes the range OA of the trajectory 2, and at the same time the abscissa OA of the point B of the trajectory 1, we have

$$0 = x\tan\phi_x - \frac{f(x)}{\cos^2\phi_x}, \quad \text{.........................}(e)$$

$$\tan(-\omega_x) = \tan\phi_x - \frac{f'(x)}{\cos^2\phi_x}, \quad \text{.........................}(f)$$

$$\frac{g}{(v_{ex}\cos\omega_x)^2} = \frac{f''(x)}{\cos^2\phi_x}. \quad \text{.........................}(g)$$

Eliminating $f(x)$ between (b) and (e), $f'(x)$ between (c) and (f), and $f''(x)$ between (d) and (g), we obtain the results

$$y = x\left(\tan\phi - \frac{\cos^2\phi_x\tan\phi_x}{\cos^2\phi}\right) = x\frac{\sin 2\phi - \sin 2\phi_x}{2\cos^2\phi},$$

$$\tan\theta = \tan\phi - \frac{\cos^2\phi_x\tan\phi_x}{\cos^2\phi} - \frac{\cos^2\phi_x\tan\omega_x}{\cos^2\phi} = \frac{y}{x} - \frac{\tan\omega_x\cos^2\phi_x}{\cos^2\phi},$$

$$\frac{v\cos\theta}{v_{ex}\cos\omega_x} = \frac{\cos\phi}{\cos\phi_x}.$$

Finally, the time of flight from O to B is given by

$$\frac{dt}{dx} = \frac{1}{v\cos\theta} = \frac{\cos\phi_x}{\cos\phi}\cdot\frac{1}{v_{ex}\cos\omega_x};$$

and then by integration, from O to A, we have

$$t=T_x\frac{\cos\phi_x}{\cos\phi}.$$

We have, therefore, the following results:

$$y=x\left(\tan\phi-\frac{\cos^2\phi_x\tan\phi_x}{\cos^2\phi}\right),\quad\text{.....................(25)}$$

$$\tan\theta=\frac{y}{x}-\frac{\tan\omega_x\cos^2\phi_x}{\cos^2\phi},\quad\text{.................................(26)}$$

$$v\cos\theta=v_{ex}\cos\omega_x\frac{\cos\phi}{\cos\phi_x},\quad\text{...................................(27)}$$

$$t=T_x\frac{\cos\phi_x}{\cos\phi}.\quad\text{..(28)}$$

Equation (25) will serve to determine the height AB or y, when the angle of departure ϕ is given.

Solved with respect to ϕ, we have

$$\tan\phi=\frac{1-\sqrt{[\cos^2 2\phi_x-2\tan E\sin 2\phi_x]}}{\sin 2\phi_x},\quad \tan E=\frac{y}{x},\quad\text{.........(29)}$$

and this serves to calculate the departure angle ϕ, when a given mark $B\,(xy)$ is to be struck. Equation (26) determines θ, and then (27) determines the velocity v, and (28) the time of flight t.

Provided the angles ϕ and ϕ_x of departure are not very different, equations (25) to (28) can be replaced approximately by the following:

$$\frac{y}{x}=\tan\phi-\tan\phi_x,\quad \tan\theta=\frac{y}{x}-\tan\omega_x,\quad v\cos\theta=v_{ex}\cos\omega_x,\quad t=T_x.$$

Since $\frac{y}{x}$ is the tangent of the angle of sight BOA, or E, as seen from O (E is the slope of the ground) the first of these equations is equivalent to

$$\tan E=\tan\phi-\tan\phi_x,$$

or with small angles, $E=\phi-\phi_x$ ("Tilting of the trajectory").

These equations, (25) to (28), provide useful approximations, at least for angles of departure up to 20°.

5. Vallier, in 1886, was the first to employ the remainder R in its integral form in ballistic calculations.

The corresponding equations for y, θ, v, and t, are expressed as follows:

$$y=x\tan\phi-\frac{gx^2}{2v_0^2\cos^2\phi}-g\int_{t=0}^{t=x}(x-t)^2\left[\frac{cf(v)}{v^4\cos^3\theta}\right]_t dt,\quad\text{.........(30)}$$

$$\tan\theta=\tan\phi-\frac{gx}{v_0^2\cos^2\phi}-2g\int(x-t)\left[\frac{cf(v)}{v^4\cos^3\theta}\right]_t dt,\quad\text{...............(31)}$$

$$v\cos\theta=v_0\cos\phi-\frac{cf(v_0)}{v_0}x+\frac{1}{2}\int\left[\frac{cf(v)}{v}\right]'\frac{cf(v)+g\sin\theta}{v\cos\theta}(x-t)\,dt,\quad\text{...(32)}$$

$$t=\frac{x}{v_0\cos\phi}+\int(x-t)\left[\frac{cf(v)}{v^3\cos^2\theta}\right]_t dt.\quad\text{.................................(33)}$$

Except in the third of these equations, the integral expresses the correction of the equation of motion in a vacuum by insertion of the term due to the space being filled with air.

A first application by Vallier relates to the deduction in a different way of the equations of § 25 for flat trajectories.

Suppose the biquadratic law of air resistance is assumed, $cf(v)=cv^4$, and sec θ is replaced in the integral by a constant mean value a (Didion's mean value). Then equation (30) becomes

$$y=x\tan\phi-\frac{gx^2}{2v_0^2\cos^2\phi}-gca^3\int_{t=0}^{t=x}(x-t)^2\,dt$$

$$=x\tan\phi-\frac{gx^2}{2v_0^2\cos^2\phi}\left(1+\frac{2}{3}ca^3v_0^2\cos^2\phi\,.\,x\right)$$

as in § 25.

A second application will be mentioned in § 29, for the calculation of an expression for an adjusting factor β.

CHAPTER V

Second group of numerical methods of approximation. Exact solution of an approximate Chief Equation

§ 23. Generalities. Comparison of the different methods.

In the preceding chapter, many methods were mentioned, by which the original differential equation, called the Chief Equation, could be treated, when written in the exact form

$$gd(v\cos\theta) = cf(v)\,v\,d\theta, \text{ or } \frac{d\theta}{\cos^2\theta} = \frac{gd(v\cos\theta)}{v\cos\theta\,.\,cf(v)\cos\theta},$$

$cf(v)$ denoting the retardation due to the air resistance; and then further integrations are made when arbitrary approximations are introduced.

The simplification of the Chief Equation is another method, and we may replace it by an approximate Chief Equation, of which the integration presents no difficulty.

The principle appears to have been employed first by Borda, 1769, and later it was further developed by St Robert, N. Mayevski, and F. Siacci. Borda replaced the air density δ, which forms a factor of c, by the approximate $\delta\frac{\cos\theta}{\cos\phi}$ (and this is exact for two points of the path) or c is replaced by $c\frac{\cos\theta}{\cos\phi}$, assuming the quadratic law,

$$cf(v) = cv^2.$$

Then, with $v\cos\theta = u$, we have

$$\frac{d\theta}{\cos^2\theta} = \frac{g\,du}{ucv^2\cos\theta} = \sim \frac{g\,du}{uc\dfrac{\cos\theta}{\cos\phi}v^2\cos\theta} = \frac{g\cos\phi}{c}\frac{du}{u^3};$$

and here the variables are separated, so that the integration is at once possible. Thus, for example,

$$dx = -\frac{v^2}{g}d\theta = -\frac{u^2}{g}\frac{d\theta}{\cos^2\theta} = -\frac{u^2}{g}\frac{g\cos\phi}{c}\frac{du}{u^3} = -\frac{\cos\phi}{c}\cdot\frac{du}{u},$$

$$\log u = -\frac{cx}{\cos\phi} + \log(v_0\cos\phi), \quad v\cos\theta = v_0\cos\phi\, e^{-\frac{cx}{\cos\phi}}, \text{ etc.}$$

As will be shown later, the solution in the last line is the same as that of Didion (compare § 29, with $\alpha = \sec\phi$). The same procedure is due to Besout.

Legendre made many suggestions as to the integration of the Chief Equation; in particular under an assumption of the quadratic law, $cf(v) = cv^2$, he replaced the air density δ by

$$\delta \frac{1 + ap^2}{\sqrt{(1+p^2)}}, \text{ or } c \text{ by } c\cos\theta\,(1 + ap^2),$$

where $p = \tan\theta$; and the factor a he assumes $= \dfrac{\cos\phi}{1+\cos\phi}$. (This c agrees with the true c in three points of the trajectory, namely for $\theta = +\phi$, $\theta = 0$, and $\theta = -\phi$.)

Thence the equation becomes the following:

$$\frac{d\theta}{\cos^2\theta} = dp = \frac{g\,d(v\cos\theta)}{v\cos\theta\,.\,cv^2\cos\theta} = \sim \frac{g\,d(v\cos\theta)}{v\cos\theta\,.\,c(1+ap^2)\,v^2\cos^2\theta}$$

$$= \frac{g\,du}{uc\,(1+ap^2)\,u^2}, \text{ or } dp\,(1+ap^2) = \frac{g\,du}{cu^3}.$$

In this equation the variables are separated again, between p and u, or $\tan\theta$ and $v\cos\theta$. And when it is integrated, and the value of v^2 as a function of θ is substituted in the general equation for dx, that is in $g\,dx = -v^2 d\theta$, then this equation can be solved in a finite form: and by help of an equation of the 3rd degree, x will be expressed in terms of p; and y also.

Français raised the objection to this procedure, that for $\theta = \frac{1}{2}\pi$, and $\tan\theta = \infty$, the air density will become infinite. He himself, on this account, replaces δ by $\delta\cos\theta \dfrac{1 + a\tan^2\theta}{\sqrt{(1 + b\tan^2\theta)}}$, where a and b must be determined accordingly.

A more general method, that is applicable to any function $cf(v)$ whatever, whether given analytically or in a tabular form, is the following:

Let the retardation due to air resistance be still denoted by $cf(v)$. The exact equation is

$$\frac{d\theta}{\cos^2\theta} = \frac{g\,d(v\cos\theta)}{v\cos\theta\, cf(v)\cos\theta} = \frac{g\,d\left(\dfrac{v\cos\theta}{\sigma}\right)}{\dfrac{v\cos\theta}{\sigma}\, cf\left(\dfrac{v\cos\theta}{(\cos\theta)}\right)((\cos\theta))} \quad \ldots(1)$$

where σ denotes a constant, to be determined hereafter; and we have in addition the equations

$$dx = -\frac{v^2}{g}\,d\theta, \quad \text{.................................(2)}$$

$$dt = -\frac{v\,d\theta}{g\cos\theta}, \quad \text{.............................(3)}$$

$$dy = -\frac{v^2\tan\theta\,d\theta}{g}. \quad \text{..........................(4)}$$

With the object of integrating equation (1) the simplification is introduced of replacing the $\cos\theta$ under the functional sign f, enclosed in a single pair of round brackets, by a constant mean value σ, considering that along the trajectory or at least along a great part of it, $\cos\theta$ varies slowly; replacing also the $\cos\theta$ between a pair of double round brackets by another constant γ.

Then the equation is approximately exact

$$\frac{d\theta}{\cos^2\theta} = \sim \frac{g\,d\left(\dfrac{v\cos\theta}{\sigma}\right)}{\dfrac{v\cos\theta}{\sigma}\,c\,f\left(\dfrac{v\cos\theta}{\sigma}\right)\gamma} = \frac{g\,du}{c\gamma u\,f(u)}, \quad \text{......(5)}$$

where $u = \dfrac{v\cos\theta}{\sigma}$. Thereby the variables θ and u are separated, and

$$(\tan\theta)_\phi^\theta = \frac{g}{c\gamma}\int_{u_0}^{u}\frac{du}{u\,f(u)}, \text{ or } \tan\theta = \tan\phi - \frac{1}{2c\gamma}[J(u) - J(u_0)],$$

where
$$J(u) = -\,2g\int\frac{du}{u\,f(u)};$$

and therefore

$$dx = -\frac{v^2 d\theta}{g} = -\frac{\sigma^2}{g}\frac{u^2 d\theta}{\cos^2\theta} = -\frac{\sigma^2 u^2}{g}\frac{g\,du}{c\gamma u f(u)} = -\frac{\sigma^2}{\gamma c}\frac{u\,du}{f(u)},$$

$$(x)_0^x = -\frac{\sigma^2}{\gamma c}\int_{u_0}^{u}\frac{u\,du}{f(u)}; \quad x = +\frac{\sigma^2}{\gamma c}[D(u) - D(u_0)],$$

where
$$D(u) = -\int\frac{u\,du}{f(u)}.$$

Further

$$dt = -\frac{v\,d\theta}{g\cos\theta} = -\frac{\sigma u}{g}\frac{d\theta}{\cos^2\theta} = -\frac{\sigma u}{g}\cdot\frac{g\,du}{c\gamma u\,f(u)} = -\frac{\sigma}{c\gamma}\frac{du}{f(u)},$$

$$t = -\frac{\sigma}{c\gamma}\int_{u_0}^{u}\frac{du}{f(u)} = +\frac{\sigma}{c\gamma}[T(u) - T(u_0)], \text{ where } T(u) = -\int\frac{du}{f(u)}.$$

Finally $\quad dy = dx\tan\theta = dx\tan\phi - \dfrac{1}{2c\gamma}[J(u) - J(u_0)]\,dx,$

or, substituting the above value of dx,

$$dy = dx\tan\phi - \frac{1}{2c\gamma}\left[-J(u)\frac{\sigma^2}{\gamma c}\frac{u\,du}{f(u)} - J(u_0)\,dx\right];$$

and integrating from $y=0$ to y, or from u_0 to u, or from $x=0$ to x,

$$y = x\tan\phi - \frac{1}{2c\gamma}\left[-\frac{\sigma^2}{\gamma c}\int_{u_0}^{u}\frac{J(u)\,u\,du}{f(u)} - x\,J(u_0)\right]$$

$$= x\tan\phi - \frac{1}{2c\gamma}\left[-\frac{\sigma^2}{\gamma c}\int\frac{J(u)u\,du}{f(u)} - \frac{\sigma^2}{\gamma c}J(u_0)\{D(u) - D(u_0)\}\right],$$

and, when $-\int\dfrac{J(u)\,u\,du}{f(u)}$ is denoted by $A(u)$,

$$y = x\tan\phi - \frac{\sigma^2}{2c^2\gamma^2}\{[A(u) - A(u_0)] - J(u_0)[D(u) - D(u_0)]\}.$$

The integral values of $D(u)$, $T(u)$, $J(u)$, $A(u)$, which are called Siacci's Primary Functions, can be evaluated exactly if we assume the monomial law, $f(v) = v^n$, or $f(u) = u^n$, and then entered in a Table. Thus, for instance, on the cubic law of resistance, $cf(v) = cv^3$, or $f(u) = u^3$,

$$J(u) = -2g\int\frac{du}{u\,.\,u^3} = +\frac{2}{3}gu^{-3},$$

$$A(u) = -\int\frac{J(u)\,u\,du}{f(u)} = -\frac{2}{3}g\int\frac{u^{-3}u\,du}{u^3} = \frac{1}{6}gu^{-4}, \text{ etc.}$$

For more complicated functions $f(u)$, the integrals must be evaluated by Simpson's rule, or by the use of an Integraph.

This is similar to the case when the coefficient of form i involved in c is replaced by a constant mean value. In such a case, where i is to be expressed as a function of v, as for example

$$\frac{1}{i} = 1{\cdot}3206 - \frac{58{\cdot}2}{v} - 0{\cdot}0001024v,$$

or, more generally $\qquad = p - \dfrac{q}{v} - rv,$

the procedure can be carried out in two ways.

Either we calculate for the purpose an adjusting factor β, and this will be explained later in § 29. Or else we proceed more exactly in the way first proposed by O. von Eberhard: the function above, for example, $J(u)$ is replaced by

$$J(u) = -2g\int\frac{\left(p - \frac{q}{u} - ru\right)du}{uf(u)} = -2gp\int\frac{du}{uf(u)} + 2gq\int\frac{du}{u^2f(u)} + 2gr\int\frac{du}{f(u)}.$$

Of these three integrals, the first and third are contained already among the Siacci Primary Functions.

But a new function arises in $\int \frac{du}{u^2 f(u)}$. Thus we have seven new functions, which may be called the Eberhard functions, as opposed to those of Siacci.

We must wait for the present for these new functions to be calculated numerically.

Collection of formulae.

(I) $x = \frac{\sigma^2}{\gamma c}[D(u) - D(u_0)]$

(II) $t = \frac{\sigma}{c\gamma}[T(u) - T(u_0)]$

(III) $\tan\theta = \tan\phi - \frac{1}{2c\gamma}[J(u) - J(u_0)]$

(IV) $y = x\tan\phi - \frac{\sigma^2}{2c^2\gamma^2}\{A(u) - A(u_0) - J(u_0)[D(u) - D(u_0)]\}$

or

(IV a) $y = x\tan\phi - \frac{x}{2c\gamma}\left\{\frac{A(u) - A(u_0)}{D(u) - D(u_0)} - J(u_0)\right\}$

Retardation due to air resistance $= cf(v)$.

σ and γ are certain constants to be determined approximately.

Notation.

$D(u) = -\int \frac{u\,du}{f(u)}$

$T(u) = -\int \frac{du}{f(u)}$

$J(u) = -2g\int \frac{du}{u f(u)}$

$A(u) = -\int \frac{J(u)\,u\,du}{f(u)}$

$u = \frac{v\cos\theta}{\sigma}$

$u_0 = \frac{v_0\cos\phi}{\sigma}$

If u in (I) is expressed in terms of x, and substituted in (II), (III), (IV), then t, θ, y are given as functions of x.

On the choice of the constants σ and γ.

These constants so far have been considered to be arbitrary; but now we must settle as to how they are to be chosen.

According to the numerous methods of solution proposed in the course of the last 150 years, it can be shown that they are essentially contained in equations (I) to (IV), but differ in the choice of σ and γ.

The corresponding methods are not really derived from equations (I) to (IV), but rather they have been developed by their authors in a totally different manner.

Still it is of interest to treat the different methods from a general standpoint, and to examine their inner connexion.

(*a*) Borda, 1769, assumes:

$$\sigma = 1,\ \gamma = \frac{1}{\cos\phi},\ \text{on the assumption } cf(v) = cv^2.$$

(*b*) J. Didion, 1848: $\sigma = \gamma = \frac{1}{\alpha}$, where α is some mean value of $\sec\theta$ between the beginning and end of the corresponding arc of the trajectory. He chose the air resistance function $cf(v) = cv^2\left(1 + \frac{v}{r}\right)$, c and r being constants; and therefore

$$x = \frac{1}{\alpha c}[D(u) - D(u_0)], \qquad u = \alpha v\cos\theta,$$

$$t = \frac{1}{c}[T(u) - T(u_0)], \qquad u_0 = \alpha v_0\cos\phi,$$

$$\tan\theta = \tan\phi - \frac{\alpha}{2c}[J(u) - J(u_0)],$$

$$y = x\tan\phi - \frac{1}{2c^2}\{A(u) - A(u_0) - J(u_0)[D(u) - D(u_0)]\}.$$

Moreover Didion did not take u as the independent variable in his method of solution, but x, and established a system of formulae for t, θ, y, $v\cos\theta$, as functions of x.

The mean value for α, employed by him, is

$$\alpha = \frac{\int_\phi^\theta \sec^3\theta\, d\theta}{\tan\theta - \tan\phi}.$$

(*c*) St Robert proposed instead, among other things, the arithmetic mean between the value of $\sec\theta$ at the starting point $\theta = \phi$ (and also at the point $\theta = -\phi$ in the descending branch) and the value at the vertex $\theta = 0$; i.e., the arithmetic mean between $\sec\phi$ and $\sec 0$, or $\alpha = \frac{1}{2}(1 + \sec\phi)$.

(*d*) Hélie in his method of solution assumed the geometric mean between $\sec\phi$ and $\sec 0$, that is $\alpha = \sqrt{(\sec\phi)}$.

(*e*) F. Siacci too in his method of 1880 (denoted briefly for convenience by "Siacci I") assumed $\sigma = \gamma = \frac{1}{\alpha}$; and employed them in his method assuming zones for the laws of air resistance.

(*f*) N. v. Wuich, 1886, also assumes $\sigma=\gamma=\frac{1}{\alpha}$, and the quadratic law $cf(v)=cv^2$, with a possible change of c along the trajectory, and with x as independent variable.

(*g*) F. Krupp (first procedure): $\sigma=\gamma=1$; and employment of the Krupp Tables of air resistance.

(*h*) F. Siacci, in his procedure of 1888 (denoted conveniently as "Siacci II"): $\sigma=\cos\phi$, $\gamma=\beta\cos^2\phi$, where β is to be taken out of a Table from X and ϕ; so that a first approximation results. Zones are to be employed for air resistance to determine the dependence on the velocity v.

The system of equations is then:

$$x=\frac{1}{c\beta}[D(u)-D(u_0)], \qquad u=\frac{v\cos\theta}{\cos\phi},$$

$$t=\frac{1}{c\beta\cos\phi}[T(u)-T(u_0)], \qquad u_0=v_0,$$

$$\tan\theta=\tan\phi-\frac{1}{2c\beta\cos^2\phi}[J(u)-J(u_0)],$$

$$y=x\tan\phi-\frac{1}{2c^2\beta^2\cos^2\phi}\{A(u)-A(u_0)-J(u_0)[D(u)-D(u_0)]\}.$$

Similar suggestions were made by J. M. Ingalls (North America) 1900, and by N. Sabudski (Russia), for flat trajectories.

(*i*) E. Vallier in 1894: here $\sigma=\cos\phi$, $\gamma=\frac{1}{m}\cos^2\phi$, and after a first approximation, m is to be calculated by a formula.

(*k*) F. Siacci, procedure of 1896 ("Siacci III"): again $\sigma=\cos\phi$, $\gamma=\beta\cos^2\phi$.

(*l*) P. Charbonnier: as a first approximation $\sigma=\gamma=1$ (as also in the former procedure of F. Krupp), and so

$$x=\frac{1}{c}[D(u)-D(u_0)], \qquad u=v\cos\theta,$$

$$t=\frac{1}{c}[T(u)-T(u_0)], \qquad u_0=v_0\cos\phi,$$

$$\tan\theta=\tan\phi-\frac{1}{2c}[J(u)-J(u_0)],$$

$$y=x\tan\phi-\frac{1}{2c^2}\{A(u)-A(u_0)-J(u_0)[D(u)-D(u_0)]\}.$$

And then we proceed again in a second approximation, taking in the ascending branch, instead of δ, $\delta(1+\frac{1}{2}\kappa_0 \tan^2\phi)$, and in the descending branch $\delta(1+\frac{1}{2}\kappa_\omega \tan^2\omega)$, and $\kappa=\frac{1}{2}\left[v\cos\theta\frac{f'(v\cos\theta)}{f(v\cos\theta)}-1\right]$. The procedure is to be employed in flat trajectories, and the air resistance taken from Krupp's tables.

§ 24. Solution of J. Didion (1848).

The retardation of air resistance is taken as $cf(v)=cv^2\left(1+\frac{v}{r}\right)$ where c and r are the constants introduced in § 10. According to § 17, $dx=-\frac{v^2}{g}d\theta$, or since $d\theta=\frac{g\,d(v\cos\theta)}{vcf(v)}$,

$$dx=-\frac{v\cos\theta\, d(v\cos\theta)}{cf(v)\cos\theta}. \quad \text{.................}(1)$$

Didion's method of approximate calculation has already been mentioned, in which $f(v)$ on the right-hand side has been replaced by $f(\alpha v\cos\theta)$ and $\cos\theta$ by $\frac{1}{\alpha}$; α is then a mean value of $\sec\theta$.

Thence, with $\alpha v\cos\theta=u$, the differential equation (1) assumes a form, in which the variables x and u are separated, so that the integration can be carried out at once; in fact $dx=-\frac{1}{\alpha c}\frac{u\,du}{f(u)}$, or

$$f(u)=u^2\left(1+\frac{u}{r}\right),$$

$$dx=-\frac{1}{\alpha c}\frac{du}{u\left(1+\frac{u}{r}\right)}. \quad \text{.......................}(2)$$

From this equation an integration will give $v\cos\theta$, and further θ, t and y as functions of x.

The equation (2) may be deduced in the manner proposed by Didion himself; but the introduction of u as independent variable in the method of solution was first employed by St Robert in 1872.

The equation of motion of the shell resolved in a horizontal direction leads to

$$\frac{dv_x}{dt}=-cf(v)\cos\theta=-cv^2\left(1+\frac{v}{r}\right)\frac{dx}{ds}=-cv\left(1+\frac{v}{r}\right)v_x, \text{ or } \frac{dv_x}{dx}=-cv\left(1+\frac{v}{r}\right).$$

Didion now replaces ds by the approximate value $a\,dx$; or v by av_x.

With $u=av_x=av\cos\theta$, $du=ad(v_x)$, the equation will become as in (2) above

$$\frac{1}{a}\frac{du}{dx}=-cu\left(1+\frac{u}{r}\right).$$

The further calculations are as follows:

$$\alpha c\,dx = \left(\frac{1}{1+\dfrac{u}{r}} - \frac{1}{\dfrac{u}{r}}\right) d\left(\frac{u}{r}\right),$$

and integrating from 0 to x, and so from u_0 to u,

$$\alpha cx = \log \frac{1+\dfrac{u}{r}}{\dfrac{u}{r}} - \log \frac{1+\dfrac{u_0}{r}}{\dfrac{u_0}{r}}.$$

Here $u = \alpha v \cos\theta$, $u_0 = \alpha v_0 \cos\phi$; and then by solution for $v\cos\theta$, we have

$$v\cos\theta = \frac{v_0\cos\phi}{(1+\kappa_0)\,e^{c\alpha x} - \kappa_0}, \qquad \text{......................(3)}$$

when $\dfrac{\alpha v_0 \cos\phi}{r}$ is replaced by κ_0.

From (3) the horizontal component $v\cos\theta$ is known for any distance x of the shell. The corresponding time t is given by integration, by means of $dt = \dfrac{dx}{v\cos\theta}$, after substitution of the value of $v\cos\theta$ in equation (3),

$$dt = \frac{1}{v_0\cos\phi}\left[(1+\kappa_0)\,e^{c\alpha x} - \kappa_0\right] dx,$$

and

$$t = \frac{1}{v_0\cos\phi}\left[(1+\kappa_0)\frac{e^{c\alpha x}-1}{c\alpha} - \kappa_0 x\right]. \qquad \text{............(4)}$$

When relation (3) is applied in a similar way to the equation for θ, viz.,

$$d\theta = -\frac{g\,dx}{v^2}, \text{ or } \frac{d\theta}{\cos^2\theta} = -\frac{g\,dx}{(v\cos\theta)^2},$$

then

$$\frac{d\theta}{\cos^2\theta} \text{ or } d(\tan\theta) = -\frac{g}{v_0^2\cos^2\phi}\left[(1+\kappa_0)\,e^{c\alpha x} - \kappa_0\right]^2 dx,$$

and integrated from ϕ to θ, and from 0 to x,

$$\tan\theta - \tan\phi = -\frac{g}{v_0^2\cos^2\phi}\left[(1+\kappa_0)^2\frac{e^{2c\alpha x}-1}{2c\alpha} - 2\kappa_0(1+\kappa_0)\frac{e^{c\alpha x}-1}{c\alpha} + \kappa_0^2 x\right]. \qquad \text{......(5)}$$

Finally, since $\tan\theta = \dfrac{dy}{dx}$, equation (5) is integrated again with respect to x, and then y is given as a function of x.

The expressions for $v \cos\theta$, t, θ, y as functions of x are stated again further on.

Didion calculates the mean value α of the actual variable $\frac{ds}{dx}$ or $\sqrt{(1+\tan^2\theta)}$ or $\sec\theta$ as approximately the ratio $\frac{s}{x}$ of the finite arc ($OM = s$) of the actual trajectory, which is under consideration, to its horizontal projection ($OM_1 = x$). This ratio, $OM : OM_1$, he takes as approximately the same as the ratio $s_1 : x_1$ of the arc of the path of flight OP in a vacuum (having the same initial and final inclinations ϕ and θ, and the same initial velocity v_0), to the horizontal projection OP_1 of this arc: that is

$$\alpha = \frac{OP}{OP_1} = \frac{\text{arc in a vacuum with the same } \phi,\ \theta, \text{ and } v_0}{\text{horizontal projection of this arc}} = \frac{s_1}{x_1}.$$

Given the values of ϕ, θ, and v_0, OP and OP_1 must then be calculated on the assumption that the resistance of the air ceases (compare § 1).

(*a*) Numerator $OP = s_1$:

Now
$$s_1 = \int_\phi^\theta \sqrt{\left[1 + \left(\frac{dy_1}{dx_1}\right)^2\right]}\, dx_1.$$

In a vacuum

$$y_1 = x_1 \tan\phi - \frac{gx_1^2}{2v_0^2 \cos^2\phi}, \quad \tan\theta_1 = \tan\phi - \frac{gx_1}{v_0^2\cos^2\phi}.$$

Put $\tan\theta_1 = \frac{dy_1}{dx_1} = p$, so that $dp = -\frac{g\,dx_1}{v_0^2\cos^2\phi}$; then

$$s_1 = -\frac{v_0^2\cos^2\phi}{g}\int_\phi^\theta \sqrt{(1+p^2)}\, dp.$$

Now

$$\int_0^\theta \sqrt{(1+p^2)}\, dp = \frac{1}{2}\left[\frac{\sin\theta}{\cos^2\theta} + \log\tan\left(\frac{1}{4}\pi + \frac{1}{2}\theta\right)\right] = \int_0^\theta \frac{d\theta}{\cos^3\theta} = \xi(\theta)$$

(Table, Vol. IV, No. 10); thence

$$s_1 = +\frac{v_0^2\cos^2\phi}{g}[\xi(\phi) - \xi(\theta)].$$

(*b*) Denominator $OP_1 = x_1$:

According to the above, $x_1 = \frac{v_0^2\cos^2\phi}{g}(\tan\phi - \tan\theta)$; so that

$$\alpha = \frac{s_1}{x_1} = \frac{\xi(\phi) - \xi(\theta)}{\tan\phi - \tan\theta}. \qquad \text{..................(6)}$$

When the arc of flight to be calculated extends from the starting point ($\theta=\phi$) up to the vertex ($\theta=0$), we have

$$\alpha=\frac{\xi(\phi)}{\tan\phi}, \text{ since when } \theta=0,\ \xi(\theta)=0,\ \tan\theta=0.$$

If the trajectory is to be calculated as accurately as possible, it is divided into several arcs, of which the parts near the vertex can be taken of greater length: thus, for example, if the angle of departure is 45°, the division would be made into four arcs: say,

(*a*) from $\theta=\phi=45°$ to $\theta=30°$; here

$$\alpha=\frac{\xi(45)-\xi(30)}{\tan 45-\tan 30}=1{\cdot}2772;$$

(*b*) from $\theta=30°$ to $\theta=0$ (vertex); here

$$\alpha=\frac{\xi(30)-\xi(0)}{\tan 30-\tan 0}=\frac{\xi(30)}{\tan 30}=1{\cdot}0531;$$

(*c*) from $\theta=0$ to $\theta=-30°$; here

$$\alpha=\frac{\xi(0)-\xi(-30)}{\tan 0-\tan(-30)}=\frac{\xi(30)}{\tan 30}=1{\cdot}0531, \text{ as in } (b);$$

(*d*) from $\theta=-30°$ to $\theta=-45°$; here

$$\alpha=\frac{\xi(-30)-\xi(-45)}{\tan(-30)-\tan(-45)}=\frac{\xi(45)-\xi(30)}{\tan 45-\tan 30}=1{\cdot}2772, \text{ as in } (a).$$

The mean value $\alpha=\dfrac{\xi(\phi)-\xi(\theta)}{\tan\phi-\tan\theta}=\dfrac{\int_\phi^\theta \sec^3\theta\,d\theta}{\int_\phi^\theta \sec^2\theta\,d\theta}$ can also be determined from another point of view. For if there are n positive fractions, $\dfrac{a_1}{b_1}, \dfrac{a_2}{b_2}, \dfrac{a_3}{b_3}, \ldots, \dfrac{a_n}{b_n}$, then the fraction $\dfrac{a_1+a_2+\ldots+a_n}{b_1+b_2+\ldots+b_n}$ is smaller than the greatest, and greater than the smallest of the separate fractions, and so is some mean value between them.

Now denote by $\theta_1, \theta_2, \theta_3, \ldots$ the angles of slope to the horizon of the tangent to the trajectory at various intermediate points. The values of $\sec\theta$ or $\dfrac{\sec^3\theta}{\sec^2\theta}$ are thus

$$\frac{\sec^3\phi}{\sec^2\phi}, \frac{\sec^3\theta_1}{\sec^2\theta_1}, \frac{\sec^3\theta_2}{\sec^2\theta_2}, \frac{\sec^3\theta_3}{\sec^2\theta_3}, \ldots \text{ up to } \frac{\sec^3\theta}{\sec^2\theta},$$

where θ and ϕ are the end-values of the angle of slope. These fractions are unequal, but a certain mean value of them is the fraction

$$\frac{\sec^3\phi+\sec^3\theta_1+\sec^3\theta_2+\ldots+\sec^3\theta}{\sec^2\phi+\sec^2\theta_1+\sec^2\theta_2+\ldots+\sec^2\theta}=\frac{\int_\phi^\theta \sec^3\theta\,d\theta}{\int_\phi^\theta \sec^2\theta\,d\theta}=\frac{\xi(\theta)-\xi(\phi)}{\tan\theta-\tan\phi},$$

as above.

Statement of the formulae for Didion's solution.

(1) $y = x\tan\phi - \dfrac{gx^2}{2v_0^2\cos^2\phi}B$ where $B = (1+\kappa_0)^2\dfrac{e^{2cax} - 2cax - 1}{\frac{1}{2}(2cax)^2} - 2\kappa_0(1+\kappa_0)\dfrac{e^{cax} - cax - 1}{\frac{1}{2}(cax)^2} + \kappa_0^2$,

(2) $\tan\theta = \tan\phi - \dfrac{gx}{v_0^2\cos^2\phi}J$ where $J = (1+\kappa_0)^2\dfrac{e^{2cax} - 1}{2cax} - 2\kappa_0(1+\kappa_0)\dfrac{e^{cax} - 1}{cax} + \kappa_0^2$,

(3) $v\cos\theta = v_0\cos\phi\dfrac{1}{V_1}$ where $V_1 = (1+\kappa_0)e^{cax} - \kappa_0$,

(4) $t = \dfrac{x}{v_0\cos\phi}D$ where $D = (1+\kappa_0)\dfrac{e^{cax} - 1}{cax} - \kappa_0$,

$$c = \frac{AR^2\pi g i\delta}{1\cdot 208\,P}, \quad\text{..........................(5)}$$

$$\kappa_0 = \frac{\alpha}{r}v_0\cos\phi, \quad\text{..............................(6)}$$

$$\alpha = \frac{\xi(\phi) - \xi(\theta)}{\tan\phi - \tan\theta}, \quad\text{.......................(7)}$$

or, approximately

$$\alpha = \frac{\xi\left(\dfrac{\phi+\theta}{2}\right)}{\tan\dfrac{\phi+\theta}{2}}, \quad\text{..........................(8)}$$

or,

$$\alpha = \frac{\xi(\phi)}{\tan\phi}. \quad\text{..........................(9)}$$

Here for (xy), the end point of the arc of the trajectory to be calculated, v is the velocity of the shell in its path, in m/sec; θ the inclination to the horizon of the tangent; t the time of flight in seconds to this point. Further v_0 is the initial velocity of the shell in m/sec; ϕ the angle of departure; $2R$ the calibre of the shell in metres; δ the air density in kg/m³; P the weight of the shell in kg; g the acceleration of gravity in m/sec²; $A = 0\cdot 0270$; $r = 435$ (from $v = 550$ m/sec, downwards; according to Didion, *Traité de balistique*, Paris 1860, p. 67); $i = 1$ for spheres. The functions B, J, V_1, D are to be put $= 1$ for a vacuum.

Calculation of some Trajectories. Examples.

(*a*) Given v_0, ϕ, c. We calculate a from (9) and κ_0 from (6), and then for any assigned x we calculate y from (1), θ from (2), v from (3) and t from (4). If the range $x=X$ is to be calculated from $y=0$, and so from

$$\frac{v_0^2 \sin 2\phi}{g} = XB\,(2caX,\ \kappa_0),$$

the value of X is to be found by trial and error, with the assistance of the Table for B.

If the problem is to be solved more accurately, the trajectory is divided into several arcs, the first one reaching, say, from $\theta=\phi=45°$ to $\theta=40°$; and then

$$a=\frac{\xi(45)-\xi(40)}{\tan 45-\tan 40};$$

and thus it follows from (2) that

$$Jx=(\tan\phi-\tan\theta)\,\frac{v_0^2\cos^2\phi}{g};$$

and since θ has been chosen arbitrarily (for instance $\theta=40°$), we find by a tentative process the abscissa x of the end of the first arc, and by means of (1), (3), (4) the corresponding values of y, v, and t. We now consider the origin of coordinates and the starting point of the first arc to be transferred to this end point, and calculate again from this for the second arc, and so on.

(*b*) Given c, ϕ, and the target (xy), to find v_0.

A first approximate value is calculated from the formula for a vacuum

$$v_0^2=\frac{gx^2}{2\cos^2\phi\,(x\tan\phi-y)};$$

or better, it is estimated with the help of a range table; thence a first approximate value is known of κ_0, and also of B, as x is given.

Further a closer value of v_0 is given by equation (1), in which y is given.

The calculation is to be repeated with this value of v_0; and in this way a still closer value is obtained of v_0; still it would be useless to repeat this very often, as the whole calculation is merely an approximation method.

(*c*) Given c, v_0, and the target (xy), to find ϕ (in flat trajectories and curved fire).

A first approximate value of ϕ is given by the equation for a vacuum; thence $\cos\phi$ is known, and a, and so also $\kappa_0=\dfrac{av_0\cos\phi}{r}$, as well as B. Then from equation (1) in which $\cos^2\phi$ is replaced by $\dfrac{1}{1+\tan^2\phi}$, the double value of $\tan\phi$ can be calculated. The calculation is then to be repeated, when necessary, so as to proceed from this value of ϕ to a second approximation.

(*d*) Given a target (xy), c and angle of impact θ; to find ϕ and v_0.

Approximate solutions for ϕ and v_0 are found from the vacuum equations $y = x \tan\phi - \frac{gx^2}{2v_0^2 \cos^2\phi}$, and $\tan\theta = \tan\phi - \frac{gx}{v_0^2 \cos^2\phi}$ (two equations with two unknowns v_0 and ϕ); thence the first approximate values are obtained of κ_0, B and J; thence corresponding to equations (1) and (2) in which x, y, θ, B, J are known, the two unknown quantities v_0 and ϕ are to be calculated.

This calculation is to be repeated with the values of v_0 and ϕ so found.

Numerical example. Given $2R = 0{\cdot}1895$ m; $\phi = 45°$, $P = 29{\cdot}37$ kg; $\delta = 1{\cdot}208$; $i = 1$; $X = 225$ m (for $y = 0$); to find v_0, and further θ_e, v_e, T. Then

$$c = 0{\cdot}000254; \quad a = \xi(45) : \tan 45 = 1{\cdot}1478;$$

$$caX = 0{\cdot}000254 \times 1{\cdot}1478 \times 225 = 0{\cdot}065.$$

In a vacuum

$$v_0 = \sqrt{\frac{gX}{\sin 2\phi}} = \sqrt{\frac{9{\cdot}81 \times 225}{1}} = 46{\cdot}98 \text{ m/sec},$$

thence

$$\kappa_0 = \frac{av_0 \cos\phi}{r} = \frac{1{\cdot}1478 \times 46{\cdot}98 \times \cos 45}{435} = 0{\cdot}0875;$$

and the Tables of Didion give $B = 1{\cdot}024$, $J = 1{\cdot}0375$, $D = 1{\cdot}0355$, $V_1 = 1{\cdot}0555$.

Thence a closer value of v_0 is obtained from equation (1), where $x = X = 225$, and $y = 0$.

We have, thus, $2v_0^2 \cos^2 45 \tan 45 = 225 \times 9{\cdot}81 \times 1{\cdot}024$; and so $v_0 = 47{\cdot}5$.

Further, with $x = X$,

$$\tan\theta_e = \tan 45 - \frac{9{\cdot}81 \times 225 \times 1{\cdot}0375}{(47{\cdot}5)^2 \cos^2 45}; \quad \theta_e = -45° \, 50',$$

$$v_e = \frac{v_0 \cos\phi}{V_1 \cos\theta_e} = \frac{47{\cdot}5 \cos 45}{1{\cdot}0555 \times \cos\theta_e} = 45{\cdot}6;$$

$$T = \frac{XD}{v_0 \cos\phi} = \frac{225 \times 1{\cdot}0355}{47{\cdot}5 \times \cos 45} = 6{\cdot}94.$$

Therefore, $v_0 = 47{\cdot}5$ m/sec, $\theta_e = -45° \, 50'$, $v_e = 45{\cdot}6$ m/sec, $T = 6{\cdot}94$ sec.

§ 25. Bernoulli-Didion Approximation Method for the Monomial Law $cf(v) = cv^n$.

On the law of retardation due to air resistance $cf(v) = cv^n$ (which becomes the quadratic law for $n = 2$, cubic for $n = 3$, biquadratic for $n = 4$, and so on) the solution can be carried out in a manner similar to that of Didion (§ 24).

The procedure in that case was, from the approximate equation $dx = -\frac{1}{\alpha c}\frac{u\,du}{f(u)}$, in which $u = \alpha v \cos\theta$, to obtain a relation between

$v \cos\theta$ and x, and then to employ it in the equations

$$dt = \frac{dx}{v\cos\theta}, \text{ and } \frac{d\theta}{\cos^2\theta} = -\frac{g\,dx}{(v\cos\theta)^2}.$$

The corresponding formulae present no difficulty in the present case, and we may take the example of the biquadratic law, $cf(v) = cv^4$:

$$dx = -\frac{u\,du}{\alpha c u^4} = -\frac{du}{\alpha c u^3}, \quad 2\alpha c x = \frac{1}{u^2} - \frac{1}{u_0^{\,2}};$$

or since $u = \alpha v \cos\theta$, $u_0 = \alpha v_0 \cos\phi$,

$$v\cos\theta = v_0\cos\phi\,(1 + 2c\alpha^3 v_0^{\,2}\cos^2\phi\,.\,x)^{-\frac{1}{2}},$$

we have
$$dt = \frac{dx}{v\cos\theta} = \frac{1}{v_0\cos\phi}\sqrt{(1 + 2c\alpha^3 v_0^{\,2}\cos^2\phi\,.\,x)}\,dx,$$

and
$$t = \frac{1}{3c\alpha^3 v_0^{\,3}\cos^3\phi}\left[(1 + 2c\alpha^3 v_0^{\,2}\cos^2\phi\,.\,x)^{\frac{3}{2}} - 1\right].$$

Further

$$\frac{d\theta}{\cos^2\theta} = -\frac{g}{v_0^{\,2}\cos^2\phi}(1 + 2c\alpha^3 v_0^{\,2}\cos^2\phi\,.\,x)\,dx,$$

$$\tan\theta - \tan\phi = -\frac{g}{v_0^{\,2}\cos^2\phi}(x + c\alpha^3 v_0^{\,2}\cos^2\phi\,.\,x^2);$$

and since $\tan\theta = \dfrac{dy}{dx}$,

$$y = x\tan\phi - \frac{g}{v_0^{\,2}\cos^2\phi}\left(\tfrac{1}{2}x^2 + \tfrac{1}{3}c\alpha^3 v_0^{\,2}\cos^2\phi\,.\,x^3\right).$$

Statement of Formulae.

$$y = x\tan\phi - \frac{gx^2}{2v_0^{\,2}\cos^2\phi}B, \quad \text{.................(1)}$$

$$\tan\theta = \tan\phi - \frac{gx}{v_0^{\,2}\cos^2\phi}J, \quad \text{.................(2)}$$

$$v\cos\theta = v_0\cos\phi\,\frac{1}{V}, \quad \text{.................(3)}$$

$$t = \frac{x}{v_0\cos\phi}D, \quad \text{.................(4)}$$

in which the functions B, J, V, D have the following values:

Retardation due to air resistance	$cf(v)=cv^2$	cv^3	cv^4	$cf(v)=cv^n$ in general
$B=$	$\dfrac{e^z-z-1}{\frac{1}{2}z^2}$	$1+\frac{2}{3}z+\frac{1}{6}z^2$	$1+\frac{1}{3}z$	$\dfrac{(1+z)^{\frac{2n-2}{n-2}}-\frac{2n-2}{n-2}z-1}{\frac{n(n-1)}{(n-2)^2}z^2}$
$J=$	$\dfrac{e^z-1}{z}$	$1+z+\frac{1}{3}z^2$	$1+\frac{1}{2}z$	$\dfrac{(1+z)^{\frac{n}{n-2}}-1}{\frac{n}{n-2}z}$
$V=$	$e^{\frac{1}{2}z}$	$1+z$	$(1+z)^{\frac{1}{2}}$	$(1+z)^{\frac{1}{n-2}}$
$D=$	$\dfrac{e^{\frac{1}{2}z}-1}{\frac{1}{2}z}$	$1+\frac{1}{2}z$	$\dfrac{(1+z)^{\frac{3}{2}}-1}{\frac{3}{2}z}$	$\dfrac{(1+z)^{\frac{n-1}{n-2}}-1}{\frac{n-1}{n-2}z}$
with the abbreviation	$z=2cax$	$z=ca^2v_0 \times \cos\phi\, x$	$z=2ca^3v_0^2 \times \cos^2\phi\, x$	$z=ca^{n-1}(n-2) \times (v_0\cos\phi)^{n-2}x$

Various transformations of the Bernoulli-Didion Approximate Solution, specially for the Quadratic and the Cubic Laws of Air Resistance.

A. In the case of the quadratic law (retardation $cf(v)=cv^2$),

$$y = x\tan\phi - \frac{gx^2}{2v_0^2\cos^2\phi}B(z),$$

$$\tan\theta = \tan\phi - \frac{gx}{v_0^2\cos^2\phi}J(z),$$

$$v\cos\theta = \frac{v_0\cos\phi}{V(z)},\quad t = \frac{x}{v_0\cos\phi}D(z),$$

where $z=2cax$. For the point of fall ($x=X$, $y=0$), let $2caX=Z$;

and at this point, from the first equation $\frac{v_0^2 \sin 2\phi}{g}$, denoted by $\mathfrak{W}, = XB(Z)$.

Further, let the function $E(z)$ be introduced in place of $J(z)$, so that $2J = B(1+E)$; $E(z)$ is thus the function

$$\frac{2J - B}{B} = \frac{e^z(z-1)+1}{e^z - z - 1}.$$

Therefore the second equation assumes the form

$$\tan\theta = \tan\phi - \frac{gxB(1+E)}{2v_0^2\cos^2\phi} = \tan\phi\left\{1 - \frac{xB(z)[1+E(z)]}{\mathfrak{W}}\right\}.$$

At the point of descent, with $z = Z$, $\theta = \theta_e = -\omega$ (ω the acute angle of descent) we have $\tan\omega = E(Z)\tan\phi$.

Besides the function $D(z)$ another function $\Theta(z)$ may further be defined, through the equation $D = \sqrt{(B\Theta)}$, and so $\Theta = \frac{D^2}{B} = \frac{2(e^{\frac{1}{2}z} - 1)^2}{e^z - z - 1}$; and then for any point on the trajectory

$$t = \frac{x}{v_0\cos\phi}\sqrt{[B(z)\,\Theta(z)]},$$

and for the point of descent,

$$T = \frac{X}{v_0\cos\phi}D(Z) = \frac{X}{v_0\cos\phi}\sqrt{[B(Z)\,\Theta(Z)]}$$

$$= \frac{X}{v_0\cos\phi}\sqrt{\left[\frac{v_0^2\sin 2\phi}{gX}\Theta(Z)\right]} = \sqrt{\left[\frac{2X\tan\phi}{g}\Theta(Z)\right]}.$$

Finally, at the vertex of the trajectory (with $\theta = 0$, $\tan\theta = 0$; $x = x_s$, $y = y_s$, $z = z_s = 2c\alpha x_s$) the second of the above equations takes the following form:

$$0 = \tan\phi - \frac{gx_s}{v_0^2\cos^2\phi}J(z_s), \text{ or } XB(Z) = 2x_s J(z_s).$$

Another relation follows for the vertex-abscissa, because the equation for $\tan\theta$ may be written in the form

$$\tan\theta = \tan\phi - \frac{g}{v_0^2\cos^2\phi}\,\frac{e^z - 1}{2c\alpha}.$$

Here $e^z - 1 = V^2 - 1$, since $V = e^{\frac{1}{2}z}$; so that at $\theta = 0$,

$$\frac{v_0^2\sin 2\phi}{2g}2c\alpha = e^{z_s} - 1 = V^2(z_s) - 1;$$

and since $\mathfrak{W} = XB(Z)$, and $Z = 2c\alpha X$, this assumes the form

$$V^2(z_s) = 1 + \tfrac{1}{2}\mathfrak{W}\,.\,2c\alpha = 1 + \tfrac{1}{2}ZB(Z).$$

Statement of the formulae.

For any point (xy) of the trajectory:

$$y = x \tan\phi - \frac{gx^2}{2v_0^2 \cos^2\phi} B(z) = x\tan\phi \left[1 - \frac{xB(z)}{XB(Z)}\right], \ldots(1)$$

$$\tan\theta = \tan\phi - \frac{gx}{v_0^2\cos^2\phi} J(z) = \tan\phi \left\{1 - \frac{xB(z)}{XB(Z)}[1 + E(z)]\right\}$$

for the inclination of the tangent to the trajectory. ....(2)

$$v\cos\theta = \frac{v_0\cos\phi}{V(z)} \text{ for the horizontal velocity } v\cos\theta. \ldots(3)$$

$$t = \frac{x}{v_0\cos\phi} D(z) = \frac{x}{v_0\cos\phi}\sqrt{[B(z)\,\Theta(z)]} \text{ for the time of flight } t. \ (4)$$

$$z = 2c\alpha . x. \ldots(5)$$

For the point of descent:

$$2c\alpha\,\mathfrak{W} = 2c\alpha\frac{v_0^2\sin 2\phi}{g} = ZB(Z), \text{ or } \frac{v_0^2\sin 2\phi}{gX} = B(Z) \ \ldots(6)$$

with $Z = 2c\alpha X$.

$$\tan\omega = -\tan\theta_e = \tan\phi\, E(Z), \text{ or } \tan\omega = \frac{gX}{v_0^2\cos^2\phi} J(Z) - \tan\phi$$

for the acute angle of descent ω. ....(7)

$$v_e = \frac{v_0\cos\phi}{\cos\omega\, V(Z)}, \text{ for the final velocity } v_e. \ldots(8)$$

$$T = \frac{X}{v_0\cos\phi} D(Z) = \sqrt{\left[\frac{2X\tan\phi}{g}\Theta(Z)\right]} \text{ for the total time of}$$

flight T. ....(9)

For the vertex:

$$XB(Z) = 2x_s J(z_s), \text{ or } 2c\alpha\mathfrak{W} = 2z_s J(z_s) \ \ldots\ldots(10)$$

$V^2(z_s) = 1 + c\alpha\mathfrak{W} = 1 + \frac{1}{2}ZB(Z)$ for $z_s = 2c\alpha x_s$, and thence the vertex abscissa x_s. ....(11)

$$y_s = x_s\tan\phi - \frac{gx_s^2}{2v_0^2\cos^2\phi} B(z_s) \text{ for the vertex ordinate } y_s. \ldots(12)$$

$$v_s = \frac{v_0\cos\phi}{V(z_s)} \text{ for the vertex velocity } v_s. \ \ldots\ldots(13)$$

$$t_s = \frac{x_s}{v_0\cos\phi} D(z_s), \text{ for the time of flight } t_s \text{ to the vertex.} \ (14)$$

Here v_0 = initial velocity in m/sec; ϕ = angle of departure; X = range in metres for $y=0$; $\alpha = \dfrac{\xi(\phi)}{\tan\phi}$ (see Vol. IV, Table 10 for ξ),

or more exactly $\alpha = \dfrac{\xi\left(\dfrac{\theta+\phi}{2}\right)}{\tan\dfrac{\theta+\phi}{2}}$, and more exactly still

$$\alpha = \frac{\xi(\phi)-\xi(\theta)}{\tan\phi - \tan\theta}; \quad c = \frac{R^2\pi g i \delta}{1{\cdot}206\,P} \times 0{\cdot}014,$$

where $2R$ = calibre of shell in m, P = weight of shell in kg, δ = air density on the day in question in kg/m³, $i=1$ for elongated shell with ogival point of 2 calibre radius (compare also § 13); the factor 0·014 holds good for velocities below the normal velocity of sound; a factor 0·039 (instead of 0·014) holds for v between 550 and 420 m/sec; sometimes a mean value of the factor is taken; but it is safer to calculate the trajectory in several parts, and to vary the numerical factor (compare § 29 and see pp. 48, 51, 52 for the factor K): B, J, V, D, E, Θ are the functions as follows,

$B(z) = \dfrac{e^z - z - 1}{\frac{1}{2}z^2}$ (Vol. IV, Table 6 *b*), $J(z) = \dfrac{e^z - 1}{z}$ (Vol. IV, Table 6 *c*),

$V(z) = e^{\frac{1}{2}z}$ (Vol. IV, Table 6 *a*), $D(z) = \dfrac{e^{\frac{1}{2}z}-1}{\frac{1}{2}z}$ (Vol. IV, Table 6 *c*),

$E(z) = \dfrac{e^z(z-1)+1}{e^z - z - 1}$; $\quad \Theta(z) = \dfrac{2\left(e^{\frac{1}{2}z}-1\right)^2}{e^z - z - 1}$

(for corresponding Tables for $E(z)$ and $\Theta(z)$, as well as for $zB(z)$, consult Heydenreich, *Lehre vom Schuss*, Berlin 1908, Vol. II, pp. 130—131).

Procedure in the solution of individual examples: (1) Given v_0, ϕ, X, R, P, δ; to determine i, v_e, T, ω, x_s, y_s, v_s and y for any given x. Calculate $\dfrac{{v_0}^2 \sin 2\phi}{gX}$ and α, thence Z from (6), and with it $c = \dfrac{Z}{2\alpha X}$, and consequently i. Then z_s follows from (11) and with it x_s, and then y_s from (12), v_s from (13), t_s from (14). Further ω from (7), then v_e from (8), T from (9). Since i has been calculated, the value of z is given for any given x, and then y from (1), θ from (2), v from (3) and t from (4).

(2) Given c, ϕ, v_0; to determine the remaining quantities. With the first approximation $\alpha = \dfrac{\xi(\phi)}{\tan\phi}$, and for any given x, the value of z is given by (5), and then y, θ, v, t from (1), (2), (3), (4).

Next from (6) the Table for $ZB(Z)$ gives the value of Z and thence of X, as well as ω from (7). With this value of ω the value of a can be calculated more accurately, and then the remaining calculations are to be repeated.

Thus v_e follows from (8), T from (9), z_s and with it x_s from (11); and so on.

(3) Given c, v_0, X; to determine ϕ (as for instance in the calculation of the error of departure); also the remaining quantities.

The value of ϕ is calculated either from the formula corresponding to a vacuum, or else from a Range Table; and then the first approximate value of a is calculated.

Thence $Z=2caX$, and from it the second approximation to ϕ from (6) and ω from (7). Then a can be found more accurately, and the calculation repeated.

Then v_e is given from (8), and T from (9), &c.

(4) Given c, ϕ, x and $v\cos\theta$, to determine v_0.

Calculate a, and then z and $V(z)$; then v_0 follows from (3).

(5) Given R, P, δ, as well as X, ϕ, T; to determine i, v_0, v_e, ω, &c.

Since T, X, ϕ are given, $\Theta(Z)$ follows from (9) and thence Z; $c=\frac{Z}{2aX}$, and thence i is found.

Afterwards v_0 is determined, for instance, from $v_0=\frac{XD}{T\cos\phi}$, ω from (7), v_e from (8), &c.

The above solution applies especially to cases, where the initial velocity v_0 of the shell is less than 300 m/sec ; on the degree of the accuracy, consult § 33.

Besides the original functions B, J, V, D, and the functions E, Θ and $zB(z)$, it is evident that others may be introduced into the system of equations.

N. v. Wuich has prepared Tables for practical use. The method introduced by Heydenreich in his *Lehre vom Schuss*, Berlin, 1908, II. p. 122 is identical with the above.

B. Siacci's "Factors of Fire" are given in Vol. IV, Table 11, and are to be used for similar purposes.

This table contains for the different values of Z the values of

$$\frac{v_0^2\sin 2\phi}{gX},\quad \frac{\tan\omega}{\tan\phi},\quad \frac{T}{\sqrt{(X\tan\phi)}},\quad \frac{v_0\cos\phi}{v_e\cos\omega},\quad \frac{x_s}{X},\quad \frac{y_s}{X\tan\phi},$$

$$\frac{\delta i \alpha X\,(2R)^2\,1000}{1{\cdot}206P},\ \text{and}\ \frac{\delta i \alpha\,1000\,(2R)^2}{1{\cdot}206P}\,\frac{v_0^2\sin 2\phi}{g};$$

these expressions are denoted respectively by $f, f_1, f_2, \ldots, f_7$; they are called the "Factors of Fire."

The tables are to be used as follows:

Given P, R, δ, i, as well as X and ϕ.

Proceed from the given f_6; look out on the horizontal line, i.e., for the given value of Z, the corresponding numerical values of $f, f_1, f_2,$

f_3, f_4, f_5. Then v_0 is given by f, ω by f_1, T by f_2, v_e by f_3, x_s by f_4, y_s by f_5.

Given v_0, X, ϕ.

Start from f, and look out the corresponding $f_1, f_2, \ldots$. Then f_1 gives ω, f_2 gives T, and so on.

Explanation of the Tables.

According to the above,

$$\frac{v_0^2 \sin 2\phi}{gX} = B(Z), \text{ denoted by } f,$$

$$\frac{\tan\omega}{\tan\phi} = E(Z) = \frac{2J(Z)}{B(Z)} - 1, \text{ denoted by } f_1,$$

$$\frac{T}{\sqrt{(X\tan\phi)}} = \sqrt{\frac{2}{g}} \cdot \frac{D(Z)}{\sqrt{[B(Z)]}} = \sqrt{\left[\frac{2}{g}\,\Theta(Z)\right]}, \text{ denoted by } f_2,$$

$$\frac{v_e\cos\omega}{v_0\cos\phi} = \frac{1}{V(Z)}, \text{ denoted by } \frac{1}{f_3},$$

$$V^2(z_s) = 1 + \tfrac{1}{2}ZB(Z);$$

to every given Z a definite z_s corresponds, so that there is a definite $\frac{z_s}{Z}$, or, a definite $\frac{x_s}{X}$; and this is f_4. Further,

$$y = x\tan\phi\left[1 - \frac{xB(z)}{XB(Z)}\right]$$

and at the vertex $y = y_s$ and $x = x_s$; and so, after division by $\tan\phi$ and X,

$$\frac{y_s}{X\tan\phi} = \frac{x_s}{X}\left[1 - \frac{x_s}{X}\frac{B(z_s)}{B(Z)}\right].$$

But here z_s is a definite function of Z, and so $B(z_s)$ is given as well as Z; and $\frac{x_s}{X} = f_4$ is a given function of Z, and $\frac{y_s}{X\tan\phi}$ is also known; and $Z = 2caX$, so that f_6 is given by Z, and in consequence of f, f_7 is given also in terms of Z.

The "Factors of Fire," $f, f_1, \ldots$, and their calculation in terms of Z, where $Z = 2caX$, may be stated as follows:

$$f = \frac{v_0^2\sin 2\phi}{gX} = B(Z),$$

$$f_1 = \frac{\tan\omega}{\tan\phi} = E(Z) = \frac{2J(Z)}{B(Z)} - 1,$$

$$f_2 = \frac{T}{\sqrt{(X\tan\phi)}} = \sqrt{\frac{2}{g}} \cdot \frac{D(Z)}{\sqrt{[B(Z)]}} = \sqrt{\left[\frac{2}{g}\,\Theta(Z)\right]},$$

$$f_3 = \frac{v_0\cos\phi}{v_e\cos\omega} = V(Z),$$

$$f_4 = \frac{x_s}{X} = \frac{z_s}{Z},$$

$$f_5 = \frac{y_s}{X\tan\phi} = \frac{x_s}{X}\left[1 - \frac{x_s}{X}\cdot\frac{B(z_s)}{B(Z)}\right],$$

$$f_6 = \frac{\delta i a X (2R)^2 1000}{1\cdot 206 P},$$

$$f_7 = \frac{\delta i a 1000 (2R)^2}{1\cdot 206 P}\,\frac{v_0^2 \sin 2\phi}{g}.$$

In this way for every value of Z the corresponding value can be calculated of $f, f_1, f_2, \ldots$. This calculation is extended in Table 11. It is seen that f is merely the former function $B(z)$ of Table 6*b*, f_3 the function $\frac{1}{V(z)}$ of Table 6*a*, and that with f_1 and f_2, the functions $E(z)$ and $\Theta(z)$ are given in a tabular form.

C. For the cubic law of air resistance the corresponding table of Factors of Fire has been constructed by F. Chapel.

This table is equivalent to the system of formulae constructed by Fr. v. Zedlitz in 1896, independently of Chapel, based on the cubic law.

When the retardation due to air resistance is taken as cv^3,

$$y = x\tan\phi - \frac{gx^2}{2v_0^2\cos^2\phi}\left(1 + \tfrac{2}{3}z + \tfrac{1}{6}z^2\right)$$

where $$z = c\alpha^2 x v_0 \cos\phi,$$

$$\tan\theta = \tan\phi - \frac{gx}{v_0^2\cos^2\phi}\left(1 + z + \tfrac{1}{3}z^2\right),$$

$$v\cos\theta = \frac{v_0\cos\phi}{1+z},$$

$$t = \frac{x}{v_0\cos\phi}\left(1 + \tfrac{1}{2}z\right).$$

Substituting $1 + \frac{1}{2}z = q$, or $z = 2(q-1)$

we have $$y = x\tan\phi - \frac{gx^2}{2v_0^2\cos^2\phi}\,\frac{1+2q^2}{3},$$

$$\tan\theta = \tan\phi - \frac{gx}{v_0^2\cos^2\phi}\,\frac{1-2q+4q^2}{3},$$

$$v\cos\theta = \frac{v_0\cos\phi}{2q-1}, \quad q = \frac{tv_0\cos\phi}{x}.$$

Here q is a parameter, which varies along the trajectory; at the end, where $y=0$, $x=X$, $v=v_e$, $\theta=-\omega$, $t=T$, the value of q may be denoted by q_e; and then

$$\sin 2\phi = \frac{gX}{v_0^2}\,\frac{1+2q_e^2}{3} \quad \text{.............................(I)}$$

$$\tan\omega = \frac{gX}{2v_0^2\cos^2\phi}\,\frac{1-4q_e+6q_e^2}{3} \quad \text{...............(II)}$$

$$v_e = \frac{v_0\cos\phi}{\cos\omega}\,\frac{1}{2q_e-1} \quad \text{.........................(III)}$$

$$T = \frac{Xq_e}{v_0\cos\phi} \quad \text{..................................(IV)}$$

The construction of a Range Table will proceed then in the following manner: For various ranges X the angle of departure will be found by experiment, and the initial velocity v_0 will be known as well. The corrections for wind, air density and so forth will be carried out; and then from the three values of v_0, ϕ, X and by the help of (I) a definite q_e will be determined. The collection of q_e-values is then shown graphically as a function of X.

On this curve the values of q_e will be shown corresponding to values of X, say, to $X = 100, 200, 300, \ldots$ metres.

For every value of q_e the angle of departure ϕ will be calculated by (I), the acute angle of descent ω by (II), the final velocity v_e from (III) and the time of flight T from (IV).

The following remarks on this system of solutions were made by Fr. v. Zedlitz, to prove that, in spite of a difference of form, it is identical with Chapel's Table of "Factors of Fire": therefore it is not included among the Tables of Vol. IV.

The system of equations (I) to (IV) shows that the "Factors of Fire"

$$\frac{v_0^2\sin 2\phi}{X},\quad \frac{\tan\omega}{\tan\phi},\quad \frac{v_e\cos\omega}{v_0\cos\phi},\quad \frac{Tv_0\cos\phi}{X},$$

are definite functions of q_e or $1+\frac{1}{2}Z$, and so of $Z \equiv c\alpha^2 X v_0 \cos\phi$; they can be calculated in consequence as functions of Z, and given in a tabular form.

Suppose $\frac{Tv_0\cos\phi}{X}$ to be divided by $\frac{v_0^2\sin 2\phi}{X}$, it will be seen that $\frac{T}{v_0\sin\phi}$ is a given function of Z. So also is the expression $\frac{y_1}{X\tan\phi}$, where y_1 is the ordinate of the trajectory for the abscissa $x = \frac{1}{2}X$.

We know that

$$y = x \tan \phi - \frac{gx^2}{2v_0^2 \cos^2 \phi} \frac{1 + 2q^2}{3},$$

and
$$y_1 = \tfrac{1}{2} X \tan \phi - \frac{gX^2}{8v_0^2 \cos^2 \phi} \frac{1 + 2q_1^2}{3},$$

from which we have

$$q_1 = 1 + \tfrac{1}{2} z_1 = 1 + \tfrac{1}{2} c\alpha^2 \frac{X}{2} v_0 \cos \phi.$$

But $q_e = 1 + \frac{1}{2} c\alpha^2 X v_0 \cos \phi$, so that $q_1 = 1 + \frac{1}{2}(q_e - 1)$: and so q_1 is also a function of q_e. Now

$$y_1 = \tfrac{1}{2} X \tan \phi \left[1 - \frac{gX}{2v_0^2 \sin 2\phi} \frac{1 + 2q_1^2}{3} \right],$$

and since q_1 and $\dfrac{v_0^2 \sin 2\phi}{X}$ are given functions of q_e, and so of Z, the quantity in the bracket is a function of Z, and $\dfrac{y_1}{X \tan \phi}$ also.

These "Factors of Fire"

$$\frac{v_0^2 \sin 2\phi}{X}, \quad \frac{\tan \omega}{\tan \phi}, \quad \frac{v_e \cos \omega}{v_0 \cos \phi}, \quad \frac{T}{v_0 \sin \phi}, \quad \frac{y_1}{X \tan \phi},$$

are those given by Chapel in his Table; and we see now the intimate connexion with the system of equations introduced by Fr. v. Zedlitz. Ronca has shown this in another way.

§ 26. Approximate solution of F. Siacci, 1880 ("Siacci I").

As opposed to Didion's procedure, Siacci introduced the following modifications, which depend less on methods of mathematical integration.

In the first place, following N. Mayevski (1872), and the proposal of St Robert (1872), he chose as independent variable in his method of solution the horizontal velocity multiplied by Didion's correction factor α, ($u = \alpha v \cos \theta$), instead of the abscissa of the trajectory; or, in other words, he chose $\sigma = \gamma = \dfrac{1}{\alpha}$ in the general system of solutions of § 23.

Secondly, in the place of Didion's law of air resistance, Mayevski's Zones are employed.

The retardation due to the air resistance, that is the air resistance divided by the mass of the shell, being $cf(v)$, the constant

$$c=\frac{(2R)^2\delta i\times 1000}{P\times 1{\cdot}206}$$

($2R$ the calibre in m; δ the air density in kg/m³; P the weight of the shell in kg; i the form coefficient, where $i=1$ for ogival shell, rounded to a radius of 2 calibres, and $n=1000\,i$ is the so-called "form value"): then for

$$\begin{aligned}
700>v>419\ (\text{m/sec}),\quad f(v)&=\frac{0{\cdot}039\,\pi g}{4000}v^2,\\
419>v>375\qquad ,,\qquad ,,\quad &=\frac{0{\cdot}000094\,\pi g}{4000}v^3,\\
375>v>295\qquad ,,\qquad ,,\quad &=\frac{0{\cdot}\overset{(9)}{0}671\,\pi g}{4000}v^5,\\
295>v>240\qquad ,,\qquad ,,\quad &=\frac{0{\cdot}\overset{(4)}{0}583\,\pi g}{4000}v^3,\\
v\gtreqless 240\qquad ,,\qquad ,,\quad &=\frac{0{\cdot}014\,\pi g}{4000}v^2.
\end{aligned}$$

Thence we have:

$$x=\frac{1}{\alpha c}(D_u-D_{u_0}),\ \ldots\ldots(1)$$

$$\tan\theta=\tan\phi-\frac{\alpha}{2c}(J_u-J_{u_0}),\ \ldots\ldots(2)$$

$$\begin{aligned}
y&=x\tan\phi-\frac{x\alpha}{2c}\left(\frac{A_u-A_{u_0}}{D_u-D_{u_0}}-J_{u_0}\right)\\
&=x\tan\phi-\frac{1}{2c^2}[A_u-A_{u_0}-J_{u_0}(D_u-D_{u_0})],\ \ldots(3)
\end{aligned}$$

$$t=\frac{1}{c}(T_u-T_{u_0}),\ \ldots\ldots(4)$$

$$u=\alpha v\cos\theta,\quad u_0=\alpha v_0\cos\phi.\ \ldots\ldots(5)$$

ϕ the angle of departure, v_0 the initial velocity; (x, y) coordinates of the end of an arc of the trajectory, θ the slope of the tangent, v the velocity of the shell at this point, t the time of flight to reach this point;

$$D_u=-\int\frac{u\,du}{f(u)},\ J_u=-2g\int\frac{du}{uf(u)},\ T_u=-\int\frac{du}{f(u)},\ A_u=-\int\frac{J_u\,u\,du}{f(u)},$$

and the values of D, J, T, A are calculated in Tables by Siacci, Braccialini, Klussmann, and others.

As to the calculation of the integrals D, J, T, A, it is seen in the first place that in the expressions for x, θ, t, these functions occur only in their difference values, calculated from some initial condition, as $D_u - D_{u_0}$, $J_u - J_{u_0}$, and so on, so that arbitrary numerical constants may be introduced.

The same is true for A, since y is obtained from $\tan\theta$ by integration with respect to u; in the passage between the zones it is necessary to preserve continuity.

The calculation is carried out in the following manner:

(*a*) First zone: v between 700 and 419 m/sec: $f(v) = qv^2$,

where $$q = \frac{0\cdot0394\,\pi g}{4000}.$$

$$D(u) = -\int \frac{u\,du}{f(u)} = -\frac{1}{q}\int\frac{u\,du}{u^2} = -\frac{1}{q}\log u + \text{an arbitrary constant } Q,$$

$$J(u) = -2g\int\frac{du}{uf(u)} = -\frac{2g}{q}\int\frac{du}{u^3} = +\frac{g}{qu^2} + \quad ,, \quad ,, \quad Q_1,$$

$$T(u) = -\int\frac{du}{f(u)} = -\frac{1}{q}\int\frac{du}{u^2} = \frac{1}{qu} \quad + \quad ,, \quad ,, \quad Q_2,$$

$$A(u) = -\int\frac{J(u)\,u\,du}{f(u)} = -\int\left(\frac{g}{qu^2} + Q_1\right)\frac{u\,du}{qu^2} = -\frac{g}{q^2}\int\frac{du}{u^3} - \frac{Q_1}{q}\int\frac{du}{u}$$

$$= \frac{g}{2q^2u^2} - \frac{Q_1}{q}\log u + \text{an arbitrary constant } Q_3.$$

The constants Q, Q_1, Q_2, Q_3 are arbitrary. Siacci takes $Q_1 = 0$, and Q, Q_2, Q_3 such that the tables for $D(u)$, $T(u)$, $A(u)$ begin at zero for $u = 700$.

(*b*) Second zone: v between 419 and 375 m/sec: $f(v) = pv^3$,

$$p = \frac{0\cdot00009404\,\pi g}{4000}.$$

$$D(u) = -\int\frac{u\,du}{f(u)} = -\frac{1}{p}\int\frac{u\,du}{u^3} = +\frac{1}{pu} \quad + \text{an arbitrary constant } C,$$

$$J(u) = -2g\int\frac{du}{uf(u)} = -\frac{2g}{p}\int\frac{du}{u^4} = +\frac{2g}{3pu^3} + \quad ,, \quad ,, \quad C_1;$$

and so on.

The constants C, C_1, C_2, C_3 are to be determined, so as to make the initial value of a function at the beginning of the second zone the same as the final value at the end of the first zone. Thus, for instance, C is such that, when $u = 419$, $-\frac{1}{q}\log u + Q = \frac{1}{pu} + C$: and so forth.

The solution of the separate trajectories by means of the system of equations (1) to (5), is as follows: Suppose, for example, v_0, ϕ, c, given, and for given x the values of y, v, t, θ are to be determined; then from (1), in which x, v_0, ϕ, c, as well as u_0 and α are known, $D(u)$ and u are calculated: the other Tables determine the corresponding values of $J(u)$, $A(u)$, $T(u)$: then θ is obtained from (2): and from it the value of α is corrected and the calculation of θ repeated; then y is found from (3), t from (4), and v from (5).

On the other hand the calculation of the elements X, v_e, ω, T of the point of descent from the same data v_0, ϕ, c, requires more work. From (3), at the point of descent, $(y = 0, x = X)$, we have

$$0 = \tan\phi - \frac{\alpha}{2c}\left(\frac{A_{u_e} - A_{u_0}}{D_{u_e} - D_{u_0}} - J_{u_0}\right);$$

and here, by successive trials, $u_e = \alpha v_e \cos\omega$ is determined: thence it follows from (1) $X = \dfrac{1}{\alpha c}(D_{u_e} - D_{u_0})$: and then $\tan\omega$ from (2). Then a closer value of α is determined, and the whole calculation is repeated.

The problem of greatest importance in the range table calculation, (viz., from given v_0, ϕ, X, P, R and δ, but with i unknown, and also c unknown, to calculate the elements v_e, ω, T of the point of descent), is not easy to carry out, because successive approximation by trial and error is required.

On this account Siacci proposed to construct other functions and tables, in addition to the primary functions D, J, T, A, with a view to lessening the labour in the solution of these problems.

Secondary functions E, N, Q, O, S, S' and the corresponding tables.

When equation (1), $D(u) = \alpha cx + D(u_0)$ is solved for the determination of u, we see that u is a function of αcx and u_0; and also $J_u - J_{u_0}$, $A_u - A_{u_0}$, $T_u - T_{u_0}$, and $\dfrac{A_u - A_{u_0}}{D_u - D_{u_0}} - J_{u_0}$, are given functions of αcx and u_0.

The following abbreviations are introduced in the use of these functions:

$$T_u - T_{u_0} = S;\quad J_u - J_{u_0} = Q;\quad \frac{A_u - A_{u_0}}{D_u - D_{u_0}} - J_{u_0} = E.$$

Another function, dependent on αcx and u_0, is further $\frac{E}{\alpha cx} = N$; also $Q - E = O$, and finally $\frac{S}{\alpha cx} = S'$.

These functions, E, N, Q, S, S', O, are called secondary functions; the corresponding tables are easily constructed by the help of the original values of D, J, A and T.

Therefore

$$D(u) = \alpha cx + D(u_0),$$

$$\tan\theta = \tan\phi - \frac{\alpha}{2c} Q(c\alpha x, u_0),$$

$$y = x\tan\phi - \frac{\alpha x}{2c} E(c\alpha x, u_0),$$

$$t = \frac{1}{c} S(c\alpha x, u_0),$$

$$u = \alpha v\cos\theta, \quad u_0 = \alpha v_0 \cos\phi.$$

At the vertex, $u = u_s = \alpha v_s$, $x = x_s$, $y = y_s$, $\theta = 0$, and so

$$D(u_s) = \alpha c x_s + D(u_0), \quad \tan\phi = \frac{\alpha}{2c} Q(c\alpha x_s, u_0),$$

$$y_s = x_s \tan\phi - \frac{\alpha x_s}{2c} E(c\alpha x_s, u_0), \quad t_s = \frac{1}{c} S(c\alpha x_s, u_0).$$

At the point of descent, $y = 0$, $x = X$, $u = u_e = \alpha v_e \cos\omega$, $\theta = -\omega$; and so

$$D(u_e) = \alpha cX + D(u_0),$$

$$\tan\omega = \frac{\alpha}{2c} Q(c\alpha X, u_0) - \tan\phi,$$

$$\tan\phi = \frac{\alpha}{2c} E(c\alpha X, u_0), \quad T = \frac{1}{c} S(c\alpha X, u_0);$$

and also

$$\tan\omega = \frac{\alpha}{2c} O(c\alpha X, u_0), \quad \tan\phi = X\frac{\alpha^2}{2} N(c\alpha X, u_0),$$

$$T = \alpha X S'(c\alpha X, u_0).$$

For instance, if the range X, the angle of departure ϕ, and the initial velocity v_0 are given, and c is to be determined (or the form coefficient i with given R, P, δ), the most convenient equation to employ would be $\tan\phi = \frac{1}{2}\alpha^2 XN$.

Here X, ϕ, v_0 are known, and α and u_0; therefore N can be calculated, and from the N table in the column of given u_0 the value of caX can be found, and so c is known. Then ω is found from $\tan\omega = \frac{\alpha}{2c} O$.

Taking this value of ω, a closer value of α can be determined, and the calculation repeated with greater accuracy. This example will show the advantages of the secondary tables.

§ 27. The approximate solution of Krupp-Gross (employed earlier by F. Krupp).

In this procedure the correction factor α of Didion and Siacci I is put = 1, and so $u = \alpha v \cos\theta = v\cos\theta$, $u_0 = v_0 \cos\phi$: and u represents simply the horizontal component of the velocity of the shell.

In other words this approximation in the integration of the Chief Equation consists in replacing $cf(v)\cos\theta$ by $cf(v\cos\theta) = cf(u)$: it is assumed then that the horizontal component of the air resistance, $mcf(v)\cos\theta$, for any velocity, is the same as the air resistance $mcf(v\cos\theta)$ for the horizontal component of the corresponding velocity of the shell.

Moreover no analytical function is assumed for the retardation $cf(v)$ of the air resistance, but the value is taken from Krupp's experimental tables, which are based on experiments.

These tables contain the following:

Put $\frac{R^2\pi\delta i}{P \,.\, 1{\cdot}206} = a$ ($2R$ the calibre of the shell in cm; P the weight of the shell in kg; δ the air density in kg/m^3: i the coefficient of shape of point, $i = 1$ for ogival shell of the early Krupp normal form, viz. with a rounding of 2 calibre radius for the ogival point).

Then $cf(v) = agf(v)$ is the retardation due to the air resistance, and so $\frac{P}{g} cf(v) = Paf(v)$ is the actual air resistance to the shell, with its long axis in the tangent of the path: $cf(u)$ or $cf(v_x)$ is taken as the horizontal component of the retardation, or according to Krupp, it is the retardation due to the horizontal component of the shell.

For every velocity u or $v\cos\theta$, from $u = 1000$ to $u = 50$ m/sec, the values are given of:

1. $f(u)$:

2. $\frac{1}{g}\frac{u}{f(u)}$, denoted by Δx,

3. $\sum_{1000}^{u} \frac{1}{g} \frac{u}{f(u)}$, denoted by $D'(u)$,

4. $\frac{1}{g} \frac{1}{f(u)}$, denoted by Δt,

5. $\sum_{1000}^{u} \frac{1}{g} \frac{1}{f(u)}$, denoted by $T'(u)$,

6. $\frac{1000}{u\,f(u)}$,

7. $\sum_{1000}^{u} \frac{1}{u\,f(u)}$, denoted by $J'(u)$,

8. $J'(u)\,\Delta x$, or $J'(u)\frac{1}{g}\frac{u}{f(u)}$,

9. $\sum_{1000}^{u} J'(u)\,\Delta x = \sum J'(u)\frac{1}{g}\frac{u}{f(u)}$, denoted by $A'(u)$.

The system Siacci I consisted of the following equations, in which the factor c occurs, where $c = \frac{(2R)^2\,\delta i\,.\,1000}{P.1{\cdot}206}$, and $cf(v)$ is the retardation:

$$dx = -\frac{1}{\alpha c}\frac{u\,du}{f(u)}, \quad x = \frac{1}{\alpha c}\int_{u_0}^{u}\frac{-u\,du}{f(u)} = \frac{1}{\alpha c}(D_u - D_{u_0}),$$

where
$$D_u = \int\frac{-u\,du}{f(u)}, \quad u = \alpha v\cos\theta;$$

$$\tan\theta = \frac{dy}{dx} = \tan\phi - \frac{\alpha}{2c}\int_{u_0}^{u}\frac{-2g\,du}{uf(u)} = \tan\phi - \frac{\alpha}{2c}(J_u - J_{u_0}),$$

where
$$J_u = \int\frac{-2g\,du}{u\,f(u)};$$

$$dy = dx\tan\phi - \frac{\alpha}{2c}\left[J_u\frac{-u\,du}{\alpha c\,f(u)} - J_{u_0}\,dx\right];$$

$$y = x\tan\phi - \frac{1}{2c^2}\left[\int_{u_0}^{u} J(u)\frac{-u\,du}{f(u)} - J(u_0)\int_{u_0}^{u}\frac{-u\,du}{f(u)}\right]$$

$$= x\tan\phi - \frac{1}{2c^2}[A_u - A_{u_0} - J_{u_0}(D_u - D_{u_0})],$$

where
$$A_u = \int J(u) \frac{-u\,du}{f(u)};$$

$$t = \frac{1}{c}\int_{u_0}^{u} \frac{-du}{f(u)} = \frac{1}{c}(T_u - T_{u_0}), \text{ where } T_u = \int \frac{-du}{f(u)}.$$

When we put $\alpha = 1$, and introduce the notation of Krupp, where

$$c = \frac{R^2\pi \,(\text{in cm}^2)\, \delta i g}{P\,(\text{in kg})\, 1{\cdot}206} = ag,$$

$$D'(u) = \int \frac{-u\,du}{g f(u)}, \quad T'(u) = \int \frac{-du}{g f(u)}, \quad J'(u) = \int \frac{-du}{u f(u)},$$

$$A'(u) = \int \frac{-J'(u) u\,du}{g f(u)},$$

then we have

$$dx = -\frac{1}{c}\frac{u\,du}{f(u)}, \text{ where } u = v\cos\theta, \text{ or } dx = -\frac{1}{a}\frac{u\,du}{g f(u)},$$

$$x = \frac{1}{a}\int_{u_0}^{u} \frac{-u\,du}{g f(u)} = \frac{1}{a}(D'_u - D'_{u_0}),$$

$$\frac{dy}{dx} = \tan\theta = \tan\phi - \frac{1}{a}\int_{u_0}^{u} \frac{-du}{u f(u)} = \tan\phi - \frac{1}{a}(J'_u - J'_{u_0}),$$

$$y = x\tan\phi - \frac{1}{a^2}\int_{u_0}^{u} \frac{-J'(u) u\,du}{g f(u)} + \frac{x}{a} J'(u_0)$$

$$= x\left[\tan\phi + \frac{J'(u_0)}{a}\right] - \frac{1}{a^2}(A'_u - A'_{u_0}),$$

$$t = \frac{1}{a}\int_{u_0}^{u} \frac{-du}{g f(u)} = \frac{1}{a}(T'_u - T'_{u_0}),$$

$$u = v\cos\theta, \; u_0 = v_0\cos\phi.$$

The integrals arising here, $D'_u - D'_{u_0}$, $J'_u - J'_{u_0}$, $A'_u - A'_{u_0}$, $T'_u - T'_{u_0}$, were calculated approximately by Krupp, by summation of the corresponding values from $u = 1000$ m/sec, downwards; and he assumes, for instance, $\Delta u = -1$ m/sec, and calculates

$$D_u - D_{u_0} \text{ or } \int_{u_0}^{u} \frac{-u\,du}{g f(u)} = \sum_{u_0}^{u} \frac{+1\,.\,u}{g f(u)} = \sum_{1000}^{u} \frac{u}{g f(u)} - \sum_{1000}^{u_0} \frac{u}{g f(u)}.$$

Thus $\dfrac{u}{g f(u)}$ is calculated for every value of the horizontal velocity u, from $u = 1000$ downwards, and each value added to the sum of the preceding values, starting from the beginning of the Table; the successive sums are the values of D'_u.

The corresponding values of T'_u, J'_u, A'_u, are obtained by a finite summation of the small finite differences. This procedure recalls in many respects the methods of calculation long employed in the insurance offices, where a similar method for the establishment of death probability by an integration of finite summations from one year of life to another, is employed.

Formulae.

For any given point (xy):

$$x = \frac{1}{a}(D'_u - D'_{u_0}), \quad u = v\cos\theta, \quad u_0 = v_0 \cos\phi, \quad \text{.........(1)}$$

$$\tan\theta = \tan\phi + \frac{1}{a}J'_{u_0} - \frac{1}{a}J'_u, \quad \text{.........(2)}$$

$$y = x\left(\tan\phi + \frac{1}{a}J'_{u_0}\right) - \frac{1}{a^2}(A'_u - A'_{u_0}), \quad \text{.........(3)}$$

$$t = \frac{1}{a}(T'_u - T'_{u_0}) \quad \text{.........(4)}$$

For the point of descent, $x = X$, $u = u_e = v_e \cos\omega$, $t = T$, $\theta = -\omega$, $y = 0$:

$$X = \frac{1}{a}(D'_{u_e} - D'_{u_0}), \quad \text{.........(5)}$$

$$\tan\omega = -\left(\tan\phi + \frac{1}{a}J'_{u_0}\right) + \frac{1}{a}J'_{u_e}, \quad \text{.........(6)}$$

$$X\left(\tan\phi + \frac{J'_{u_0}}{a}\right) = \frac{1}{a^2}(A'_{u_e} - A'_{u_0}), \quad \text{.........(7)}$$

$$\tan\phi + \frac{1}{a}J'_{u_0} = \frac{1}{a}\frac{A'_{u_e} - A'_{u_0}}{D'_{u_e} - D'_{u_0}}, \quad \text{.........(7a)}$$

$$T = \frac{1}{a}(T'_{u_e} - T'_{u_0}), \quad \text{.........(8)}$$

$$v_e = \frac{u_e}{\cos\omega} \quad \text{.........(9)}$$

At the vertex, $x = x_s$, $y = y_s$, $\theta = 0$, $u = u_s = v_s \cos 0 = v_s$, $t = t_s$:

$$J'_{u_s} = a\tan\phi + J'_{u_0}, \quad \text{.........(10)}$$

$$x_s = \frac{1}{a}(D'_{u_s} - D'_{u_0}), \quad \text{.........(11)}$$

$$y_s = \frac{1}{a} x_s J'_{u_s} - \frac{1}{a^2}(A'_{u_s} - A'_{u_0}), \quad \ldots\ldots\ldots\ldots(12)$$

or

$$y_s = \frac{1}{a^2}[J'_{u_s}(D'_{u_s} - D'_{u_0}) - (A'_{u_s} - A'_{u_0})], \quad \ldots(12a)$$

$$t_s = \frac{1}{a}(T'_{u_s} - T'_{u_0}). \quad \ldots\ldots\ldots\ldots (13)$$

The notation is: (x, y) the coordinates of any point where the velocity is v m/sec, slope of tangent θ, time of flight t sec, (x_s, y_s) the coordinates of the vertex, where the velocity is v_s, time of flight t_s; $(X, 0)$ the coordinates of the point of descent, at an acute angle of descent ω, total time of flight T, final velocity v_e. Further v_0 is the initial velocity, ϕ the angle of departure, $a = \frac{R^2\pi\delta i}{P.1{\cdot}206}$, $R^2\pi$ the cross section of the shell in cm²; P the weight of the shell in kg; δ the air density in kg/m³; D'_u, J'_u, T'_u, A'_u the corresponding tabular values (Krupp Tables, Vol. IV, No. 8); air resistance in kg $= \frac{P}{g} c f(v) = Pa f(v)$, and $f(v)$ given in the Table from $u = 1000$ to $u = 50$ m/sec;

$$D'_u = \int_{1000}^{u} \frac{-u\,du}{g f(u)}; \quad T'_u = \int_{1000}^{u} \frac{-du}{g f(u)}; \quad J'_u = \int_{1000}^{u} \frac{-du}{u f(u)};$$

$$A'_u = \int_{1000}^{u} \frac{-J'_u u\,du}{g f(u)}.$$

1. *Numerical example.* Given $2R = 24$ cm, $P = 215$ kg, $i = 0{\cdot}89$, $\frac{\delta}{1{\cdot}206} = 1{\cdot}005$, $v_0 = 609$ m/sec, $\phi = 20°$, $X = 12810$ m:

(*a*) To determine the final horizontal velocity $u_e = v_e \cos\omega$. We find

$$\frac{1}{a} = 0{\cdot}5313, \quad u_0 = v_0 \cos\phi = 572{\cdot}3, \quad aX = 24109;$$

with $u_0 = 572{\cdot}3$, the Table gives $D'_{u_0} = 15956$; and so, from (5),

$$D'_{u_e} = 24109 + 15956 = 40065,$$

and then $u_e = v_e \cos\omega = 280{\cdot}8$.

(*b*) To determine the time of flight T; from the Table

$$T'_{u_0} = 21{\cdot}17, \quad T'_{u_e} = 86{\cdot}14,$$

and so, from (8), $\quad T = 0{\cdot}5313\,(86{\cdot}14 - 21{\cdot}17) = 34{\cdot}52$ sec.

(According to Krupp's measurement, $T = 34{\cdot}23$ sec.)

(*c*) To determine ω the angle of descent:

$$J_{u_0} = 0{\cdot}2829, \quad J_{u_e} = 2{\cdot}0792, \quad \tan\phi = 0{\cdot}3640,$$

and so, from (6), $\quad \tan\omega = 0{\cdot}5904, \quad \omega = 30°\,34'.$

(d) To determine v_s, x_s, t_s, y_s, at the vertex:

From (10), $$J'_{u_s}=\frac{0\cdot3640}{0\cdot5313}+0\cdot2829=0\cdot9680,$$

$$u_s=v_s=351\cdot6 \text{ m/sec.}$$

From (11), $$x_s=7058 \text{ m},$$

from (13), $$t_s=16\cdot02 \text{ sec};$$

and further, $$A'_{u_s}=9387,\quad A'_{u_0}=1838,$$

and so from (12), $$y_s=1498 \text{ m.}$$

2. *Example.* Application to the calculation of the muzzle velocity v_0, when the velocity of the shell has been measured near the muzzle, by screens and the Boulengé chronograph.

Suppose, for example, the velocity has been measured at 50 m from the muzzle and found to be $v_{50}=544$ m/sec.

That is, the horizontal projection of the velocity, measured between two parallel vertical screens, is

$$v\cos\theta=v_{50}=544.$$

Further, take

$$\delta=1\cdot206 \text{ kg/m}^3,\quad i=1,\quad R^2\pi : P=4\cdot76 \text{ cm}^2\text{/kg}.$$

Then from equation (1),

$$D'(v_0\cos\phi)=D'(v_{50})-50a,$$

where $$a=\frac{R^2\pi\,(\text{cm}^2)\,\delta i}{P\,(\text{kg})\,1\cdot206}=4\cdot76,$$

$$50a=238,\quad v_{50}=544,\quad D'(v_{50})=17277,$$

$$D'(v_0\cos\phi)=17277-238=17039,$$

$$v_0\cos\phi=549.$$

Assume an angle $\phi=8°$ in the experiment, then

$$v_0=549\sec 8°=554 \text{ m/sec.}$$

For such problems, which frequently occur in practice, the Krupp tables are of great use.

Gross has given a table, for different values of u_e and u_0, for the expression $\dfrac{A'_{u_e}-A'_{u_0}}{D'_{u_e}-D'_{u_0}}=E$, from which, for given u_0 and E, the value of u_e can be calculated by (7a).

Finally W. Olsson has calculated a convenient table, in which the values of $v_e\cos\omega$, T, X, ω, for $a=1$, are given for different values of ϕ and $v_0\cos\phi$.

§ 28. Methods of solution of Siacci 1888 (Siacci II) and 1896 (Siacci III).

In respect to the methods of integration, we here have (see § 23), $\sigma=\cos\phi$, $\gamma=\beta\cos^2\phi$, where ϕ is the angle of departure and β is a certain correction factor to be described more closely later, but determined like Didion's factor α, to smooth out the errors arising in the integration.

The system of solution is the following:

$$x=\frac{1}{c\beta}(D_u-D_{u_0}),$$

$$\tan\theta=\tan\phi-\frac{1}{2c\beta\cos^2\phi}(J_u-J_{u_0}),$$

$$y=x\tan\phi-\frac{1}{2c^2\beta^2\cos^2\phi}[A_u-A_{u_0}-J_{u_0}(D_u-D_{u_0})],$$

$$t=\frac{1}{c\beta\cos\phi}(T_u-T_{u_0}),$$

$u=\dfrac{v\cos\theta}{\cos\phi}$, $u_0=v_0$; retardation $cf(v)$; β a tabular value.

For the determination of the air resistance as depending on the velocity, Siacci 1888 chose the zone laws:

retardation	$=c_1v^2$,	for $v=700$ to 420 m/sec	
„	$=c_2v^3$,	„ $v=420$ „ 343 „	
„	$=c_3v^6$,	„ $v=343$ „ 282 „	
„	$=c_4v^3$,	„ $v=282$ „ 240 „	
„	$=c_5v^2$,	„ $v=240$ m/sec downwards.	

The corresponding tables were calculated by Berardinelli, from $u=700$; later by von Mola to $u=983$; secondary functions were calculated by Braccialini.

Similar primary and secondary tables, but with somewhat different division into zones (viz., with those of Mayevski-Sabudski), are to be found in the work of von Heydenreich, *Die Lehre vom Schuss*, Berlin 1908, Part II, to which reference should be made.

Siacci's procedure of 1896 (Siacci III) differs from Siacci II merely in being based on other laws of air resistance: here also Siacci has recalculated the primary Tables D, J, T, A, as well as a Table of β.

The factor of correction β of Siacci II and III.

The point of difficulty in the solution lies obviously in the factor β.

The exact Chief Equation was

$$\frac{d\theta}{\cos^2\theta} = \frac{g}{c}\,\frac{d\,(v\cos\theta)}{v f(v)\cos^2\theta} = \frac{g}{c_y}\,\frac{d\left(\dfrac{v\cos\theta}{\cos\phi}\right)}{\dfrac{v\cos\theta}{\cos\phi}\, f\left(\dfrac{v\cos\theta}{\cos\phi}\right)(\cos\theta)},$$

in which c depends among other things on the air density, and is variable in fact with the height y of the shell above the ground; this may be represented by the suffix y in c_y.

The approximate Chief Equation is

$$\frac{d\theta}{\cos^2\theta} = \sim \frac{g}{c_0}\,\frac{d\left(\dfrac{v\cos\theta}{\cos\phi}\right)}{\dfrac{v\cos\theta}{\cos\phi}\, f\left(\dfrac{v\cos\theta}{\cos\phi}\right)\beta\cos^2\phi},$$

and here c is provided with the suffix 0, to denote that the c, really variable, may be represented approximately by its value at the height of the muzzle of the gun. Introducing $u = \dfrac{v\cos\theta}{\cos\phi}$, the approximation consists in putting $c_y f(v)\cos\theta = \sim c_0 f(u)\,\beta\cos^2\phi$. So that since

$$\beta = \frac{c_y}{c_0}\,\frac{f(v)\cos\theta}{f\left(\dfrac{v\cos\theta}{\cos\phi}\right)\cos^2\phi} \quad\text{.....................}(1)$$

on the quadratic law for instance, $f(v) = v^2$,

or $$f\left(\frac{v\cos\theta}{\cos\phi}\right) = \left(\frac{v\cos\theta}{\cos\phi}\right)^2,$$

β has the following values:

at the point of departure $O\,(\theta = \phi)$, $\beta = \dfrac{v_0^2\cos\phi}{\left(\dfrac{v_0\cos\phi}{\cos\phi}\right)^2\cos^2\phi} = \sec\phi$;

at the vertex S, $(\theta = 0)$, $\beta = \dfrac{v_S^2\,.\,1}{\left(\dfrac{v_S\,.\,1}{\cos\phi}\right)^2\cos^2\phi} = 1$;

in the descending branch, where $\theta = -\phi$, $\beta = \dfrac{v_1^2\cos(-\phi)}{\left(\dfrac{v_1\cos(-\phi)}{\cos\phi}\right)^2\cos^2\phi} = \sec\phi$.

So that in these cases, β is always $\geqq 1$, and differs less from 1 as ϕ is smaller.

On the cubic law, the corresponding values of β in the series are $\sec\phi$, $\cos\phi$, $\sec\phi$, and so respectively >1, <1, >1. On the biquadratic law, they are $\sec\phi$, $\cos^2\phi$, $\sec\phi$; and so on.

For the quadratic law, $cf(v)=cv^2$, β is identical with Didion's adjusting factor α. Because in Didion's procedure,

$$f(v)=\sim\frac{f(\alpha v\cos\theta)}{\alpha\cos\theta}=\frac{(\alpha v\cos\theta)^2}{\alpha\cos\theta}=\alpha v^2\cos\theta.$$

In Siacci II and III on the other hand,

$$f(v)=\sim f\left(\frac{v\cos\theta}{\cos\phi}\right)\frac{\beta\cos^2\phi}{\cos\theta}=\left(\frac{v\cos\theta}{\cos\phi}\right)^2\frac{\beta\cos^2\phi}{\cos\theta}=\beta v^2\cos\theta,$$

and so the β is the same as α.

On Siacci's method, we multiply equation (1) on both sides by the factor

$$\frac{c_0}{c_y}\Phi(\tan\phi+\tan\theta)^2\frac{d\theta}{\cos^2\theta},\text{ where }\Phi=\frac{f\left(\dfrac{v\cos\theta}{\cos\phi}\right)}{f(v)}\frac{f\left(\dfrac{V_0\cos\phi}{\cos\theta}\right)}{f(V_0)}.$$

Here V_0 denotes that initial velocity which with equal ϕ will give the same range in a vacuum as that which was given in air; that is $\frac{V_0^2\sin 2\phi}{g}=X$. This gives

$$\beta\frac{c_0}{c_y}\Phi(\tan\phi+\tan\theta)^2\frac{d\theta}{\cos^2\theta}$$

$$=\frac{1}{\cos^2\phi}\frac{f\left(\dfrac{V_0\cos\phi}{\cos\theta}\right)}{f(V_0)}(\tan\phi+\tan\theta)^2\frac{d\theta}{\cos\theta},\quad(2)$$

and this equation is to be integrated from $\theta=+\phi$ to $\theta=-\phi$, and so from the origin O of the trajectory to the point O_1, lying near the point of fall on the descending branch.

In the integration on the left-hand side, a constant mean value of β, as well as of Φ and $\frac{c_0}{c_y}$ is to be taken outside the integral sign: this mean value is to be distinguished by a suffix m; and then

$$\beta_m\left(\frac{c_0}{c_y}\right)_m\Phi_m\int_{+\phi}^{-\phi}(\tan\phi+\tan\theta)^2\frac{d\theta}{\cos^2\theta}$$

$$=\frac{1}{f(V_0)\cos^2\phi}\int_{+\phi}^{-\phi}f\left(\frac{V_0\cos\phi}{\cos\theta}\right)(\tan\phi+\tan\theta)^2\frac{d\theta}{\cos\theta}.\quad(3)$$

The integral on the left-hand side is $-\frac{8}{3}\tan^3\phi$; and further it is seen that, for $cf(v) = cv^n$, the fraction $\Phi_m = 1$: (and herein lies the reason of its introduction by Siacci); Siacci then puts it equal to unity.

On the right-hand side, the limits are inverted, and $\int_{-\phi}^{\phi}$ replaced by $2\int_0^{\phi}$; in this manner, as is seen easily by the expansion of $(\tan\phi + \tan\theta)^2$, the concluding formula is

$$\beta_m = \left(\frac{c_y}{c_0}\right)_m \frac{3}{2\sin 2\phi\, f(V_0)} \int_0^{\phi} f\left(\frac{V_0\cos\phi}{\cos\theta}\right)\left(1 + \frac{\tan^2\theta}{\tan^2\phi}\right)\frac{d\theta}{\cos\theta}. \quad (4)$$

Except for the fraction $\left(\frac{c_y}{c_0}\right)_m$, this expression contains only V_0 and ϕ; the integral was calculated by Siacci by approximate quadrature for different values of ϕ and $V_0\cos\phi$; so that, putting $\left(\frac{c_y}{c_0}\right)_m$ aside, a Table of β can be drawn up that gives it for all possible values of ϕ and $V_0\cos\phi$ in practice; or for all possible ϕ and X, since $V_0 = \sqrt{\frac{gX}{\sin 2\phi}}$.

Siacci's β Table gives these values.

How far this factor β will adjust the errors of integration, requires further consideration (consult §§ 32, 33).

It was seen in the calculation of β that various quantities were neglected, that might affect the accuracy of the complete calculation of the trajectory.

Siacci sought to make the procedure more accurate by calculating with different values of β for $x = 0, \frac{1}{4}X, \frac{1}{2}X, \frac{3}{4}X, X$; and also with different β for x, y, t, θ.

Parodi has extended this method for practical purposes (see Note no. 28).

§ 29. The approximate solution of E. Vallier, 1894.

The choice of σ and γ, on Vallier's method, is the same as in Siacci II and III. But the calculation of β is somewhat different.

Moreover a different law of air resistance is assumed: that is, in the value of the air resistance as dependent on the velocity v of the shell, for $v > 330$ m/sec the Chapel-Vallier laws have been employed; and for $v < 330$, the two zone-laws of Hojel (compare § 10).

When the retardation due to the air resistance is represented, as before, by $cf(v)$, and $c=\frac{\delta_y R^2 i}{1\cdot 206\,P}$ (δ_y is the air density at a height y, measured in kg/m³; R the half calibre of the shell in cm, P weight of shell in kg, $i=\frac{n}{1000}$, the coefficient of form), then, as in § 10, 8; for

$$\begin{array}{lll} v \geqq 330 \text{ m/sec}, & & f(v)=0\cdot 125\,(v-263), \\ 330 > v \geqq 300 & ,, & ,, \quad =0\cdot\overset{(11)}{0}21692\,v^5, \\ v < 300 & ,, & ,, \quad =0\cdot\overset{(5)}{0}33814\,v^{\frac{5}{2}}. \end{array}$$

Here $i=1$ for rotating elongated shell with ogival head of semi-angle of opening $\gamma=41^\circ\cdot 5$.

But Vallier on the other hand considers i as slightly variable, and puts

$$i=\frac{\gamma\,[v-(180+2\gamma)]}{41\cdot 5\,(v-263)}, \text{ for } v \geqq 330 \text{ m/sec};$$

but for $v < 330$ m/sec,

$$i= 0\cdot 67, \quad 0\cdot 72, \quad 0\cdot 78, \quad 1\cdot 10,$$

for

$$\gamma=31^\circ, \quad 33^\circ\cdot 6, \quad 36^\circ\cdot 9, \quad 48^\circ\cdot 2.$$

The adjusting factor β has been calculated by Vallier very systematically, employing the Taylor-Maclaurin expansion and the remainder in the integral form. A finite formula for β is thus obtained.

This formula is really not so simple to manage as the β Table of Siacci, given in Siacci's work, *Balistique extérieure*, Paris 1892, or in the *Lehre vom Schuss* of Heydenreich, Berlin 1908, II. p. 30. Vallier's formula is hardly suitable for tables.

On the other hand it has the advantage of greater generality; the β Tables of Siacci and Heydenreich fail frequently with guns of great calibre. The formula for β is calculated in the following manner. Vallier starts with the Maclaurin expansion of § 22 a, with the remainder term in the integral form, as in equation (30) in § 22 a; and thus

$$y=x\tan\phi-\frac{gx^2}{2v_0^2\cos^2\phi}-g\int_{t=0}^{t=x}(x-t)^2\left[\frac{cf(v)}{v^4\cos^3\theta}\right]_t dt. \quad \ldots\ldots(1)$$

The method of approximation, employed by Vallier, as well as in Siacci II and III, in the integration of the Chief Equation, consists in replacing

$$\frac{R^2 i(v)\,\delta_y f(v)\cos\theta}{1\cdot 206\,P}$$

approximately by

$$\frac{R^2 i(v_0)\,\delta_0 f\left(\frac{v\cos\theta}{\cos\phi}\right)\beta\cos^2\phi}{1{\cdot}206\,P},$$

in which δ_y and $i(v)$ denote the air density and the form coefficient i for the actual height y above the horizon of the muzzle; δ_0 and $i(v_0)$, on the other hand, are the corresponding values at the point of departure of the trajectory.

Further $cf(v)\cos\theta$ is replaced approximately by the function $c_1 f(u)$, and $u=\frac{v\cos\theta}{\cos\phi}$; further $\delta_y=\delta_0(1-0{\cdot}00011y)$. ....(2)

The following relations are contrasted:
correct:

$$y=x\tan\phi-\frac{gx^2}{2v_0^2\cos^2\phi}-g\int_{t=0}^{t=x}(x-t)^2\left[\frac{cf(v)\cos\theta}{v^4\cos^4\theta}\right]_t dt, \quad \text{...(3)}$$

$$c=\frac{R^2 i(v)\,\delta_0(1-0{\cdot}00011y)}{1{\cdot}206\,P};$$

incorrect:

$$y=x\tan\phi-\frac{gx^2}{2v_0^2\cos^2\phi}-g\int_{t=0}^{t=x}(x-t)^2\left[\frac{c_1 f(u)}{v^4\cos^4\theta}\right]_t dt, \quad \text{......(4)}$$

$$c_1=\frac{R^2 i(v_0)\,\delta_0\beta\cos^2\phi}{1{\cdot}206\,P}.$$

In this approximation, the error ϵ in relation to the ordinate y of the path, corresponding to the abscissa x, is thus the difference of the two expressions in (3) and (4),

$$\epsilon=g\int_{t=0}^{t=x}(x-t)^2\left[\frac{cf(v)\cos\theta-c_1 f(u)}{v^4\cos^4\theta}\right]_t dt.$$

Let us consider the complete trajectory, in so far as it lies above the horizontal through the muzzle.

Then $x=X$; and $y=0$ in (3), assuming that there is no error in the employment of the function $cf(v)$: on the other hand y in (4) will be different from zero. Denote then the error in the height y of the trajectory at the end of the range by $\mathfrak{E}$.

Let the variable in the definite integrals be x instead of t; and let the other variables remain as before, then the error is

$$\mathfrak{E}=g\int_{x=0}^{x=X}(X-x)^2\,\frac{cf(v)\cos\theta-c_1 f(u)}{v^4\cos^4\theta}\,dx. \quad \text{.........(5)}$$

The factor β in question occurs in c_1: it is required then to determine β so as to make this error $\mathfrak{E}$ zero. If this could be carried out exactly, the solution of the problem of the trajectory with this value of β would be exact.

But here it is evident that other approximations must arise in a further manipulation of the expression in (5).

Therefore a new variable is introduced, $z = \frac{x}{X}$, so that $dx = X\,dz$, $(X - x)^2 = X^2(1 - z^2)$.

Vallier assumes further that it is permissible to represent the fraction $\frac{cf(v)\cos\theta - c_1 f(u)}{v^4 \cos^4\theta}$ as a function of x and therefore of z, denoted by $\phi(z)$, and to expand it in a convergent series in ascending powers of z, of which only the first two terms need be employed.

He assumes therefore that the curve $\phi(z)$ can be replaced by a straight line, $\phi(z) = a_0 + a_1 z$ (first assumption). Then

$$\mathfrak{E} = gX^3 \int_{z=0}^{z=1} (1 - z)^2 (a_0 + a_1 z)\, dz = gX^3 \left(\frac{a_0}{3} + \frac{a_1}{12}\right).$$

If $\mathfrak{E} = 0$, we have

$$4a_0 + a_1 = 0, \quad \text{.............................(6)}$$

and this is the relation, from which β is to be calculated.

The calculation of a_0 and a_1 follows from the relations of the trajectory at the vertex: in the first place $x = 0$, $z = 0$, and here $\phi(z) = a_0 + a_1 z = a_0$; denote it by $\phi(0)$.

Moreover Vallier assumes that $z_s = \frac{x_s}{X}$ can be taken as constant, for all trajectories, and $= 0{\cdot}55$ (second assumption): the corresponding value of $\phi(z)$ is denoted by $\phi(s)$; so that $\phi(s) = a_0 + 0{\cdot}55\, a_1$. Thence two equations for the determination of a_0 and a_1. Substitute these values in (6), and the condition becomes

$$6\phi(0) + 5\phi(s) = 0. \quad \text{.........................(7)}$$

But now

$$\phi(z) = \frac{R^2\delta_0}{1{\cdot}206P} \frac{i(v)(1 - 0{\cdot}00011\,y) f(v)\cos\theta - i(v_0) f(u)\,\beta\cos^2\phi}{(v\cos\theta)^4},$$

and so

$$\phi(0) = \frac{R^2\delta_0}{1{\cdot}206\,P} \frac{i(v_0) f(v_0)\cos\phi - i(v_0) f(v_0)\,\beta\cos^2\phi}{(v_0\cos\phi)^4},$$

since $u_0 = v_0$; and

$$\phi(s) = \frac{R^2\delta_0}{1{\cdot}206\,P}\;\frac{i(v_s)(1-0{\cdot}00011\,y_s)f(v_s) - i(v_0).f(u_s)\,\beta\cos^2\phi}{(v_s\,.\,1)^4 \text{ or } u_s^4\cos^4\phi}.$$

Substitute these values of $\phi(0)$ and $\phi(s)$ in (7), and we have

$$\beta\left[6\frac{f(v_0)}{v_0^4} + 5\frac{f(u_s)}{u_s^4}\right]\sec^2\phi\, i(v_0)$$
$$= 6i(v_0)\frac{f(v_0)}{v_0^4}\sec^3\phi + 5i(v_s)(1-0{\cdot}00011\,y_s)\frac{f(v_s)}{v_s^4}. \quad \ldots\ldots(8)$$

This is the formula for the determination of β that was to be constructed.

It is obvious that this value of β can only be employed after finding a first approximate value of β (such as $\beta = 1$ or better, according to Vallier's procedure, $\beta = \cos\frac{2}{3}\phi$) in a provisional calculation of the trajectory in respect of the vertex, and provisional values of v_s, u_s, y_s. A repeated application of this procedure does not always lead to a more accurate result. Vallier denotes $\beta i(v_0)$ by $\frac{1}{m}$.

Another somewhat more exact formula for β, given by Vallier, may be stated here without proof.

Suppose a preliminary calculation of the ballistic elements $x_1 y_1 v_1 u_1 \theta_1$ has been provisionally made for the point $(x_1 y_1)$ with abscissa $x_1 = 0{\cdot}225\,x_s$, then

$$\beta\left[9\frac{f(u_1)}{u_1^4} + 4\frac{f(u_s)}{u_s^4}\right] i(v_0)\sec^2\phi$$
$$= 9i(v_1)\frac{f(v_1)}{v_1^4}(1-0{\cdot}00011\,y_1)\sec^3\theta_1$$
$$+ 4i(v_s)\frac{f(v_s)}{v_s^4}(1-0{\cdot}00011 y_s). \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

The author, in 1910, has proposed another form of the Vallier formula for β.

In the preceding calculation the value $\frac{x_s}{X} = 0{\cdot}55$, has been assumed.

In the provisional calculation of a trajectory, always required in the application of Vallier's β, besides v_s, u_s, and y_s, we can calculate x_s and X as well.

The ratio $x_s : X$ is thus known more accurately; and then the preceding process is carried out afresh: for instance, the formula (8) becomes the following:

$$\beta\left[\left(4\frac{x_s}{X} - 1\right)\frac{f(v_0)}{v_0^4} + \frac{f(u_s)}{u_s^4}\right]\sec^2\phi\, i(v_0)$$
$$= \left(4\frac{x_s}{X} - 1\right)\frac{f(v_0)}{v_0^4}\sec^3\phi\, i(v_0) + i(v_s)(1-0{\cdot}00011\,y_s)\frac{f(v_s)}{v_s^4}, \quad \ldots(10)$$

in which the special feature is that, with constant i, the factors $i(v_0)$ and $i(v_s)$ cancel, as $i(v_0)=i(v_s)$.

The β formulae above can obviously be applied to the i values of v. Eberhard (§ 10).

Further it might be possible to have a suitable value for the function $\phi(z)$, instead of the linear $a_0+a_1 z$, which might apply to a definite group of values of c, v_0, ϕ, for rifles, field guns, howitzers, etc.

Finally it may be noted that the choice of the mean value $\sigma=\cos\phi$ in $u=\frac{v\cos\theta}{\sigma}=\frac{v\cos\theta}{\cos\phi}$ was arbitrary. More generally, we may put $\sigma=\cos^p\psi$, and obtain p from calculated trajectories (§ 32) or from observation; here ψ denotes a mean of the values of θ at the ends of the corresponding arc of the trajectory.

§ 30. Approximate solutions of P. Charbonnier.

Charbonnier attempts the calculation of trajectories, equally by the help of expansion in series, but yet in a manner essentially different from Siacci and Vallier. His method may be explained here, as applied to flat trajectories.

A first approximate solution is made on Krupp's method, or according to Siacci I, with $\alpha=1$, based on the system

$$x=\frac{1}{c}(D_u-D_{u_0}),\quad \tan\theta=\tan\phi-\frac{1}{2c}(J_u-J_{u_0}),$$

$$y=x\tan\phi-\frac{x}{2c}\left(\frac{A_u-A_{u_0}}{D_u-D_{u_0}}-J_{u_0}\right),\quad t=\frac{1}{c}(T_u-T_{u_0}),$$

in which $u=v\cos\theta$, $u_0=v_0\cos\phi$, and $cf(v)$ denotes the retardation due to air resistance.

The exact Chief Equation is

$$\frac{d\theta}{\cos^2\theta}=\frac{g}{cvf(v)}\frac{d(v\cos\theta)}{\cos^2\theta},$$

or, with $v\cos\theta=u$, and

$$\frac{1}{vf(v)}=\phi(v)=\phi\left(\frac{v\cos\theta}{\cos\theta}\right)=\phi\left(\frac{u}{\cos\theta}\right),$$

it may be written

$$\frac{d\theta}{\cos^2\theta}=\frac{g}{c}\phi\left(\frac{u}{\cos\theta}\right)\frac{du}{\cos^2\theta}. \quad \text{..................(1)}$$

Employ on the right-hand side of the equation the expansion

$$\frac{1}{\cos\theta}=1+\frac{\theta^2}{2!}+\frac{5\theta^4}{4!}+\dots,\quad \frac{1}{\cos^2\theta}=1+\frac{\theta^2}{1}+\frac{2\theta^4}{1.3}+\dots,$$

so that

$$\frac{d\theta}{\cos^2\theta} = \frac{g}{c}\,\phi\left(u + \frac{u\theta^2}{2\,!} + \frac{5u\theta^4}{4\,!} + \ldots\right)\left(1 + \frac{\theta^2}{1} + \frac{2\theta^4}{1\,.\,3} + \ldots\right) du,$$

and here the function $\phi\left(u + \frac{u\theta^2}{2\,!} + \ldots\right)$ is expanded by Taylor's Theorem, and written

$$= \phi(u) + \frac{u\theta^2}{2}\,\phi'(u) + \theta^4\left[\frac{5\phi'(u)}{4\,!} + \frac{u^2\phi''(u)}{8}\right] + \ldots.$$

Multiply together the two series, and the equivalent of the Chief Equation in (1) may be given in the form

$$\frac{d\theta}{\cos^2\theta} = \frac{g}{c}\left\{\phi(u) + \theta^2\left[\frac{u\phi'(u)}{2} + \phi(u)\right]\right.$$
$$\left. + \theta^4\left[\tfrac{2}{3}\phi(u) + \tfrac{17}{24}u\phi'(u) + \frac{u^2}{8}\,\phi''(u) + \ldots\right] + \ldots\right\} du. \quad \ldots(2)$$

If θ is so small that only the first term in the square brackets need be retained, we have

$$\frac{d\theta}{\cos^2\theta} = \frac{g}{c}\,\phi(u)\,du = \frac{g\,d(v\cos\theta)}{cv\cos\theta\,f(v\cos\theta)}, \quad \ldots\ldots\ldots\ldots(3)$$

and this is the approximate equation, which formed the basis of the previous Krupp solution, and that of Siacci I, with $\alpha = 1$.

A second approximation is reached when the first two terms of (2) are employed. The simplified Chief Equation in this case leads to

$$\frac{d\theta}{\cos^2\theta} = \frac{g}{c}\left\{\phi(u) + \theta^2\left[\frac{u\phi'(u)}{2} + \phi(u)\right]\right\} du,$$

or since

$$\phi(u) = \frac{1}{uf(u)}, \quad \phi'(u) = \frac{-f(u) - uf'(u)}{u^2 f^2(u)},$$

$$\frac{d\theta}{\cos^2\theta} = \frac{g\,du}{cuf(u)} + \frac{g}{c}\,\theta^2\psi(u)\,du, \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where

$$\psi(u) = \frac{1}{2uf(u)} - \frac{f'(u)}{2f^2(u)}.$$

Even this differential equation between θ and u will lead to difficulties in an exact solution, because θ^2 occurs on the right-hand side; for this reason, Charbonnier replaces θ^2 on the right by $\tan^2\theta$ as an approximation; and as a further approximation takes the expression $\tan\theta = \tan\phi - \frac{1}{2c}(J_u - J_{u_0})$ from a former equation (Siacci I with $\alpha = 1$).

Substitute then in (4) for θ^2 the expression $\left(q - \frac{1}{2c} J(u)\right)^2$, where q denotes $\tan\phi + \frac{1}{2c} J(u_0)$, and an equation is obtained containing θ on the left, but u only on the right. This is integrable at once; that is

$$\frac{d\theta}{\cos^2\theta} = \frac{g\,du}{c\,u\,f(u)} + \frac{g}{c}\left(q - \frac{1}{2c} J(u)\right)^2 \psi(u)\,du, \quad \text{.........(5)}$$

and this gives, with $\qquad J(u) = -\int \frac{2g\,du}{uf(u)}$,

$$\tan\theta - \tan\phi = -\frac{1}{2c} J(u) + \frac{1}{2c} J(u_0)$$
$$+ \frac{g}{c}\int_{u_0}^{u} \left[q^2 \psi(u) + \frac{1}{4c^2} J^2(u)\,\psi(u) - \frac{q}{c} J(u)\,\psi(u)\right] du.$$

Suppose further that three new primary tables are set out, namely for $\int \psi(u)\,du$, $\int J(u)\,\psi(u)\,du$, and $\int J^2(u)\,\psi(u)\,du$, and their values denoted by J', J'', J''' respectively, then

$$\tan\theta = q - \frac{1}{2c} J(u) + \frac{g}{c}\left[q^2 (J'_u - J'_{u_0})\right.$$
$$\left. + \frac{1}{4c^2}(J_u''' - J_{u_0}''') - \frac{q}{c}(J_u'' - J_{u_0}'')\right]. \quad \text{...(6)}$$

In the corresponding case the value of t can be calculated more exactly: for

$$dt = -\frac{v\cos\theta}{g}\,\frac{d\theta}{\cos^2\theta},$$

so that $$dt = -\frac{u}{g}\left[\frac{du}{uf(u)} + \frac{g}{c}\,\theta^2\,\psi(u)\,du\right],$$

or on the same procedure

$$dt = -\frac{1}{c}\,\frac{du}{f(u)} - \frac{1}{c}\left(q - \frac{1}{2c} J(u)\right)^2 u\psi(u)\,du.$$

For the calculation of t, further integrals are seen to be required:

$$\int u\psi(u)\,du, \quad \int J(u)\,u\psi(u)\,du, \quad \int J^2(u)\,u\psi(u)\,du,$$

and these require equally the primary tables to be calculated, for T', T'', T'''.

Finally y will be given by

$$dy = -\frac{(v\cos\theta)^2}{g}\tan\theta\,\frac{d\theta}{\cos^2\theta}.$$

Considering that the construction of further tables is laborious, and that even when these tables are at hand, the calculation of trajectories is only slightly simplified, the following method is proposed by Charbonnier.

Equation (4) can be written in the form:

$$\frac{d\theta}{\cos^2\theta} = \frac{g\,du}{cuf(u)}[1 - \theta^2\kappa(u)]$$

where $$\kappa(u) = \frac{uf'(u)}{2f(u)} - \tfrac{1}{2};$$

or approximately:

$$\frac{d\theta}{\cos^2\theta} = \frac{g}{c(1+\kappa\theta^2)}\,\frac{du}{uf(u)}. \qquad \text{(7)}$$

Contrasted with equation (3) $\dfrac{d\theta}{\cos^2\theta} = \dfrac{g\,du}{cuf(u)}$, this equation is more accurate.

Charbonnier next operates with a mean value of the factor $1 + \kappa\theta^2$, which is different for the ascending and descending branches of the trajectory. On the first branch the value of that factor is $1 + \phi^2\kappa(u_0)$ at the point of departure, at the vertex it is 1, so that the mean is $1 + \frac{1}{2}\phi^2\kappa(u_0)$. On the descending branch the mean value is

$$1 + \tfrac{1}{2}\omega^2\kappa(u_e),$$

where ω is the acute angle of descent, and $u_e = v_e\cos\omega$. So that the procedure is as follows:

Starting from the equation $\dfrac{d\theta}{\cos^2\theta} = \dfrac{g\,du}{cuf(u)}$, on a system of solution similar to that of Siacci I (with $\alpha = 1$) or to the earlier one of Krupp, a first provisional estimate is made, determining in particular the vertex and point of descent; the calculation is then repeated, and c is replaced by $c(1 + \frac{1}{2}\kappa_0\phi^2)$ in the ascending branch, where $\kappa_0 = \dfrac{u_0 f'(u_0)}{2f(u_0)} - \frac{1}{2}$, $u_0 = v_0\cos\phi$; and in the descending branch c is replaced by $c(1 + \frac{1}{2}\kappa_e\omega^2)$, where $\kappa_e = \dfrac{u_e f'(u_e)}{2f(u_e)} - \frac{1}{2}$, $u_e = v_e\cos\omega$; and ϕ^2 or ω^2 is then replaced by $\tan^2\phi$, or $\tan^2\omega$, respectively.

It must be added that Charbonnier's plan contains a rational principle for increasing the accuracy of the calculation of a trajectory. Nevertheless it is somewhat laborious, in spite of the employment of tables, which Charbonnier has calculated recently (see Note). This method of approximation, which provides a separate calculation for the two branches, is tested, partially at least, in §§ 32, 33.

§ 30 a. On the secondary ballistic functions, and on ballistic curves.

1. *The Secondary functions.*

The Bernoulli-Didion solution in § 25 may first be examined. There the functions B, J, V, D enter in the equations (1) to (4), serving for the calculation of y, θ, v, and t. Subsequently the functions E and Θ are derived from these.

So also in the Siacci I procedure of § 26, the functions E, N, Q, O, S, S' are introduced as supplementary.

Corresponding relations hold for the system of solutions of Siacci II and III, § 28, and of Vallier, § 29. Writing c' for $\frac{1}{c\beta}$, this system of equations is as follows:

$$\frac{x}{c'} = D(u) - D(v_0), \quad \text{......(1)}$$

$$t = \frac{c'}{\cos\phi}[T(u) - T(v_0)], \quad \text{......(2)}$$

$$\tan\theta = \tan\phi - \frac{c'}{2\cos^2\phi}[J(u) - J(v_0)], \quad \text{......(3)}$$

$$y = x\tan\phi - \frac{c'^2}{2\cos^2\phi}\{A(u) - A(v_0) - J(v_0)[D(u) - D(v_0)]\}$$

$$= x\tan\phi - \frac{c'x}{2\cos^2\phi}\left(\frac{A(u) - A(v_0)}{D(u) - D(v_0)} - J(v_0)\right), \quad \text{......(4)}$$

$$v = \frac{u\cos\phi}{\cos\theta}, \quad v_0 = u_0; \quad \text{......(5)}$$

and thus the elements x, t, θ, y of any chosen trajectory are expressed in the parameter u.

The functions occurring here, D, T, J, A, are called the primary ballistic functions (tables for them in Vol. IV, Table 12 *a* and Table 13).

Denote $\frac{x}{c'}$ in equation (1) by ξ, and for the end point of the path, where $x = X$, let $\frac{X}{c'}$ be denoted by ξ_e; then equation (1) shows that u is a function of ξ and v_0.

Let (2), (3), (4), be written respectively:

$$T(u) - T(v_0) = H(v_0, \xi),$$

$$J(u) - J(v_0) = L(v_0, \xi),$$

$$\frac{A(u) - A(v_0)}{D(u) - D(v_0)} - J(v_0) = E(v_0, \xi).$$

It is easily seen, how by help of the primary tables, the secondary tables for H, L and E, can be established. A definite value of v_0 and ξ is chosen, and then, for example, from equation (1) we get the value of u and thence of $J(u)$, and thence the value of L.

Tables may also be calculated for $\frac{E}{\xi} = N$, and $L - E = M$.

The system of equations is given then in the form:

$$\xi = D(u) - D(v_0), \quad \text{.............................(6)}$$

$$t = \frac{c'}{\cos\phi} H(v_0, \xi), \quad \text{.............................(7)}$$

$$\tan\theta = \tan\phi - \frac{c'}{2\cos^2\phi} L(v_0, \xi), \quad \text{..................(8)}$$

$$= \tan\phi \left[1 - \frac{c'}{\sin 2\phi} L(v_0, \xi)\right], \quad \text{..............(9)}$$

$$y = x\tan\phi - \frac{c'x}{2\cos^2\phi} E(v_0, \xi), \quad \text{...............(10)}.$$

$$= \frac{xc'}{2\cos^2\phi}\left[\frac{\sin 2\phi}{c'} - E(v_0, \xi)\right]. \quad \text{...........(11)}$$

Referring specially to the point of descent on the horizon through the muzzle, here $y = 0$, $x = X$, $\theta = -\omega$, $v = v_e$, $u = u_e$, $\xi = \xi_e$.

Thence, from (11), $\sin 2\phi = c'E(v_0, \xi_e)$, and since $c' = \frac{X}{\xi_e}$, and $\frac{E(v_0, \xi_e)}{\xi_e} = N(v_0, \xi_e)$, it follows that

$$\frac{\sin 2\phi}{X} = N(v_0, \xi_e).$$

Equation (10) gives, when $y=0$, $\tan\phi = \dfrac{c'}{2\cos^2\phi} E(v_0, \xi_e)$; and so from (8):

$$-\tan\omega = \frac{c' E_e}{2\cos^2\phi} - \frac{c' L_e}{2\cos^2\phi} = -\frac{c' M_e}{2\cos^2\phi}.$$

Therein L_e is to be written for $L(v_0, \xi_e)$, E_e for $E(v_0, \xi_e)$, and so on; and so too, L_s for $L(v_0, \xi_s)$, E_s for $E(v_0, \xi_s)$, and so on.

For the point of descent, equation (9) is written:

$$-\tan\omega = \tan\phi\left(1 - \frac{c' L_e}{\sin 2\phi}\right), \text{ or since } \frac{\sin 2\phi}{c'} = E_e,$$

$$-\tan\omega = \tan\phi\left(1 - \frac{L_e}{E_e}\right) = -\tan\phi\frac{M_e}{E_e}.$$

At the vertex of the trajectory, $\theta = 0$, $\tan\theta = 0$, so equations (8) and (9) become

$$\tan\phi = \frac{c' L_s}{2\cos^2\phi}, \text{ and } 1 = \frac{c' L_s}{\sin 2\phi}.$$

Finally it follows from equation (11) that

$$y_s = \frac{c' x_s}{2\cos^2\phi}\left(\frac{\sin 2\phi}{c'} - E_s\right) = \frac{x_s \tan\phi}{L_s}(L_s - E_s) = x_s \tan\phi \frac{M_s}{L_s}.$$

Hence the following formulae:

$$\left.\begin{aligned}
\xi_e &= \frac{X}{c'}, && \ldots\ldots(12)\\
\sin 2\phi &= XN(v_0, \xi_e) = c' E(v_0, \xi_e), && \ldots\ldots(13)\\
v_e &= \frac{u_e \cos\phi}{\cos\omega}, && \ldots\ldots(14)\\
\xi_e &= D(u_e) - D(v_0), && \ldots\ldots(15)\\
T &= \frac{c'}{\cos\phi} H(v_0, \xi_e), && \ldots\ldots(16)\\
-\tan\theta_e = \tan\omega &= \frac{c'}{2\cos^2\phi} M(v_0, \xi_e) && \\
&= \tan\phi \frac{M(v_0, \xi_e)}{E(v_0, \xi_e)}. && \ldots\ldots(17)
\end{aligned}\right\} \text{ for the point of descent, } y = 0,\ x = X$$

$$\left.\begin{aligned} \frac{\sin 2\phi}{c'} &= L(v_0, \xi_s), & \text{......(18)}\\ t_s &= \frac{c'}{\cos\phi} H(v_0, \xi_s), & \text{......(19)}\\ \xi_s &= \frac{x_s}{c'} = D(u_s) - D(v_0), & \text{......(20)}\\ y_s &= \frac{c' x_s}{2\cos^2\phi} M(v_0, \xi_s) & \\ &= x_s \tan\phi \frac{M(v_0, \xi_s)}{L(v_0, \xi_s)}, & \text{......(21)}\\ v_s &= u_s \cos\phi. & \text{......(22)} \end{aligned}\right\} \text{ for the vertex, } x = x_s,\ y = y_s,\ \theta = 0$$

The functions introduced here, E, N, H, L, M, are called the secondary ballistic functions. The corresponding tables are given in Vol. IV, Tables 12 *b* to 12 *f*.

The use of these secondary functions is evident at once, when the problem to be solved is thus: the range X being measured, and the angle of departure ϕ, and the initial velocity v_0; to find the time of flight T, the velocity v_e at descent, the angle of descent ω, the abscissa x_s and ordinate y_s of the vertex.

The solution is laborious with the use of the primary functions, D, T, J, A (compare Chapter VIII for the solution of particular problems of trajectories); on the other hand with the secondary functions the solution is completed very simply:

In equation (13), $\sin 2\phi = XN(v_0, \xi_e)$, ϕ and X are known, and consequently N, and since v_0 is known also, ξ_e can be calculated, and c' also, by (12).

Equation (16) gives T, and (17) gives ω. And $D(u_e)$ follows from (15), and also u_e. And v_e can be calculated from (14). Further $\xi_s = \frac{x_s}{c'}$ is given by (18), and x_s; and then the value of y_s from (21).

It is evident moreover that there is nothing to prevent the introduction of other secondary functions, in addition to E, N, H, L, M. For instance it follows from (12) and (16) that

$$T = \frac{X}{\xi_e \cos\phi} H(v_0, \xi_e);$$

and consequently, if another function R is introduced for $\frac{H}{\xi}$, and a table is calculated, we have

$$T = \frac{X}{\cos\phi} R(v_0, \xi_e);$$

and so forth.

2. *The Ballistic Curve.* (Vol. IV, diagrams III *a* to III *g*.)

Equations (I) to (IV) in § 23, give the point of descent ($x=X$, $y=0$, $\theta=-\omega$, $v=v_e$, $t=T$, $u=u_e$): then

$$\frac{X\gamma c}{\sigma^2}=D(u_e)-D(u_0), \quad\text{.......................................(1)}$$

$$T=\frac{\sigma}{c\gamma}[T(u_e)-T(u_0)], \quad\text{..............................(2)}$$

$$-\tan\omega=\tan\phi-\frac{1}{2c\gamma}[J(u_e)-J(u_0)], \quad\text{......................(3)}$$

$$0=\tan\phi-\frac{1}{2c\gamma}\left[\frac{A(u_e)-A(u_0)}{D(u_e)-D(u_0)}-J(u_0)\right], \quad\text{.........(4)}$$

$$u_e=\frac{v_e\cos\omega}{\sigma}, \quad\text{...(5)}$$

$$u_0=\frac{v_0\cos\phi}{\sigma}. \quad\text{...(6)}$$

Let $\frac{\gamma cX}{\sigma^2}$ be denoted by ξ.

Equation (1) shows that, from u_0 and ξ, u_e also is given; consequently from (2) $\frac{c\gamma}{\sigma}T$ is a function of u_0 and ξ; and from (3) the same holds for $(\tan\phi+\tan\omega)\,2c\gamma$; and, from (4), $2c\gamma\tan\phi$ is a function of u_0 and ξ; or,

$$T=\frac{\sigma}{c\gamma}F_1(u_0,\xi); \quad \tan\omega=\frac{1}{2c\gamma}F_2(u_0,\xi)-\tan\phi;$$

$$\tan\phi=\frac{1}{2c\gamma}F_3(u_0,\xi),$$

and it follows also that

$$\tan\omega=\frac{1}{2c\gamma}F_4(u_0,\xi).$$

Assuming the two quantities u_0 and ξ are known, it follows that these also are given:

firstly: $$\frac{\tan\omega}{\tan\phi}=\frac{F_4}{F_3};$$

secondly: $$\frac{v_e\cos\omega}{v_0\cos\phi}=\frac{u_e}{u_0},$$ and u_e is given by u_0 and ξ;

thirdly: $$\frac{v_0^2\sin 2\phi}{X},$$

$$=\frac{u_0^2\sigma^2}{\cos^2\phi}\,\frac{2\sin\phi\cos\phi}{X}=2u_0^2\frac{\sigma^2}{2c\gamma X}F_3(u_0,\xi)=\frac{u_0^2}{\xi}F_3(u_0,\xi),$$

in which only known quantities are present;

fourthly: $$\frac{T}{\sqrt{(X \tan \phi)}} = \frac{\sigma}{c\gamma} F_1(u_0, \xi) \frac{\sqrt{(2c\gamma)}}{\sqrt{X}\sqrt{[F_3(u_0, \xi)]}} = \frac{\sqrt{2}}{\sqrt{\xi}} \frac{F_1(u_0, \xi)}{\sqrt{[F_3(u_0, \xi)]}}.$$

The equations (I) to (IV) give in a similar way the vertex of the trajectory, for $\theta = 0$, $x = x_s$, $y = y_s$, $t = t_s$, $v = v_s$, $u = u_s = \frac{v_s}{\sigma}$; and it is easily seen that $\frac{x_s}{X}$ and $\frac{y_s}{X \tan \phi}$ are given with u_0 and ξ.

It has already been convenient to choose $\sigma = \cos \phi$ and $\gamma = \beta \cos^2 \phi$ (Siacci II and III); then $u_0 = v_0$, and $\xi = c\beta X$, and the results can be expressed in the following manner: Suppose v_0 and $c\beta X$ given, then the following elements of the trajectory are known:

$$\frac{v_0^2 \sin 2\phi}{X} = A_1, \quad \frac{\tan \omega}{\tan \phi} = A_2,$$

$$\frac{v_e \cos \omega}{v_0 \cos \phi} = A_3, \quad \frac{T}{\sqrt{(X \tan \phi)}} = A_4, \quad \frac{x_s}{X} = A_5, \quad \frac{y_s}{X \tan \phi} = A_6,$$

and finally $A_7 = \xi A_1$.

In practice, the initial velocity is usually given, and also the range X to which the gun fires with angle of departure ϕ, so that A_1 and v_0 are given. Then A_2 is given too, and with it the acute angle of descent ω; and A_3 and consequently the final velocity v_e; A_4 and the time of flight T; A_5 and the abscissa of the vertex x_s; A_6 and the height of the vertex y_s; finally $c\beta X$ and thence $c\beta$, and consequently the product βi, given the calibre $2R$, weight of shell P, and air density δ.

It is clear from the above that these factors $A_1, A_2, A_3, \ldots$ can be calculated with the appropriate tables. For instance Schatte took the Ballistic Tables, No. 13 of Siacci 1896, and the Tables of Fasella, in which several of the factors are calculated; and with their help, on the suggestion of the author, he constructed the six curves, Tables III *a—f*, which are printed in Vol. IV.

If the elements of the trajectory ϕ, X, ω, v_e, x_s, y_s, T could be observed directly in numerous trajectories (Method of Neesen) it would be possible to construct such a curve empirically, without a law of air resistance, together with primary and secondary tables; and for β the value in Vallier's formula would be assumed.

On the assumption that the Range Tables are of purely experimental nature—which is known not to be the case—an empirical curve could be drawn from a number of Range Tables; it appears that such tables can give good service for a convenient solution of trajectories.

The form of the factors $A_1, A_2, A_3, \ldots$ is the same as that of those employed by Siacci and Chapel in their Tables of Factors of Fire (§ 25), on the assumption of the quadratic and the cubic laws of air resistance.

The above remarks have shown that these factors have a far-reaching generalised meaning, as they can be established for any given law of air resistance.

But the assumptions must always be known, on which the solution is based; thus it would be a mistake if any one were to apply these curves to a high angle trajectory.

Numerical examples are given in § 42.

§ 31. Graphical method of solution of the ballistic problem.

Given the initial velocity v_0 and the angle of departure a for a given shell: to determine in a graphical manner the elements, such as range, time of flight, ordinate for any given abscissa, velocity in the trajectory, and so forth, on the assumption of a given law of air resistance, and retardation due to air resistance $cf(v)$. For instance, for $v < 330$ m/sec,

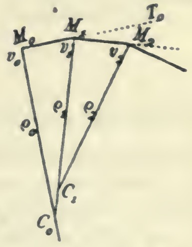
$$cf(v) = \frac{0{\cdot}014 \times R^2 \pi \delta i g}{P \times 1{\cdot}206} v^2;$$

air resistance $W(\text{kg}) = mcf(v) = \dfrac{0{\cdot}014 \times R^2 \pi \delta i}{1{\cdot}206} v^2.$

A. *Method of Poncelet-Didion* (1828).

This may be treated as a graphical development of the first group of numerical methods of approximation. It depends on the application of the law of kinetic energy.

Let M_0 be the point of departure, $M_0 T_0$ the direction of departure of the shell, v_0 the initial velocity. We suppose at first the trajectory divided into elements $M_0 M_1$, $M_1 M_2$, $M_2 M_3$, ... so small, that each arc may be considered straight, as if the trajectory were a polygon with straight sides.

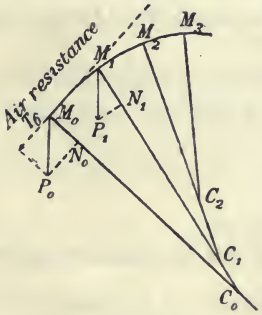


The air resistance at M_0 is along the initial tangent, and directed along $M_1 M_0$; a component of gravity acts also in the same direction, and gravity is supposed to be resolved into the direction of the tangent and normal at every point of the curve; so that along the tangent at M_0, air resistance + gravity component $N_0 P_0 = T_0 = W(v_0) + P \sin\phi$; and this can be calculated from v_0, ϕ, P, c Now along the tangent the decrease of kinetic energy of the shell is equal to the work done by T_0, in the short distance M_0M_1; assume the force T_0 constant along this arc, and we have

$$\tfrac{1}{2}mv_0^2-\tfrac{1}{2}mv_1^2=M_0M_1 \,.\, T_0;$$

and hence v_1 can be calculated; for we know T_0, v_0 and m the mass of the shell, and M_0M_1 can be chosen, as small as desired.

The arc M_0M_1 can be described in the following manner: The component vector M_0N_0 (or N_0) along the normal M_0C_0 gives zero work. The force N_0 is thus employed in curving the path, and so has the magnitude $N_0=m\frac{v_0^2}{\rho_0}$. Thence the radius of curvature M_0C_0 or ρ_0 is known at M_0, and so the point of intersection C_0 of the two consecutive normals M_0C_0 and M_1C_0. Round C_0 describe a short arc M_0M_1 with radius M_0C_0, very nearly coincident with the chord M_0M_1; and so the point M_1 is reached, at which the new tangent M_1M_2 is the tangent of the circular arc at M_1, and is perpendicular to M_1C_0.



Proceeding from M_1 the same procedure is followed; we calculate

$$T_1=W(v_1)+P_1N_1, \text{ and } N_1=m\frac{v_1^2}{\rho_1};$$

and this value of T_1 is employed in

$$\tfrac{1}{2}mv_1^2-\tfrac{1}{2}mv_2^2=T_1 \,.\, M_1M_2,$$

to determine the new velocity v_2; while the component M_1N_1 (or N_1) gives the new radius of curvature ρ_1 or M_1C_1; and so on.

This is the procedure of the construction by points; after passing the vertex, the two forces, air resistance and tangential component of gravity, are obviously opposed to each other; and for that reason care must be paid to the sign in the calculation of T.

Finally the time of flight is found as follows: in describing the arc M_0M_1 the shell takes a small time t: assume the force T_0 constant along M_0M_1; this force is then the mass m of the shell, multiplied by the ratio of the diminution of the velocity v_0-v_1 to the time t in which this takes place; and so

$$T_0=m\frac{v_0-v_1}{t}, \text{ or } t=\frac{mv_0-mv_1}{T_0};$$

or since $mv_0^2-mv_1^2=2M_0M_1 \,.\, T_0$, thence $t=\frac{M_0M_1}{\frac{1}{2}(v_0+v_1)}$. This is a relation which can in fact be deduced by noting that the arc M_0M_1, actually described by the shell with diminishing velocity, can also be considered as described with a constant velocity, equal to the arithmetic mean of the initial and final velocities, v_0 and v_1, at M_0 and M_1.

The whole time of flight is then the sum of all these small elements of time.

The vertex also, which is the point of least velocity, and the point of shortest radius of curvature, can be determined graphically in this manner.

In very flat trajectories, the radii of curvature, ρ_0, ρ_1, ... are very great; so that the points C_0, C_1, ... are at a great distance.

Didion proposed in this case to consider the arcs M_0M_1, M_1M_2, ... as circular or even parabolic arcs. On the first assumption we have for instance, taking M_0T_0 as abscissa axis and the direction of M_0C_0 as ordinate axis, (x, y) the coordinates of M_1, the equation $x^2+y^2-2\rho_0 y=0$, from which ρ_0 or y follows. For further details, consult Didion.

B. *Graphical solutions of approximation, by the author* (1896/7).

The procedure can in particular be useful in cases, in which it is required to determine without trouble at numerous points of the trajectory the ballistic elements, such as the coordinates, the velocity of the shell, time of flight, and slope of the tangent; on the assumption of direct fire and knowledge of form coefficient. The solutions rest on the mechanical principle of independence and on the application of the empirical tables, for example, of Krupp (see Vol. IV Table 8).

First consider the constructions of the trajectory in a vacuum (figs. 1 to 4).

In fig. 1, let $OB_1B_2...$ be the straight line drawn from the origin O in the direction of the angle of departure with the horizon: and take equal lengths $OB_1=B_1B_2=...$, representing to a given scale the initial velocity v_0 or some constant part of it.

From B_1, B_2, ... the vertical distances B_1O_1, B_2O_2, ... are drawn downward, equal to $\frac{1}{2}g\,.\,1^2$, $\frac{1}{2}g\,.\,2^2$, $\frac{1}{2}g\,.\,3^2$, ..., so that O, O_1, O_2, ... are points on the path.

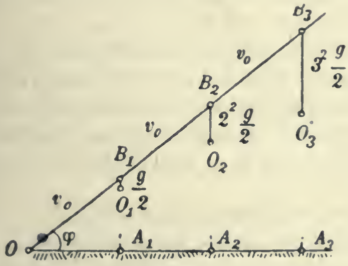


Fig. 1

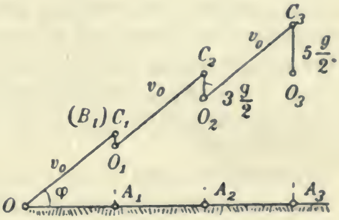


Fig. 2

A similar construction, as is easily seen, is given in fig. 2: draw OC_1 equal to v_0 in the initial direction, and $C_1O_1=\frac{1}{2}g$: then O_1C_2 equal and parallel to OC_1 and $C_2O_2=3\,.\,\frac{1}{2}g$; further O_2C_3 equal and parallel to O_1C_2 and $C_3O_3=5\,.\,\frac{1}{2}g$, and so on.

A modification of the original method of fig. 1 is shown also in the construction by chords in fig. 3: draw $OD_1=v_0$ in the original direction and $D_1O_1=\frac{1}{2}g$; next $OO_1=O_1D_2$ and $D_2O_2=2\,.\,\frac{1}{2}g$; then OO_2 prolonged to D_3, so that $A_3A_2=A_2A_1=A_1O$, and D_3O_3 is equal to $3\,.\,\frac{1}{2}g$; and so on.

In the last construction the following convenient method can be employed. Draw (in fig. 4), $OA_1 = A_1A_2 = A_2A_3 = \ldots$, and verticals through A_1, A_2, ...; and let OE_1 be the initial tangent of the trajectory. Make $E_1O_1 = \frac{1}{2}g$, where OE_1 represents the initial velocity in m/sec, and then E_1O_1 is the corresponding distance of

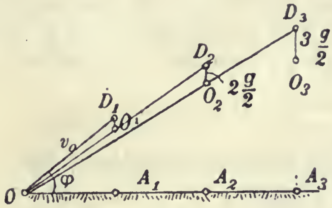


Fig. 3.

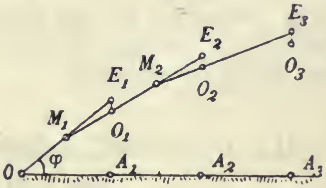


Fig. 4.

fall in the first time element. Join O_1 with the midpoint M_1 of OE_1, and let the prolongation of M_1O_1 cut the vertical through A_2 in E_2. Make $E_2O_2 = E_1O_1$, and draw M_2O_2 from M_2, the midpoint of O_1E_2, and so on. Then in this case the trajectory passes through the points O, O_1, O_2, O_3, ... and OM_1, M_1O_1, M_2O_2, ... determine the tangents of the trajectory. This method has the advantage in construction, that the equal lengths $\frac{1}{2}g = E_1O_1 = E_2O_2 = \ldots$ can be set off with the compass.

All these constructions can be adapted to motion in the air, and so are of practical interest.

1. Begin with the construction in fig. 1, and treat it in the following manner: The movement of the shell is supposed to be divided up into a large number of small equal time elements Δt (in fig. 5, Δt is taken equal to one second); the motion of the shell under the impulse of firing, in these elements of time, will be along the initial tangent; suppose then $OB_1 = B_1C_1 = C_1D_1 = \ldots$.

Next consider the problem without taking gravity into account, on the assumption of a definite law of air resistance, with the laws of Chapel-Vallier or the laws of Siacci. The corresponding differential equations determine then the loss of velocity Δv experienced by the shell in each separate element of time Δt; denote half the loss of velocity in the 1, 2, 3, ... time elements by s_1, s_2, s_3, ... respectively.

Where will the shell be found at the end of the first element of time?

According to the principle of independence, the result (for an infinitely small element of time) is the same as if the three influences in operation, the powder impulse, the air resistance, and gravity come into operation one after the other, and independently.

Due to the initial velocity, received by the shell from the powder pressure, the shell would proceed from O to B_1: through the air resistance it will move back a step s_1 from B_1 to B_2 (fig. 5), where it is assumed that the element of time t is taken so small that the air resistance may be assumed to act along the direction B_1O. Lastly, under gravity alone the shell would drop from B_2 to O_1 a distance $\frac{1}{2}g\Delta t_1^2$; and so at the end of the element of time Δt_1 the shell is found actually at O_1.

So too in the second time element Δt_2 the shell will have reached a point O_2; for in the two elements of time owing to the initial impulse the shell would have reached C_1 from O; then it comes back from C_1 to C_3 under the air resistance alone, i.e., first from C_1 to C_2 parallel to the initial tangent OB_1 a distance

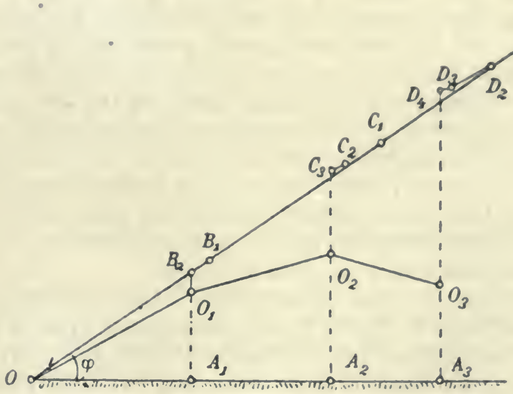


Fig. 5

C_1C_2 equal to $3s_1$, and then from C_2 to C_3 a distance C_2C_3, s_2, parallel to the direction OO_1, which may be considered to be the direction of the air resistance in the second element of time; finally under gravity alone the shell descends vertically from C_3 to O_2, a distance equal to $\frac{1}{2}g\,(3\,\Delta t_1^2 + \Delta t_2^2)$ or $\frac{1}{2}g\,(2\Delta t)^2$, when the time elements are taken equal.

In a corresponding way the next point O_3 is determined on the trajectory, by drawing $D_1D_2 = 5s_1$ parallel to OB_1, $D_2D_3 = 3s_2$ parallel to OO_1, and $D_3D_4 = s_3$ parallel to O_1O_2, and then from D_4 a distance D_4O_3 is drawn vertically, equal to $\frac{1}{2}g\,(3\Delta t)^2$; and so on.

To avoid the prolongation of the line $OB_1C_1D_1$ from falling outside the drawing, a modification, analogous to fig. 2, may be employed; and so a line O_1C_1' is drawn through O_1 equal and parallel to B_2C_1, and then from C_1' backward the polygonal path $C_1'C_2'C_3'$ congruent with $C_1C_2C_3$ is drawn, and from C_3' vertically down the distance $C_3'O_2$ is taken equal to $3\frac{g}{2}$ (in the case where each element of time is taken as one second), and so forth.

2. The procedure is more simple, if the construction of fig. 4 is employed. In this case it is necessary to determine the horizontal projection of the motion of the shell, either through the integration of the differential equation, or by employing the figures in Krupp's empirical table. This table is based on much experimental work, and gives the following quantities for all horizontal velocity components u, from 1000 m/sec downward, by decrements of one metre per second down to 50 m/sec; firstly, the air resistance in kg per square centimetre of the cross section of the shell; secondly the advance Δx in metres, during the fall of velocity of one metre per second; thirdly, the sum $\Sigma\Delta x$ of these distances from the beginning of the table; fourthly the time Δt in seconds, corresponding to the fall of velocity of 1 m/sec; and finally the sum $\Sigma\Delta t$. These numbers relate to a

sectional load 1, that is to say, in employing the table for a special case, the corresponding numbers $\Sigma\Delta x$ and $\Sigma\Delta t$ of the table must be multiplied by the factor $a = \frac{R^2 \pi \delta i}{P \times 1 \cdot 206}$; where P is the weight of the shell in kg, $R^2\pi$ the cross section in sq cm, δ the weight of a cubic metre of air in kg; i is the form coefficient, varying with the shape of the shell; for Krupp's original normal shell, $i=1$, or nearly so; but it is best determined by experiment, by taking the horizontal components of the initial and the final velocity, which relate to a given range, and then comparing with the tabular results for the corresponding shell. The Tables of Krupp provide then the complete horizontal projection of the motion of the shell. Given the successive points $O, A_1, A_2, A_3, \dots$ on the horizontal axis through the origin O, the horizontal component u of the velocity v in the trajectory is known at these points, and also the times $\Delta t_1, \Delta t_2, \dots$ which are required in the horizontal motion from O to A_1, from A_1 to A_2, and so on.

The angle of departure, ϕ, is supposed to be given.

A second trajectory can then be found, that is the one whose projection is A_1, by drawing vertically downward from B_1 the line $B_1 O_1$ equal to $\frac{1}{2} g \Delta t_1^2$. The tangent at O_1 of the trajectory may be taken as $M_1 O_1 B_2$, joining M_1, the middle point of OB_1, to O_1; and then the construction may be started afresh from O_1, proceeding in a similar manner; the vertical in A_2 meets $M_1 O_1$ in B_2, and from B_2 the line $B_2 O_2 = \frac{1}{2} g \Delta t_2^2$ is drawn vertically downward; then $M_2 O_2$ is drawn from the middle point of $O_1 B_2$, giving O_2 a third point on the trajectory, and $M_2 O_2$ the tangent at O_2; and so on.

In this procedure the trajectory is shown as an envelope of the tangents; and we thus describe a trajectory of a number of arcs of different parabolas with

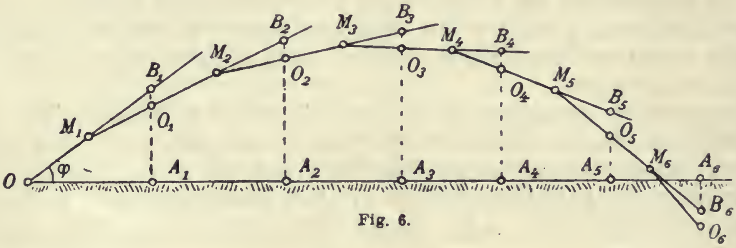


Fig. 6.

vertical axes, corresponding to the number of the lengths OA_1, A_1A_2, ... on the abscissa axis.

In fact, when the first arc of a parabola between O and O_1 is taken (fig. 7), the two points O and O_1 are given on it, the vertical direction of the axis, and the tangent OB_1 at the point O.

Now if $M_1 O_1$ is to be the tangent of such a parabola in the same point O_1, M_1 must be the middle point of OB_1.

Because if the sides of a triangle ABC (fig. 8) touch a conic section in A_1, B_1, C_1, and AA_1, BB_1, CC_1 are drawn; then by Brianchon's Theorem, these three lines meet in a point, M, and P, B_1, N, C_1 are four harmonic points. Now let the side CB, and with it A_1, recede to infinity; the conic section becomes a parabola, with vertical axis: the vertical line $M_1 Q$ (fig. 7) joining the point of intersection of the two tangents $M_1 O$ and $M_1 O_1$ to the middle point of OO_1 will be a diameter of the parabola, and so vertical and parallel to $A_1 B_1$.

At the same time it is evident that the middle point of $M_1 Q$ is another point of the parabola. Therein lies a very simple way of constructing any number of trajectories and of drawing the tangent at any arbitrary point. Join M_1 with the middle point Q of OO_1 and bisect $M_1 Q$ in P, then P is another point of the trajectory, and the tangent at this point is the line joining P with the middle point of OM_1; and so on.

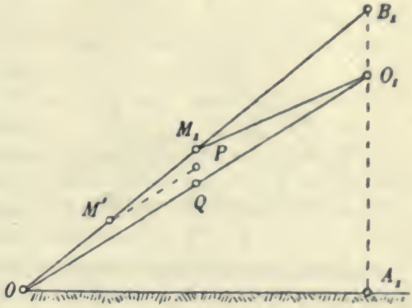


Fig. 7.

The next question is to settle the way in which the range, the part of the abscissa axis between the departure O and point of descent, should be divided, into OA_1, A_1A_2, A_2A_3, ... The construction will naturally be more exact, if a number of points, A_1, A_2, A_3, ... are taken.

For this purpose, we can either

(*a*) Select the divisions so that the time elements Δt_1, Δt_2, ... are all equal: this has the advantage that the lengths $B_1 O_1$, $B_2 O_2$, ... can all be drawn with the same setting of the compass; on the other hand there is the inconvenience that interpolation is required in Krupp's Table.

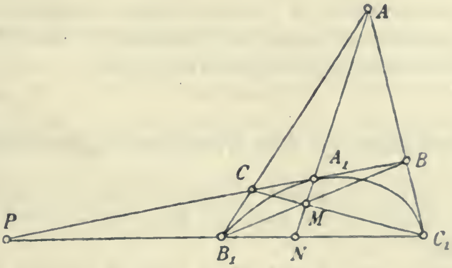


Fig. 8

(*b*) Or the distances OA_1, A_1A_2, A_2A_3, ... can be made equal, and the corresponding velocity and time intervals Δt_1, Δt_2, ... can then be taken out of Krupp's Table.

(*c*) Finally, the intervening points A_1, A_2, A_3, ... can be chosen on the horizontal projection at any desired interval, and the corresponding time interval Δt taken from Krupp's Table. This procedure is the one which leads to the result in the simplest and easiest manner if Krupp's Tables do not require to be used with interpolation, by choosing suitable, but nearly equal lengths for OA_1, A_1A_2, A_2A_3, ...

Examples are given below.

If we are to avoid the intersection of the tangents in a flat trajectory at too small an angle, the scale of the ordinates alone is increased; the range X on the drawing is not altered thereby.

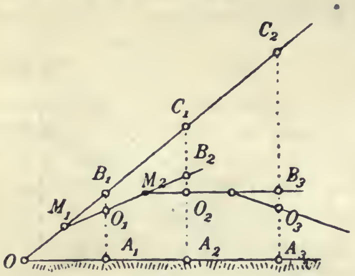


Fig 9

Further if we do not wish to measure out with the compass the small lengths B_1O_1, B_2O_2, ... and to set them out on the diagram, in which case the errors accumulate, it is easy to calculate the distances B_1O_1, C_1O_2, C_2O_3, ... that are all measured from the initial tangent, and then combine them in one measurement (fig. 9).

Suppose then, with equal lengths $OA_1 = A_1A_2 = \dots$ the distances B_1O_1, B_2O_2, B_3O_3, ... are denoted respectively by $s_1, s_2, s_3, \dots$; it is easy to show that

$$B_1O_1 = s_1, \quad C_1O_2 = 3s_1 + s_2, \quad C_2O_3 = 5s_1 + 3s_2 + s_3, \quad C_3O_4 = 7s_1 + 5s_2 + 3s_3 + s_4,$$

and so on.

These lengths are given by mere addition, as shown in the scheme

a	a	a	a
b	$a+b$	$2a+b$	$3a+b$
c	$a+b+c$	$3a+2b+c$	$5a+3b+c$
d	$a+b+c+d$	$4a+3b+2c+d$	$7a+5b+3c+d$
...	....	....	....

But, in this procedure there is the disadvantage, irrespective of the trouble of the preliminary calculation, that for a prolongation of the initial tangent OB_1 a very large sheet of drawing paper must be employed.

Examples of the graphical procedure.

1. *Example.* Given the initial velocity v_0 m/sec, the angle of departure $\phi°$; further P (kg) the weight of the shell and $R^2\pi$ (cm²) its cross section, as well as the form coefficient i.

To determine the range X, the acute angle of descent ω, the coordinates (x_s, y_s) of the vertex, the whole time of flight T, the final velocity v_e; and for any given horizontal distance x, to determine the ordinate y of the trajectory, the time of flight t, and the slope θ of the tangent to the horizon.

The horizontal range is supposed to be divided into a number of (nearly equal) parts OA_1, A_1A_2, A_2A_3, ..., and after application of the factor $a = \frac{R^2\pi\delta i}{P \cdot 1{\cdot}206}$, we use Krupp's Table to calculate the horizontal velocity at the corresponding points A_1, A_2, A_3, ... and the times of flight $t = \Sigma\Delta t$. From the corresponding time interval Δt, the fall under gravity $\frac{1}{2}g\Delta t^2$ is calculated, and thence the distances B_1O_1, B_2O_2, B_3O_3, ... (fig. 6). The trajectory from O, the point of departure, is then constructed, bit by bit, as follows: Millimetre ruled paper is taken, and on it to a large scale (for instance, for infantry weapons, 1 mm = 2 m to 5 m: for artillery, 1 mm = 5 m to 20 m) the intervals Δx chosen are drawn, or

OA_1, A_1A_2, A_2A_3, ... ; the line OB_1 is drawn at the angle of departure ϕ, cutting the vertical through A_1 in B_1. Make B_1O_1 equal to $\frac{1}{2}g\Delta t_1^2$, and join O_1 to the middle point M_1 of OB_1; then O_1 is a second point on the trajectory and M_1O_1 the tangent at it. In the same way make B_2O_2 equal to the second drop $\frac{1}{2}g\Delta t_2^2$, and join O_2 to the middle point M_2 of O_1B_2; then O_2 is the point on the trajectory with projection at A_2, and M_2O_2 is the tangent at O_2.

Proceed in this way till the ground is reached again; and continue with the construction somewhat beyond this point.

If it is necessary to obtain extra points on the trajectory, we can do so by the construction of fig. 7; thus for instance to insert points between O and O_1, draw OO_1, midpoint Q; then P, the midpoint of M_1Q, is a point on the trajectory, and the tangent at P is the line joining P with the midpoint M' of OM_1. In a similar manner another point P with its tangent can be inserted between O and P, and so on. Proportional interpolation is often sufficient.

The vertex of the trajectory and the danger zone are determined on a sheet of squared paper.

The times of flight and the horizontal velocities are obtained from Krupp's Tables; the actual velocity v in the trajectory is obtained from $v=u\sec\theta$, where θ is taken from the drawing.

A knowledge of the correct form coefficient i is of special importance for the accuracy of the result.

When the trajectory is very flat, the scale of the ordinates requires to be chosen much larger than that of the abscissae; and then it is possible to draw B_1O_1, B_2O_2, ... with greater accuracy, and the successive tangents do not cut at too small an angle in the drawing.

The choice of the size of the intervals OA_1, A_1A_2, A_2A_3, ... is made, so that they are nearly equal, but not so that interpolation is required in Krupp's Table; and these distances (8—15) will be measured out, so that the points M_1 and O_1, M_2 and O_2, do not lie too close for a drawing of the joining line to be uncertain.

If the decrease in air density δ with increase in height y is to be considered, this can be allowed for in a simple manner by taking a mean value between δ at the point of departure and at the vertex, and the construction is then repeated.

(*a*) Example: Initial velocity $v_0=441$ m/sec; angle of departure $\phi=15°\ 16\frac{1}{2}'$ and $\tan\phi=\frac{273{\cdot}2}{1000}$. Calibre $2R=8{\cdot}8$ cm; weight of shell $P=7{\cdot}5$ kg, $\delta=1{\cdot}206$, $i=1{\cdot}23$, and so $\frac{1}{a}=\frac{P}{R^2\pi i}=0{\cdot}100_4$; initial horizontal velocity

$$u_0=v_0\cos\phi=425 \text{ m/sec.}$$

The lengths OA_1, A_1A_2, ... or Δx on the horizontal axis may be chosen so that they are about 500 m long; then the values of $\Sigma\Delta x$ in Krupp's Table proceed at $\frac{500}{0{\cdot}100_4}=5000$ m intervals continuously, but so that no interpolation is required.

Here the construction of fig. 6 is to be applied; $OA_1=491$ m, $OA_2=1002$ m, $OA_3=1512$ m, and so on; $B_1O_1=7{\cdot}8$ m, $B_2O_2=11{\cdot}5$ m, and so on.

The drawing on a scale of 1 mm = 5 m for abscissa axis, and 1 mm = 2 m for ordinate axis is shown in the figure on page 196; but in the reproduction the original drawing was reduced to one-third, and the millimetre lines omitted.

Consequently the following numbers are those required to be taken from the Table.

	Horizontal velocity u	Krupp's Table		Multiplied by 0·100, the actual values, from O		Corresponding fall
		$\Sigma\Delta x$	$\Sigma\Delta t$	$\Sigma\Delta x$ m	$\Sigma\Delta t$ sec	$\frac{1}{2}g\Delta t^2$ m
At O	425	23711	36·93	0	0	0
,, A_1	358	28619	49·57	491	1·264	7·8
,, A_2	316	33734	64·87	1002	1·530	11·5
,, A_3	287	38830	81·78	1512	1·691	14·0
,, A_4	263	43858	100·07	2015	1·829	16·4
,, A_5	242	48730	119·39	2502	1·932	18·3
,, A_6	223	53655	140·61	2994	2·122	22·1
,, A_7	206	58750	164·40	3504	2·379	27·8
,, A_8	191	63961	190·69	4025	2·629	33·9
,, A_9	179	68725	216·47	4501	2·578	32·6
,, A_{10}	168	73624	244·74	4991	2·827	39·2

The results of the graphical solution are as follows:

Range $X=4501$ m,

Angle of descent $\omega=24°\ 53'\tfrac{1}{2}\ \left(\tan\omega=\dfrac{232}{500}\right)$,

Total time of flight $T=18{\cdot}0$ sec,

Final horizontal velocity $=179$ m/sec,

Final velocity $v=\dfrac{179}{\cos 24°\ 53'\frac{1}{2}}=197{\cdot}3$ m/sec,

Abscissa of the vertex $=2600$ m,

Ordinate ,, ,, $=412$ m.

Moreover the drawing can be measured directly to give the slope of the tangent, and the height of flight:

$x=$ 0 m	$\theta=\phi=15°\ 17'$	$\tan\theta=0{\cdot}2732$	$y=$ 0
491	$\theta=$ 12° 38′	112 : 500	126·0 m
1002	11° 31′	102 : 500	239·5
1512	8° 45′	77 : 500	329·5
2015	4° 55′	43 : 500	388·0
2502	0° 52′	7·5 : 500	411
2994	− 4° 50′	− 42·5 : 500	394·5
3504	− 11° 5′	− 98 : 500	325·5
4025	− 18° 0′	− 162·5 : 500	188·5
4501	− 24° 53′	− 232 : 500	0 to about 0·2 m

In illustration of the remarks on the insertion of additional points on the trajectory, the additional point P is constructed in the drawing between O_8 and O_9; O_8O_9 is drawn, bisected in Q, and P is the middle point of M_9Q.

(*b*) Example: $2R=24$ cm; $P=215$ kg; $v_0=640$ m/sec; δ, on the ground, $=1{\cdot}206$, $\phi=22°$, $i=1$, $\dfrac{1}{a}=0{\cdot}476$.

Allowing for the alteration of air density, we have

$$X = 13650 \text{ m}, \quad \omega = 27^\circ \, 38' \tfrac{1}{2}, \quad v_e = 307{\cdot}5 \text{ m/sec}, \quad T = 37{\cdot}3 \text{ sec}.$$

From calculation (Table, Vol. IV, no. 12*a*) the results obtained were

$$X = 14170 \text{ m}, \quad \omega = 33^\circ \, 36', \quad v_e = 323 \text{ m/sec}, \quad T = 38{\cdot}1 \text{ sec}.$$

(*c*) Example: $v_0 = 465$ m/sec, $\phi = 9^\circ \tfrac{7}{16}$, $\dfrac{1}{a} = 0{\cdot}1548$, $X = 4075$ m (observed 4000 m), $T = 12{\cdot}75$ sec (observed 13·0 sec), $v_e = 253$ m/sec (range table 255·5), $x_s = 2250$ m, $y_s = 214$ m.

2. *Problem.* Given the range X, initial velocity v_0, as well as the shape and mass of the shell, in P, $2R$, i. To find the angle of departure ϕ, and the other elements.

As before, a graphical solution is the best to employ, taking a provisional value of ϕ, selected by a comparison with the range table. A certain range X_1 is obtained, not agreeing exactly with the given X. But the trajectory is revolved or swung like a rigid line about the origin O, until the range becomes the given X. The angle $\Delta\phi$, through which the trajectory must be turned downward, must then be subtracted from the angle of departure ϕ_1 (or must be added). Thereby ϕ is obtained, and as in No. 1 the other results, all referred to the true range as abscissa axis.

Take the same numerical example, with $X = 4300$ m; to find ϕ.

The trajectory is first constructed with $\phi = 15^\circ \, 17'$, as was the case in the example above, when the range was found to be 4501 m. The trajectory is now turned about O till the range is 4300 m: thus a circular arc is described about a centre O with radius $OW_1 = 4300$ m, cutting the trajectory drawn already in W, and OW is drawn, which is the true abscissa axis.

The angle W_1OW or $\Delta\phi$, through which the turn is made, is given by $\tan \Delta\phi = \tfrac{97}{4000}$, $\Delta\phi = 1^\circ \, 23'$: and this angle is to be subtracted from the previous angle of departure.

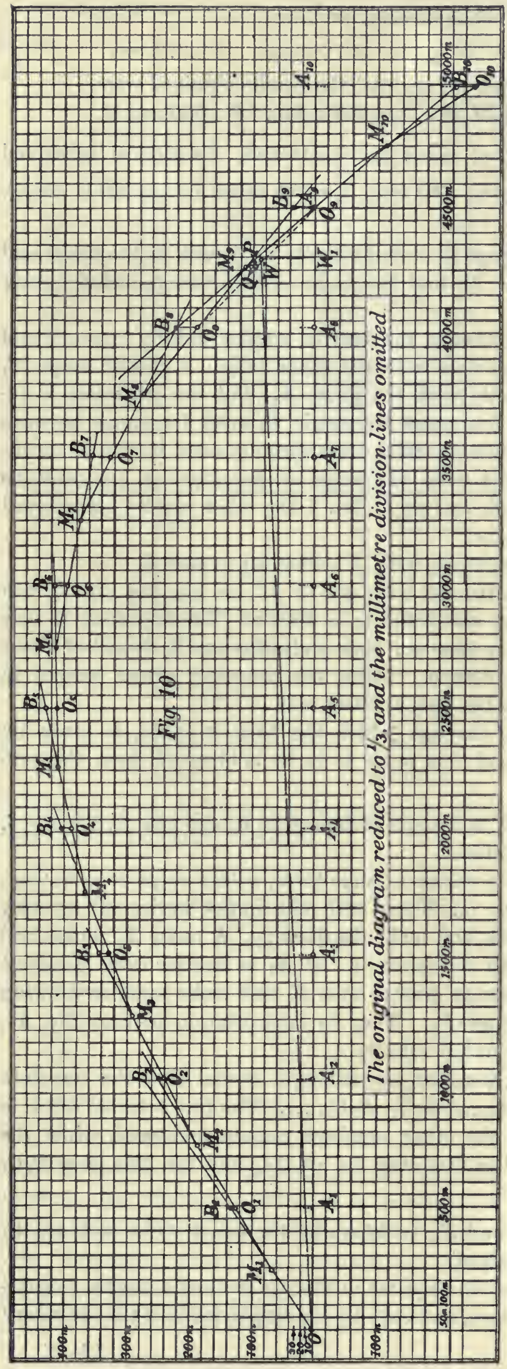
3. *Problem.* Given range X and angle of departure ϕ, and also P, $2R$, and i. To find v_0 and the other quantities.

The simplest procedure is to choose a range table, with a value of v_0 as close as can be found to the given v_0. Then with the initial horizontal velocity $v_0 \cos \phi$, employ Krupp's Tables, and make the drawing. A range X_1 is obtained thereby, but not identical with the given X. If X_1 is smaller than X, select another value of v_0, to give a range greater than X, and make another drawing which determines a second range X_2. Then by interpolation between X_1, X_2, and X, a value is found of the initial velocity v_0.

A saving of labour results from not calculating a second time the complete list of the values of OA_1, A_1A_2, ..., B_1O_1, B_2O_2, ... The first interval alone is increased or diminished, and the other numbers remain unaltered, while the horizontal velocity at O, and the first step Δx_1 or OA_1 and the first drop B_1O_1 are different.

4. *Problem.* Given v_0, ϕ, and X; to determine the factor a (for instance, to find i, when P, δ, and $2R$ are given).

This important problem, which requires in Siacci's method a double calculation and subsequent interpolation, and only by the employment of the secondary



functions can be carried out with ease, (but then solely for the case of a small angle of departure), requires here a double construction of the trajectory. The most convenient procedure, after comparison with a suitable range table, is to assume a value of a, to construct the trajectory, and to determine a value X_1 of the range; then assume another value of a and determine another X_2; then interpolate.

It is evident then that this graphical procedure is not suitable for the calculation of a range table, but it is advantageous for a case, where v_0, ϕ, a are known, and a series of intermediate points of the trajectory are required to be found.

And in problem (2) an investigation must be made as to whether the tilting of the trajectory is permissible or not.

The method should not be employed for an angle of departure over 30°.

The method can be considered to be a graphical development of the second group of methods of solution. It can be applied to each of the corresponding tables. So also σ and γ (§ 23) can here be chosen as desired.

Remarks.

1. The work of A. Indra (1886) on "Graphic Ballistics" is based on the following ideas:

In fig. 3 above, and in a vacuum, the points $O_1 O_2 O_3$ on the trajectory are the intersections of a pencil of rays OO_1, OO_2, OO_3, ... with the parallel rays D_1O_1, D_2O_2, D_3O_3, ... When the space is supposed to be filled with air, the focus of these parallel rays is supposed to move from an infinite to a finite distance. But there is no proof that this fanciful analogy has any real bearing on the question of air-resistance.

2. A graphical solution has been published by Dr Rothe, 1911, in which the relation between v and θ has been taken as the starting point (similar to the planimetric procedure in §§ 33 and 37). To the details of Rothe's solution some objections have been raised by H. Rohne and von Narath (see notes).

3. Graphical representation of the ballistic auxiliary functions.

Didion (1848) exhibited in diagrams in a convenient manner the functions arising in his analytical solution, and they have the advantage that by their use the interpolation is simplified. Since those days graphical representations in Mathematics and technical work have been developed, more especially by M. d'Ocagne, R. Mehmke, C. Runge, R. Rothe; and in ballistics these methods are valuable.

The representation of a function between two variables can be given often with advantage in "logarithmic" and other "scalar functions."

And functions of three variables can be represented in a family of curves on a coordinate system. But frequently it is better to represent it by a table. See Note (d'Ocagne, Mehmke, Schultz, Schrutka, Schilling, J. E. Mayer, Soreau, v. Pirani). The procedure was first applied in Ballistics by G. Pesci, G. Ronca, Garbasso, R. von Portenschlag-Ledermayr, A. Nowakowski. Nowakowski has lately described a method to determine the abscissa x of the trajectory corresponding to a given height of flight y, by means of the application of a logarithmic scale, to suit all trajectories of given v_0.

He has given a description of a Range Table, in which the trajectories are laid out on a curved surface, and this surface is cut up into lines of section by horizontal planes.

CHAPTER VI

Investigation of modern methods of calculation, and of their accuracy

§ **32.** The calculations of a trajectory according to the formulae in use are subject to a double error.

The first part of the error arises in that the Chief Equation in the ballistic problem cannot be integrated exactly, but is treated on a method of approximation, and this is different in the various systems.

The second part of the error arises from the fact that the function for the air resistance, comprising coefficient of form, density of air and its variation, etc., is not known with accuracy.

But at the present time it is not yet known which of these errors is the more serious.

The different approximate methods of calculation must therefore be tested, in order to see the magnitude of error arising from the method of integration, and also how far the errors in the integration counterbalance one another.

The modern methods of solution have the following in common.

As before, v denotes the velocity in the trajectory, θ the slope to the horizon of the tangent of the path, t the time of flight, $cf(v)$ the air-resistance retardation, at any point (xy) on the trajectory.

The exact Chief Equation of the problem is then

$$d\theta = \frac{g\,d\,(v\cos\theta)}{vcf\,(v)}, \text{ or } \frac{d\theta}{\cos^2\theta} = \frac{g\,d\,(v\cos\theta)}{v\cos\theta\; cf\left(\dfrac{v\cos\theta}{[\cos\theta]}\right)\left[[\cos\theta]\right]}.$$

As an approximation, the two values of $\cos\theta$, contained in the square brackets, are treated as constant along the trajectory, and replaced by their mean values, σ and γ respectively: so that

$$\frac{d\theta}{\cos^2\theta} = \frac{g\,d\,(v\cos\theta)}{v\cos\theta\; cf\left(\dfrac{v\cos\theta}{\sigma}\right)\gamma} = \frac{g\,d\left(\dfrac{v\cos\theta}{\sigma}\right)}{\dfrac{v\cos\theta}{\sigma}\; cf\left(\dfrac{v\cos\theta}{\sigma}\right)\gamma} = \frac{g}{\gamma c}\,\frac{du}{u\,f(u)},$$

where $u = \dfrac{v\cos\theta}{\sigma}$.

This differential equation between u and θ is integrable; the variables are separated. Therefore we obtain

$$x = \frac{\sigma^2}{\gamma c}(D_u - D_{u_0}); \ t = \frac{\sigma}{\gamma c}(T_u - T_{u_0});$$

$$\tan\theta = \tan\phi - \frac{1}{2c\gamma}(J_u - J_{u_0});$$

$$y = x\tan\phi - \frac{\sigma^2}{2c^2\gamma^2}[A_u - A_{u_0} - J_{u_0}(D_u - D_{u_0})],$$

where

$$J_u = -2g\int\frac{du}{uf(u)}, \ A_u = -\int\frac{J(u)\,u\,du}{f(u)}, \ D_u = -\int\frac{u\,du}{f(u)},$$

$$T_u = -\int\frac{du}{f(u)}, \quad u = \frac{v\cos\theta}{\sigma}, \ u_0 = \frac{v_0\cos\phi}{\sigma}.$$

This system of solutions, depending on the above mentioned general procedure, may be called the Modern System (compare Chapter V, 2nd group of solutions).

In all of them an error is present, because σ and γ have been assumed constant, whereas in reality they are variable along the trajectory.

They differ moreover in the choice of the value, more or less approximate, of σ and γ.

When the object is merely to settle the corresponding error, and to classify the systems according to the error, the comparison must be made on the same law of air resistance, including the form coefficient i and air density δ, and further the same angle of departure ϕ and initial velocity v_0; and then the investigation must determine the magnitude of the error in the neighbourhood of the vertex, and at the end of the trajectory.

Concerning the method of investigation, Cauchy's Law on the approximate solution of a differential equation must first be stated. Suppose a differential equation $dy = F(x, y)\,dx$, between the variables x and y. Starting from a given point $x_0 y_0$, take an arbitrary small increment Δx, and calculate the corresponding $\Delta y = F(x_0, y_0)\,\Delta x$; a second point $(x + \Delta x, y + \Delta y)$ is obtained on the curve.

Proceed from this point in the same way to another third point, and so on, building up in succession the integral curve as a polygon of small finite straight elements.

Then if ϵ is the greatest value of the different Δx increments, further if A, B, C denote the greatest numerical values that occur of F, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ respectively, between the starting and the end point of the corresponding part of the curve, and if n is the number of elements employed, then the error of y at the end of this part compared with the true value of y is always less than $\frac{B+AC}{2} \cdot \frac{(1+C\epsilon)^n - 1}{C} \epsilon$.

This procedure was employed by St Robert in an example with small initial velocity, where he started with the function $\frac{f(v)}{v^2}$, which changes only slowly; and he has thus obtained an upper limit to the error.

This method involves great labour if the accuracy is to be such as is required in the present case.

On this account the following method was introduced by the author (1909), in order to obtain several "Normal solutions" of the trajectory problem, for purpose of comparison.

Mayevski's Law of "Zones" was assumed as a basis; this takes a resistance function v^n, with n an integer, for velocities from $v = 550$ m/sec, downward. In such cases the Chief Equation can be integrated exactly: care must, however, be taken that the constants of integration are adjusted, so as to make the change continuous from one zone to another.

The relation between v and θ is thus known; in particular the vertex velocity v_s is known, where $\theta = 0$.

The functions of θ, $\frac{v^2}{g}$, $\frac{v^2}{g} \tan \theta$, $\frac{v \sec \theta}{g}$, can then be calculated in terms of θ. These functions, after they have been determined for a number of values of θ, are to be shown graphically on a large scale.

Finally the summation of

$$x = -\frac{1}{g}\int v^2 d\theta, \quad y = -\frac{1}{g}\int v^2 \tan\theta d\theta, \quad t = -\frac{1}{g}\int v \sec\theta d\theta,$$

is carried out either by a planimeter or integraph.

Since the probable error of measurement by a planimeter can be determined as a percentage for the actual curve by measuring a circle or square of similar area, the probable errors w_1, w_2, w_3,... of the individual portions of curved-line areas in the measurement are known.

The probable error of the sum of the parts of the area is then

$$\sqrt{(w_1^2 + w_2^2 + \ldots)} = w.$$

If a part is measured 10 times, the error of the final result, for example of x, or y, or t, is probably $\dfrac{w}{\sqrt{10}}$.

Six different normal trajectories have been calculated in this manner, on the basis of the same laws of air resistance.

The corresponding zone-laws are the following. The retardation $cf(v)$ of the air resistance is:

for $v = 550$—419 m/sec,
  retardation $= 0{\cdot}0394\, cv^2$;
for $v = 419$—375 m/sec,
  retardation $= 0{\cdot}\overset{(4)}{0}9404\, cv^3$;
for $v = 375$—295 m/sec,
  retardation $= 0{\cdot}\overset{(9)}{0}6709\, cv^5$;
for $v = 295$—240 m/sec,
  retardation $= 0{\cdot}\overset{(4)}{0}5834\, cv^3$;
for $v = 240$—0 m/sec,
  retardation $= 0{\cdot}014\, cv^2$.

where

$$c = \frac{R^2 \pi i g \delta}{P \,.\, 1{\cdot}206},$$

$2R$ = calibre in m,
i = form coefficient,
P = weight of shell in kg,
δ = air density in kg/m³,
$g = 9{\cdot}81$,
i a constant, taken $= 1$,
δ also constant.

These solutions were compared with the most important methods of approximation employed during the last 70 years (excluding the graphical methods, and those belonging to the first group, such as Bashforth for instance, and others).

These methods are characterised by the following differences:

1. $\sigma = \gamma = \dfrac{1}{\alpha}$; so that $u = \alpha v \cos\theta$, and then $\alpha = \dfrac{\xi(\phi)}{\tan\phi}$;

$$\xi(\phi) = \tan\phi \sec\phi + \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi),$$

(employed by Didion, Siacci 1880, and N. v. Wuich; but with different laws of air resistance).

2. Calculation of the trajectory in two parts: for the ascending branch, $\sigma = \gamma = \dfrac{1}{\alpha_1}$, where $\alpha_1 = \dfrac{\xi(\phi)}{\tan\phi}$; for the descending branch, $\sigma = \gamma = \dfrac{1}{\alpha_2}$, where $\alpha_2 = \dfrac{\xi(\omega)}{\tan\omega}$, ω = acute angle of descent (Didion, Mayevski, Siacci 1880, v. Wuich).

3. $\sigma = \gamma = \frac{1}{\alpha}$, where $\alpha = \dfrac{\xi \frac{\phi + \omega}{2}}{\tan \frac{\phi + \omega}{2}}$ (Didion, Mayevski, v. Wuich).

4. $\sigma = \gamma = \frac{1}{\alpha}$, where α is the arithmetic mean of the values of $\sec \theta$ at the point of departure, and at the vertex ($\theta = 0$); so that

$$\alpha = \tfrac{1}{2}(\sec \phi + 1)$$

(St Robert).

5. $\sigma = \gamma = \frac{1}{\alpha}$, where α is the geometric mean of the values mentioned above, so that $\alpha = \sqrt{(\sec \phi)}$ (Hélie).

6. $\sigma = \gamma = 1$, so that $u = v \cos \theta$ (simplified procedure of Siacci, employed also by F. Krupp).

7. $\sigma = \cos \phi$, $\gamma = \beta \cos^2 \phi$, so that $u = \dfrac{v \cos \theta}{\cos \phi}$; and therein

$$\beta = \frac{3}{2 \sin 2\phi f(V_0)} \int_0^\phi f\left(\frac{V_0 \cos \phi}{\cos \theta}\right)\left(1 + \frac{\tan^2 \theta}{\tan^2 \phi}\right) \frac{d\theta}{\cos \theta},$$

where V_0 is defined by $V_0 = \sqrt{\left(\dfrac{gX}{\sin 2\phi}\right)}$, and X is the horizontal range.

This is Siacci's procedure 1888 (Siacci II) and 1896 (Siacci III); in both these solutions the principle of compensation of the integration error is the same; different laws of air resistance are used.

8. The same as (7), but $\beta = 1$; and so

$$\sigma = \cos \phi, \quad \gamma = \cos^2 \phi, \quad u = \frac{v \cos \theta}{\cos \phi}.$$

9. $\sigma = \cos \phi$, $\gamma = \beta \cos^2 \phi$, and $u = \dfrac{v \cos \theta}{\cos \phi}$; where $\beta = \cos \frac{2}{3} \phi$ as a first approximation: it is then calculated more accurately through the relation

$$\beta \left[\frac{6f(v_0)}{v_0^4 \cos^2 \phi} + \frac{5f(u_s) \cos^2 \phi}{v_s^4}\right] = \frac{6f(v_0)}{v_0^4 \cos^3 \phi} + \frac{5f(v_s)}{v_s^4}:$$

v_0 = initial velocity, v_s = vertex velocity, u_s = value of u at the vertex, so that $u_s = \dfrac{v_s}{\cos \phi}$; $f(v)$ is the variable part of the function of air resistance, and is taken from the corresponding zone law.

With constant i and δ, this is Vallier's method (Vallier I).

10. $\sigma = \cos\phi, \gamma = \beta\cos^2\phi$, $u = \dfrac{v\cos\theta}{\cos\phi}$: and for β at first an approximate value, $\beta = \cos\frac{2}{3}\phi$: thence the elements are calculated, x_s, y_s, u_s, v_s at the vertex, as well as u_1, v_1, θ_1 at the point (x_1, y_1) of the trajectory, where the abscissa $x_1 = 0{\cdot}225\,x_s$; then β is recalculated from

$$\beta\left[\frac{9f(u_1)}{u_1^4\cos^2\phi} + \frac{4f(u_s)}{u_s^4\cos^2\phi}\right] = \frac{9f(v_1)}{v_1^4\cos^3\theta_1} + \frac{4f(v_s)}{v_s^4}.$$

This is another procedure of E. Vallier (Vallier II).

11. Procedure of Charbonnier for flat trajectories: First let $\sigma = \gamma = 1$, and $u = v\cos\theta$. Thence the acute angle of descent ω and the final horizontal velocity $v_{x_e} = v_e\cos\omega$ are calculated. Afterwards the calculation of the path proceeds in two parts. In the ascending branch c is replaced by $c(1 + \frac{1}{2}\kappa_0\tan^2\phi)$, in which

$$\kappa = \frac{1}{2}\left[v_x\frac{f'(v_x)}{f(v_x)} - 1\right];$$

v_x is the horizontal velocity, $v\cos\theta$, in the path; so that in the special case

$$\kappa_0 = \frac{1}{2}\left[v_{x_0}\frac{f'(v_{x_0})}{f(v_{x_0})} - 1\right], \quad v_{x_0} = v_0\cos\phi.$$

In the descending branch c is to be replaced by $c(1 + \frac{1}{2}\kappa_\omega\tan^2\omega)$, where

$$\kappa_\omega = \frac{1}{2}\left[v_{xe}\frac{f'(v_{xe})}{f(v_{xe})} - 1\right].$$

§ **33.** All these trajectory calculations were made with the same shell, with the same v_0, ϕ, i, δ, and with the same zone laws as above; the Tables for $D(u)$, $J(u)$, $T(u)$, $A(u)$, and β of Siacci II and III, are to be found in the *Lehre vom Schuss* of W. Heydenreich, 2nd edition, Berlin 1908.

The details of the calculations are not here given; only the first case is explained in full.

1. *Example.* Let the initial velocity $v_0 = 465$ m/sec; the angle of departure $\phi = 34\frac{1}{8}$ degrees, the weight of the shell $P = 6{\cdot}85$ kg, the calibre $2R = 0{\cdot}077$ m, the coefficient of form is constant, $i = 1$, the air density δ is constant and $= 1{\cdot}200$ kg/m^3.

First zone, from $v=v_0=465$ to $v=v_1=419$. Here the relation between v and θ is the following

$$c_1 v^2=\frac{g(1+p^2)}{B_1-p\sqrt{(1+p^2)}-\log[\sqrt{(1+p^2)}+p]},\quad p=\tan\theta,$$

$$c_1=\frac{(0{\cdot}077)^2\pi}{4}\times\frac{9{\cdot}81}{6{\cdot}85}\times\frac{1{\cdot}200}{1{\cdot}206}\times 0{\cdot}0394;\ \log c_1=\bar{4}{\cdot}41738,$$

$$B_1=\tan\phi\sqrt{(1+\tan^2\phi)}+\log[\sqrt{(1+\tan^2\phi)}+\tan\phi]+\frac{g(1+\tan^2\phi)}{c_1 v_0^2}=1{\cdot}74052;$$

so that at $v=v_1=419$, we have $\theta=\theta_1=33°46'$.

We are here at the end of the first and beginning of the second zone.

Second zone, from $v=v_1=419$ to $v=v_2=375$. Here v and θ are connected by the relation

$$\frac{1}{v^3\cos^3\theta}=-\frac{3c_2}{g}\left(\tan\theta+\frac{1}{3}\tan^3\theta\right)+B_2;$$

$$c_2=\frac{(0{\cdot}077)^2\pi}{4}\times\frac{9{\cdot}81}{6{\cdot}85}\times\frac{1{\cdot}200}{1{\cdot}206}\times 0{\cdot}\overset{(4)}{0}9404;\ \log c_2=\bar{7}{\cdot}79519;$$

B_2 is determined, then, from the relation $\theta=33°\,46'$, $v=419$; and so $B_2=0{\cdot}\overset{(6)}{0}170269$.

When $v=v_2=375$, we find $\theta=\theta_2=32°\,35'$; this is the beginning of the third zone.

Third zone: $$\frac{1}{v^5\cos^5\theta}=-\frac{5c_3}{g}\left(\tan\theta+\frac{2}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta\right)+B_3,$$

and there $\log c_3=\overline{12}{\cdot}64854$; and at the beginning of the third zone, $B_3=0{\cdot}\overset{(11)}{0}2211$.

For $v=v_3=295$, $\theta=\theta_3=27°\,21'\,50''$: this is the beginning of the fourth zone.

Fourth zone: $\dfrac{1}{v^3\cos^3\theta}=-\dfrac{3c_4}{g}\left(\tan\theta+\dfrac{1}{3}\tan^3\theta\right)+0{\cdot}\overset{(6)}{0}122358$, and $\log c_4=\bar{7}{\cdot}58785$:

and for $v=v_4=240$, $\theta=\theta_4=17°\,36'$; and this is the beginning of the last zone.

Fifth zone: $$c_5 v^2=\frac{g(1+p^2)}{2{\cdot}66310-p\sqrt{(1+p^2)}-\log[\sqrt{(1+p^2)}+p]},$$

$p=\tan\theta$; $\log c_5=\bar{5}{\cdot}96801$: and at $\theta=0$, $v=v_s=199{\cdot}13$ (vertex velocity); at $\theta=-15°$, $v=187{\cdot}91$: at $\theta=-49°{\cdot}5$, $v=214{\cdot}06$ (near the end point of the horizontal range).

The function $\dfrac{v^2}{g}\tan\theta$ must now be calculated for a great number of values of the angle θ, and drawn graphically on a large scale in several parts, from $\theta=\phi=34°\,37'{\cdot}5$ to $\theta=-50°$, with θ for abscissa, and $\dfrac{v^2}{g}\tan\theta$ as ordinate.

The area of the curve, $y=-\displaystyle\int\frac{v^2}{g}\tan\theta\,d\theta$, is then to be measured by the planimeter, in four parts. In the first three parts, one square cm represents 1·1636 m, in the last part one square cm represents 3·4907 m. The curved boundary of each part is measured 10 times by the planimeter. At first the planimeter measurement is to be carried out up to the point where $\theta=-49°{\cdot}5$, in the neighbourhood of the point of descent to the muzzle horizontal; here the ordinate y is still 34·1 m: afterwards to $y=0$ where $x=X$.

As to the probable error in the determination of y, derived from the square of the errors, this had the value $\pm 0{\cdot}13\,\%$: the probable error of y as far as

$\theta = 18° 17'$ has the magnitude $\pm 0{\cdot}739$ m; to $\theta = 0$ (vertex) the magnitude $\pm 0{\cdot}803$ m: and to $\theta = -49°{\cdot}5$, the magnitude $\pm 1{\cdot}71$ m.

So too $\frac{v^2}{g}$ is to be drawn as a function of θ: the integral $x = -\int \frac{v^2}{g} d\theta$ is evaluated mechanically as far as $\theta = -49°{\cdot}5$, and also to the muzzle horizon.

The measurement by planimeter was made in 5 parts: in the first three parts, 1 square cm of the drawing sheet denoted $0{\cdot}58178$ m; in the 4th part, 1 square cm $= 1{\cdot}16355$ m; in the 5th part, 1 square cm $= 3{\cdot}49070$ m.

As far as $\theta = 18° 17'$, $x = 2765{\cdot}7 \text{ m} \pm 0{\cdot}288$ m; up to $\theta = 0$ (vertex),

$$x = x_s = 4308{\cdot}0 \text{ m} \pm 0{\cdot}370 \text{ m};$$

and on to $\theta = -49°{\cdot}5$, $x = 7634{\cdot}8 \text{ m} \pm 0{\cdot}743$ m.

Analogous procedure for t:

$$\text{to } \theta = 18° 17', \quad t = 10{\cdot}001 \pm 0{\cdot}00227 \text{ sec};$$

$$\text{to } \theta = 0, \quad t = t_s = 17{\cdot}2117 \pm 0{\cdot}00235 \text{ sec};$$

$$\text{to } \theta = -49°{\cdot}5, \quad t = 37{\cdot}0407 \pm 0{\cdot}00381 \text{ sec}.$$

In this way the calculation of this trajectory was carried out as a specimen normal trajectory, since the probable extent of the errors of x, y, t is known sufficiently exactly, and since v has been determined as a function of θ.

The values of x, t, y, v are found for the same angle $\theta = -49°{\cdot}5$, according to the various methods of approximation, as well as the elements at the vertex, x_s, y_s, t_s, v_s.

For instance the original method of Krupp may be taken, in which ($\sigma = \gamma = 1$):

$$x = \frac{1}{c}(D_u - D_{u_0}), \quad t = \frac{1}{c}(T_u - T_{u_0}), \quad \tan\theta = \tan\phi - \frac{1}{2c}(J_u - J_{u_0}),$$

$$y = x\tan\phi - \frac{1}{2c^2}[A_u - A_{u_0} - J_{u_0}(D_u - D_{u_0})], \quad u = v\cos\theta, \quad u_0 = v_0\cos\phi;$$

c is the constant factor in the retardation of the air resistance (and in the employment of the Tables of J, A, D, T in Heydenreich's *Lehre vom Schuss*,

$$c = \frac{(2R)^2\, 1000\, \delta}{1{\cdot}206\, P}, \quad \log c = \bar{1}{\cdot}93512).$$

As $\theta = -49°{\cdot}5$, the third equation determines u and J_u, the first then determines x, the second t, the fourth y, and the fifth v.

Finally, the calculations are extended to $y = 0$, i.e., to the muzzle horizon, and to $x = X$, by trial-and-error methods. (Calculation and measurement carried out by J. Schatte.)

The 12 trajectories, found by this method, differ merely in the procedure that is employed. The same laws of air resistance are assumed, the same calibre $2R$, weight of shell P, form coefficient i, air density δ, angle of departure ϕ, and initial velocity v_0.

According to Table I on p. 207, these 12 trajectories can be compared in three points: first at the vertex, where $\theta = 0$; next at the point where the descending slope is the same, $\theta = -49°{\cdot}5$, lying in the descending branch near the point of descent: thirdly at the actual point of descent on the horizontal plane through the muzzle, where $y = 0$.

2. *Example.* $P=6{\cdot}9$ kg, $2R=0{\cdot}077$ m, $v_0=550$ m/sec, $\delta=1{\cdot}206$ kg/m³, $i=1$, $\phi=20°$.

In Table II the results are given for the vertex velocity v_s, abscissa x_s, ordinate y_s, time of flight t_s: further the elements v, x, y, t for a point near the point of fall, where $\theta=-31°\ 10'$: lastly the final velocity v_e, acute angle of descent ω, range X, time of flight T for the point of descent, where $y=0$.

All these elements refer to the "normal trajectory," and to all trajectories calculated by the various methods for the same law of air resistance.

The measurement by planimeter for y and x was made in 7 parts.

3. *Example*, as before, but with $\phi=45°$ (Table III). Planimeter measurement in 14 parts.

4. *Example*, as before, but with $\phi=70°$ (Table IV).

The first zone ($v=550$ to 419) reached to $\theta=69°\ 12'\ 56''$; the second zone ($v=419$ to 375) ended at $\theta=68°\ 44'\ 48''$; the third ($v=375$ to 295) at $\theta=66°\ 50'\ 50''$; the fourth ($v=295$ to 240) at $\theta=63°\ 51'\ 48''$; the fifth ($v=240$ and less) to $\theta=-77°\ 17'\ 5''$; this zone contains the vertex and the point of minimum velocity, $v=82{\cdot}398$ m/sec at $\theta=-4°$; the sixth zone ($v=240$ to 295) reached from $\theta=-77°\ 17'\ 5''$ up to the end.

The summation of the values of dy by planimeter was carried out in the ascending branch in 21, and in the descending branch in 18 steps: and for abscissa, θ, the scale was 1° to 6 cm; for ordinate, $\frac{v^2}{g}\tan\theta$, 1000 m to 5 cm.

Summation for x in $15+17$ parts, to the same scale as that of the drawing. Summation of the t values in $34+30$ parts; scale of abscissa, θ, 1° to 6 cm; scale of ordinate, $\frac{v}{g\cos\theta}$, one second to 5 cm.

This example was chosen for $\phi=70°$ because the Siacci β tables in his *Ballistik* (1892) extend from an initial angle of 60°, and this was extended to 70° in the *Lehre vom Schuss* of W. Heydenreich (1908, Tables p. 32): and therefore the application of the method to such steep trajectories was obviously considered.

Two other examples were worked out, relating to the same initial velocity $v_0=550$ m/sec, but to shells of greater weight.

5. *Example.* $P=41$ kg, $2R=0{\cdot}15$ m, $\delta=1{\cdot}206$, $i=1$, $\phi=45°$; planimetric operations in $10+7$ parts. Minimum velocity, $v=189{\cdot}49$ m/sec, was at $\theta=-12°\ 30'$.

6. *Example.* $P=82$ kg, $v_0=550$ m/sec, $2R=0{\cdot}21$ m, $\delta=1{\cdot}206$, $i=1$, $\phi=20°$.

The corresponding series of figures show that the accuracy of the "normal" trajectories is satisfactory.

Comparing, however, the different methods of solution, contained in 2 to 12 for the range X, final velocity v_e, angle of descent ω, time of flight T, height of vertex y_s, &c., with the corresponding figures of the "normal" trajectory 1, and expressing the difference as a percentage of the normal trajectory, we find for instance in y_s an extreme error of 13 °/ₒ, and in the range of about 29 °/ₒ with $\phi=45°$; and so on.

TABLE I

Nature of solution	Vertex of the Trajectory				Point of Trajectory with constant downward slope $\theta=-49°\cdot5$ of the tangent; point in the neighbourhood of the point of descent in the descending branch				Point of descent $y=0$
	Vertex velocity v_s in m/sec	Vertex abscissa x_s in m	Vertex ordinate y_s in m	Time of flight t_s in sec	Velocity v m/sec	Abscissa x m	Ordinate y m	Time t sec	Total range X m
1. Normal solution	199·13 ±0	4308·0 ±0·37	1754·8 ±0·80	17·2117 ±0·0023	214·06 ±0	7634·8 ±0·74	+ 34·1 ±1·71	37·041 ±0·0038	7663·8 ±0·7
2. $a=\xi(\phi):\tan\phi$	202·5	4590·4	1878·6	17·727	221·7	8001·5	+100·2	37·804	8085·2
3. Ascending branch $a=\xi(\phi):\tan\phi$ Descending „ $a=\xi(\omega):\tan\omega$	202·5	4590·4	1878·6	17·727	215·7	7893·9	+175·1	37·477	8040·0
4. $a=\xi\left(\frac{\phi+\omega}{2}\right):\tan\left(\frac{\phi+\omega}{2}\right)$ ...	—	—	—	—	216·4	7624·7	+113·2	36·867	7719·0
5. $a=\frac{1}{2}(\sec\phi+1)$	198·8	4445·7	1825·2	17·431	218·0	7738·6	+110·5	37·154	7829·6
6. $a=\sqrt{(\sec\phi)}$	199·4	4467·1	1833·3	17·480	218·0	7779·1	+108·1	37·261	7869·2
7. Siacci I, simplified	211·4	4953·6	2010·4	18·441	230·7	8656·3	+ 84·2	39·351	8727·0
8. Siacci II & III; β from the Table	203·3	4532·5	1864·4	17·599	230·9	8122·5	− 56·7	38·256	8090·7
9. The same, $\beta=1$	197·6	4345·2	1787·9	17·222	222·7	7699·5	+ 22·6	37·143	7718·1
10. Vallier I	192·5	4187·9	1726·5	16·888	214·6	7332·7	+ 83·0	36·180	7404·0
11. Vallier II	199·5	4405·6	1813·4	17·344	225·8	7839·2	+ 0·3	37·504	7839·5
12. Charbonnier	204·4	4748·5	1941·3	18·046	210·1	7985·9	+295·8	37·547	8224·0

($v_0=465$ m/sec, $P=6\cdot85$ kg, $2R=0\cdot077$ m, $\delta=1\cdot200$ kg/m^3, $i=1$, $\phi=34°\,{}^{10}_{18}$).

TABLE II

Nature of solution	Vertex of the Trajectory				Point of Trajectory ($\theta = -31° 10'$) near the point of descent				Point of descent ($y=0$)			
	v_s in m/sec	x_s in m	y_s in m	t_s in sec	v	x	y	t	v_e	ω	X	T
1. Normal solution	254·8	3878·3 ±0·34	853·2 ±0·2	11·7791 ±0·00050	222·65	6846·4 ±0·4	+40·7 ±0·3	25·263 ±0·00053	222·8	31° 55′·2	6912·9	25·613$_5$
2. $a=\xi(\phi):\tan\phi$	254·8	3919·6	865·0	11·831	224·2	6883·7	+52·7	25·300	224·6$_5$	32° 7′·3	6969·2	25·742
3. $a=\xi(\phi):\tan\phi$ $a=\xi(\omega):\tan\omega$	254·8	3919·6	865·0	11·831	222·0	6848·9	+63·5	25·228	222·3	32° 23′·3	6948·6	25·831
4. $a=\xi\left(\frac{\phi+\omega}{2}\right):\tan\left(\frac{\phi+\omega}{2}\right)$	252·3	3869·5	856·5	11·747	222·2	6776·1	+60·2	25·082	222·5	32° 16′·5	6859·7	25·572
5. $a=\frac{1}{2}(\sec\phi+1)$	253·1	3885·7	859·4	11·774	222·8	6810·7	+57·0	25·152	223·3	32° 12′·6	6903·0	25·641
6. $a=\sqrt{(\sec\phi)}$	253·2	3886·5	859·3	11·777	222·8$_5$	6814·1	+57·0	25·161	223·3$_5$	32° 11′·4	6904·3	25·638
7. Method Siacci I, simplified	258·4	3996·4	879·1	11·957	227·1	7042·0	+44·6	25·608	227·5	31° 57′·0	7114·2	25·986
8. Siacci II and III: β from the Table	257·0	4003·6	886·8	11·954	229·5	7061·0	+42·0	25·642	229·9	31° 53′·5	7129·3	25·986
9. The same, $\beta=1$	252·0	3874·2	860·3	11·750	223·3	6790·7	+57·6	25·115	223·8	32° 11′·8	6885·0	25·610
10. Vallier I	252·4	3888·3	863·4	11·773	223·9	6815·7	+57·0	25·158	224·5	32° 10′·6	6908·4	25·645
11. Vallier II...	250·9$_5$	3852·6	855·0	11·714	222·1	6744·7	+59·7	25·030	222·8	32° 16′·6	6842·1	25·546
12. Charbonnier	256·4	3946·0	869·8	11·879	221·8	6896·4	+64·2	25·313	221·9	32° 23′	6995·8	25·858

($v_0=550$, $P=6·9$, $2R=0·077$, $\delta=1·206$, $i=1$, $\phi=20°$).

TABLE III

Nature of solution	Vertex of the Trajectory				Point of Trajectory ($\theta = -62° 10'$) near the point of descent				Point of descent ($y = 0$)			
	v_s in m	x_s in m	y_s in m	t_s in sec	v	x	y	t	v_e	ω	X	T
1. Normal solution	168·6	4776·1 ±0·31	2900·0 ±0·90	21·53 ±0·0011	229·5	8378·5 ±0·57	0	47·67 ±0·0020	229·5	62° 10′	8378·5	47·67
2. $a = \xi(\phi) : \tan\phi$	173·9	5393	3299·4	22·97	248·7	9184	+174·8	49·82	251·8	62° 49′·7	9276	50·60
3. $a = \xi(\phi) : \tan\phi$, ascending branch; $a = \xi(\omega) : \tan\omega$, descending ,,	173·9	5393	3299·4	22·97	232·7	8915	+479·8	48·79	244·8	64° 32′·5	9154	51·00
4. $a = \xi\left(\frac{\phi+\omega}{2}\right) : \tan\left(\frac{\phi+\omega}{2}\right)$	164·9	4913	3040·5	21·80	$237·1_5$	8341	+210·5	47·40	242·4	63° 11′·3	8448	48·36
5. $a = \frac{1}{2}(\sec\phi + 1)$	168·6	5109	3139	22·30	242·0	8687	+193·0	48·40	246·9	63° 5′·5	8786	49·28
6. $a = \sqrt{(\sec\phi)}$	170·2	5189	3187	22·49	244·4	8831	+191·5	48·80	248·8	63° 4′·5	8932	49·70
7. Siacci's method I, simplified ...	188·6	6263	3786	24·80	267·9	10688	+166·4	53·80	271·1	62° 50′·3	10774	54·50
8. Siacci II and III : β from the Table	172·5	5189	3213	22·45	262·0	9173	−134·2	50·01	258·0	61° 30′	9100	49·40
9. The same, $\beta = 1$	169·1	5040	3117	22·12	254·6	8840	− 73·1	49·09	252·8	61° 50′·8	8803	48·76
10. Vallier I	162·3	4754	2952	21·45	241·5	8217	+ 76·4	47·14	243·5	62° 32′·7	8259	47·50
11. Vallier II	161·2	4684	2912	21·30	238·1	8063	+103·7	46·68	240·9	62° 41′·4	8116	47·16
12. Charbonnier (flat trajectory method)	174·5	5680	3491	23·01	Method of division into arcs; calculation in 4 parts				—	—	9297	60·30

($v_0 = 550$, $P = 6·9$, $2R = 0·077$, $\delta = 1·206$, $i = 1$, $\phi = 45°$).

TABLE IV Nature of solution	Vertex of the Trajectory				Point of Trajectory ($\theta = -78° 44'$) near the point of descent				Point of descent ($y=0$)			
	v_s	x_s	y_s	t_s	v	x	y	t	v_e	ω	X	T
1. Normal solution	82·6	2912·0 ±0·13	4831·9 ±0·11	27·76 ±0·00024	251·7	5232·3 ±0·17	+35·3 ±0·22	61·60 ±0·0004	253·4	78° 47′·37	5239·3 ±0·17	61·70 ±0·0004
2. $a=\xi(\phi)$: $\tan\phi=1·7777$ ...	89·5	4135·2	6976·6	33·18	298·8	6727·6	+1394	69·50	328·8	80° 11′·9	6982·1	73·9
3. (a) $a=\xi(\phi)$: $\tan\phi=1·7777$ (b) $a=\xi(\omega)$: $\tan\omega=2·8013$ ...	89·5	4135·2	6967·6	33·18	$259·4_5$	6269·6	+2459·5	65·30	312·9	80° 40′·2	6749·3	75·0
4. $a=\xi\left(\frac{\phi+\omega}{2}\right)$: $\tan\left(\frac{\phi+\omega}{2}\right)$ $=2·1388$	81·0	3420·3	5817·4	30·24	271·5	5553	+1190·5	63·00	300·6	80° 17′·1	5778·6	67·0
5. $a=\frac{1}{2}(\sec\phi+1)=1·9619$ ...	85·0	3754·1	6358·1	31·72	284·1	6095	+1306	66·00	314·1	80° 15′·9	6338·0	70·5
6. $a=\sqrt{(\sec\phi)}=1·7099$	91·8	4242·1	7197·8	33·52	304·6	6974	+1391	70·70	Tables do not apply			
7. Siacci I, simplified	111·7	5619·6	9015	39·21	Tables do not apply							
8. Siacci II and III : β from the Table $=1·519$	$82·3_5$	3270·6	5561·3	29·55	299·7	5689	+157·5	64·50	303·7	78° 55′·9	5721·3	65·1
9. The same, $\beta=1$	91·7	3876·4	6499·5	32·29	348·8	7009·4	−606·5	72·20	334·9	78° 7′·1	6883	70·3
10. Vallier I, $\beta=1·9391$	76·9	2945·4	5049·8	27·96	$272·3_5$	4989·7	+537	60·01	285·7	79° 27′·6	5089·6	62·0
11. Vallier II, $\beta=1·9953$	76·2	2908·7	4994·8	27·78	269·2	4911·1	+574	59·59	283·7	79° 31′·7	5019·0	61·6
12. Charbonnier	Flat trajectory method not applicable											

($v_0=550$, $P=6·9$, $2R=0·077$, $\delta=1·206$, $i=1$, $\phi=70°$).

These errors arise entirely in the procedure of integration, and can be very serious.

So far as the ballistic quantities are concerned, by which the merit of the system of calculation is to be measured, we may assume that the total range X and the height of vertex y_s are those that serve best; because these are the most important quantities in practice, and moreover exhibit the greatest percentage of error, the trajectory being very "susceptible" to change in regard to range and height of vertex.

This susceptibility is much less in respect of the angle of descent ω: thus for example, when $2R = 7{\cdot}7$ cm and $\phi = 45°$ (Example 3) the absolute values of the ω errors according to the successive methods are 1·1, 3·8, 1·6, 1·5, 1·5, 1·1, 1·1, 0·5, 0·6, 0·8 per cent.; and at $\phi = 70°$, 1·8, 3·7, 1·9, 1·9, —, —, 0·2, 0·9, 0·9, 0·9 per cent.

So too the numbers for the time of flight T, at $\phi = 45°$, give the following differences with respect to the "normal" solution: 6·2, 7·0, 1·4, 3·4, 4·3, 14·3, 3·6, 0·2, 0·3, 1·1 per cent.; and for the final velocity v_e for $\phi = 45°$, and $2R = 7{\cdot}7$ cm, respectively 9·7, 6·7, 5·6, 7·6, 8·4, 18·4, 12·8, 10·1, 6·1, 5·0 per cent.

On this account only the percentage errors for the total range X and the vertex height y_s will be worked out for the different methods of solution, and from this will be found the average percentage of error, that is the sum of the absolute values divided by the number under consideration ($n = 6$, or 5, or 4; some do not allow of calculation, because the Siacci Tables, in *Lehre vom Schuss* of W. Heydenreich, do not go far enough).

This method of averaging is suitable to the case, because in practice the same formulae are mostly employed for very different values of angle of departure and weight of shell. (Obviously, instead of the average error, the mean quadratic error can be employed as an absolute measure of the mean error.)

On this account the method of Vallier (France) is better than the others for adjusting the errors of integration; and then follows the Wuich method, employed in Austria, which represents a slight modification of the Didion method.

But the simplified method of Siacci I with $\alpha = 1$ is quite useless, concerning which P. Charbonnier has correctly remarked, that it must always give too long trajectories: it is seen that on this method, with an angle of departure 45°, the range will be 29 °/₀ too long.

Both modifications (Vallier I and II) of Vallier's procedure are about equal in accuracy. In practice the simpler formula of Vallier I will be taken, and β is then calculated from the equation

$$\beta\left\{6\frac{f(v_0)}{v_0^4}+5\frac{f(u_s)}{u_s^4}\right\}\sec^2\phi=6\frac{f(v_0)}{v_0^4}\sec^3\phi+5(1-\lambda y_s)\frac{f(v_s)}{v_s^4},$$

v_0 the initial velocity, v_s the vertex velocity, $u=\dfrac{v\cos\theta}{\cos\phi}$; $u_s=v_s\sec\phi$, ϕ angle of departure, $f(v)$ the part of the expression for the air resistance which is a function of the velocity v of the shell; y_s vertex height; and the function $1-\lambda y_s$ in place of 1 represents the alteration of air density in consequence of the height, $\lambda=0{\cdot}00008$ according to St Robert and Vallier, $\lambda=0{\cdot}00011$ on Charbonnier's calculations.

Compared with the Siacci β table, which is more convenient for use than this formula for β, the formula has the advantage of generality, and of greater accuracy.

In the employment of this formula even with very steep high angle trajectories, the error can still be kept within moderate bounds. With $v_0=500$ m/sec, $2R=3{\cdot}7$ cm, $P=680$ gr, $i=1$, $\delta=1{\cdot}206$ on the ground, $\phi=80°$, the trajectory was calculated planimetrically, taking into account the alteration of air density; it was found that

$$y_s=3571{\cdot}2\text{ m},\quad v_s=34{\cdot}51\text{ m/sec}.$$

On the other hand, a single application of the formula gave $y_s=4164$ m (error of 16 °/$_\circ$), $v_s=34{\cdot}4$ m/sec (an error of 0·3 °/$_\circ$).

At $\phi=75°$, the normal solution gave

$$x_s=1518{\cdot}1\text{ m},\quad y_s=3439{\cdot}5\text{ m};$$

while the formula gave

$$x_s=1742\text{ m},\qquad y_s=4138\text{ m}.$$

Usually in practice the formula for β will be employed for an angle of departure not exceeding 50°.

The result can be expressed in the following manner:

The best value of β is found when the ballistic coefficient c', and the β implied in it, are determined experimentally; this is the case for instance, when, besides v_0, ϕ, ($2R$, P, δ and i), the range X or the time of flight T, or the final velocity v_e, or the angle of descent ω is known and thence c' is determined.

But then in all cases, where we must obtain the β value theoretically, the greatest accuracy is obtained on the average, if the Vallier system of formulae is employed.

Still this conclusion is not necessarily true in all cases, because the number of calculated normal trajectories was not very large: moreover the conclusion relates only to initial velocities of about 500 m/sec: and Siacci has calculated his β tables, in 1892 and 1896, on the basis of the same integration formula, but with different methods of approximation.

It is desirable then that other calculations with different zone-laws, different initial velocities, and different weapons, should be carried out, and also with different methods of calculation, especially those of Sabudski, Charbonnier, Siacci-Parodi, and others.

§ **34.** In this way the best methods will be discovered. But we must now consider the choice of the most appropriate law of air resistance.

In 1909 the only modern laws were: the Laws of the Zones of Chapel-Vallier-Hojel (Law I), the Zone-Laws of Mayevski-Sabudski (Law II), and the new monomial law in Siacci III (Law III).

The method of test is the following: the calculation is carried out by the best method on the basis of the three different laws of air resistance: the calculated ranges are then to be compared with the observed ranges.

Given then the initial velocity v_0 m/sec, the weight of the shell P kg, the calibre $2R$ m, the air density δ on the ground, the form of head (ogival, two calibre radius of rounding) and the ϕ angle of departure. Thence the range X is to be calculated. This was done with the formula of Vallier, viz., Vallier II. In the first place β was put $=\cos\frac{2}{3}\phi$, and thence u_s, v_s, y_s, as well as x_1, y_1, u_1, v_1, θ_1 were calculated; then the formula Vallier II gave a closer value of β. Then we must consider the diminution of air density with the height, because we have now to compare the three laws of air resistance with the actual air resistance, which varies with the air density as it changes with the height.

Then β will be calculated from

$$\beta\left[9\frac{f(u_1)}{u_1^4}\sec^2\phi+4\frac{f(u_s)}{u_s^4}\sec^2\phi\right]$$
$$=i\left[9\frac{f(v_1)}{v_1^4}(1-\lambda y_1)\sec^3\theta_1+4\frac{f(v_s)}{v_s^4}(1-\lambda y_s)\right],$$

and λ is taken from Vallier's value: $i=1$ is assumed in Laws I and II; but in Law III, according to Siacci, $i=0{\cdot}896$ for ogival shell of 2 calibre rounding.

The results of four such calculations are given in the following Table, in which ϕ lies between 6° and 36°.

Range X in m	By observation; error in m	Error percentage
Calculated according to Law I: 4061·2	−49	−1·2
" " " " II: 4049·2	−61	−1·5
" " " " III: 4173·0	+63	+1·5
Calculated according to Law I: 4919·1	−92	−1·8
" " " " II: 4959·1	−52	−1·0
" " " " III: 5064·8	+54	+1·1
Calculated according to Law I: 6369·7	−78	−1·2
" " " " II: 6472·3	+24	+0·4
" " " " III: 6596·4	+148	+2·3
Calculated according to Law I: 7669·4	−156	−2·5
" " " " II: 7773·2	−52	−0·7
" " " " III: 7870·3	+45	+0·6

In these examples, the results of those worked on the Law I of Chapel-Vallier-Hojel required to be increased (assuming the accuracy of the observations), but on the Law III of Siacci, the results gave ranges too great.

The results are not so very different, and the errors are relatively small.

We have to reckon with an error arising from the integration procedure of 5·6 %, when the calculation is made on the system of Siacci II (*Lehre vom Schuss* of Heydenreich); so that in this case the error, arising from the β Table, was greater than that arising from the air resistance law.

The choice of the law of air resistance in the three specific instances is possibly of somewhat slighter importance than the choice of the system of calculation. The three laws of air resistance are good enough to employ in practical work; still they need improvement.

An improvement can be reached as a preliminary, when something else is chosen for i instead of unity, in Law I, as well as in Law III, for the normal shape of shell.

The problem with the β of Vallier can thus be solved on the basis of Law I as well as Law II, and the mean taken; later on, however, the question of experiments on air resistance must be considered.

Remark. The preceding holds merely for guns of the calibre mentioned. The statement is not to be considered to hold good for rifle bullets. The calculations relating to such cases have not been attempted, as no experiments are known to the author, in which the values of ϕ, v_0, X and i are trustworthy (for ϕ especially, in consequence of the error of departure and jump).

As soon as the new Krupp-Eberhard values of air resistance (§ 10) are used for the construction of the primary and secondary functions, it will obviously be desirable to use them for such problems.

CHAPTER VII

The high angle trajectory. The method of swinging the trajectory

I. Calculation of a high angle trajectory. Vertical fire.

§ 35. Motion of a shell in a vertical line.

Suppose a shell projected from O with initial velocity v_0 vertically upwards; the velocity of the shell, under the influence of gravity and air resistance, will diminish more and more, and after a certain time t_1 will be zero; and the shell will have reached its maximum height Y.

After that it begins to fall again with a zero initial velocity; the velocity increases and approaches more and more to a limiting value v_1, determined by the equality of air resistance and gravity, so that the movement tends to become uniform. On the other hand, however, the velocity of a meteoric stone, starting from space with very much greater velocity, over 30,000 m/sec on the average, and penetrating the atmosphere and striking the Earth, diminishes much more and will approach such a velocity v_1 as an inferior limit.

Before the shell can reach this superior limit v_1 of its velocity, it strikes the ground again after t_2 seconds, reckoned from the highest point, and its fall is Y and its velocity may be denoted by v_e.

The motion of the shell must be calculated separately for the ascent and descent, as these two parts of the motion are not symmetrical: on the contrary in the first part of the ascent, air resistance and gravity act in the same direction, both retarding the motion; in the second part of the descent, air resistance and gravity act in opposite directions, the air resistance retarding, but gravity accelerating the motion.

Ascending Motion.

Let the coordinate y be reckoned positive upward from the origin O.

Suppose the shell to have reached a height y above O after t

seconds, and to have velocity v; denote the retardation of air resistance by $cf(v)$; then the differential equation of motion is

$$\frac{dv}{dt} = -g - cf(v), \quad \text{......................(1)}$$

and thence

$$t = -\int_{v_0}^{v} \frac{dv}{g + cf(v)}, \quad \text{......................(2)}$$

from which the velocity v can be calculated after any time t.

The whole time t_1 of the ascending motion is given by

$$t_1 = +\int_0^{v_0} \frac{dv}{g + cf(v)}. \quad \text{......................(3)}$$

When the value of $v = \frac{dy}{dt}$ is obtained in (2) as a function of t, and integrated again, from $t = 0$, $y = 0$, the value of the instantaneous y at any time t is obtained; and substituting the value of t_1 from (3), we get the maximum height Y.

The variable in (1) is changed from t to y by the relations $dy = v\,dt$, $v\frac{dv}{dy} = -g - cf(v)$; then

$$y = -\int_{v_0}^{v} \frac{v\,dv}{g + cf(v)}, \quad \text{......................(4)}$$

where v is the velocity at any height y of the shell above the ground; then

$$Y = +\int_0^{v_0} \frac{v\,dv}{g + cf(v)}, \quad \text{......................(5)}$$

giving Y, the maximum height ascended.

Tables are given in Volume IV, for the whole time t_1 of the upward motion, and for the height Y of ascent (Table 17); these were calculated on the assumption of Siacci's 1896 laws of air resistance. Here

$$t_1 = M(0) - M(v_0), \quad Y = Q(0) - Q(v_0);$$

where $M(0)$ is the value of the function M, as shown in the tables, for $v=0$: $M(v_0)$, $Q(0)$, $Q(v_0)$ denote similar values of M and Q, and the ballistic coefficient c is defined in Volume IV, as

$$c = \frac{(2R)^2 . \delta . i . 896}{1{\cdot}206P},$$

δ the average air density, in kg/m³; and diagrams for the functions $M(v)$ and $Q(v)$ are given in Volume IV.

Example. Given $c=4$, $v_0=700$ m/sec. Taken either from the Tables, Volume IV, or from the diagrams:

$$M(0)=18\cdot21,\quad M(v_0)=0\cdot52,\quad Q(0)=2850,\quad Q(v_0)=480,$$

so that

$$t_1=18\cdot21-0\cdot52=17\cdot7 \text{ seconds},$$

$$Y=2850-480=2370 \text{ m}.$$

Descending Motion.

The coordinate y is here reckoned from the highest point O_1 downwards, and then

$$v\frac{dv}{dy}=+g-cf(v),\quad y=\int_0^v \frac{v\,dv}{g-cf(v)}, \qquad \text{(6)}$$

$$Y=+\int_0^{v_e}\frac{v\,dv}{g-cf(v)}, \qquad \text{(7)}$$

thus the velocity of descent is given after Y has been calculated by (5).

Moreover

$$\frac{dv}{dt}=+g-cf(v),\quad t=\int_0^v \frac{dv}{g-cf(v)}, \qquad \text{(8)}$$

and

$$t_2=\int_0^{v_e}\frac{dv}{g-cf(v)}, \qquad \text{(9)}$$

giving t_2 the time of descent.

An elimination of v between (6) and (8), or another integration of the equation (6), will give the instantaneous distance of the shell from the ground, after any time t in the descending motion.

If a law of air resistance is assumed, giving the retardation $cf(v)$ in a monomial form cv^n, and n is an integer, the integration is at once possible. When zone laws of this monomial form are assumed, the integration must be carried out in each zone. When an analytical formula is taken, of which the integration is not possible in a finite form, or when the law is expressed graphically in a tabular form, it is convenient to employ the Abdank-Abakanowitz integraph, or a planimeter.

Diagrams are given in Volume IV, for finding Y from equation (7) and t_2 from (9), for the values $c=0\cdot1$, $0\cdot2$, $0\cdot5$, 1, 3, 6.

Assumption of the quadratic law, $cf(v) = cv^2$.

(For c and the value of K consult § 10.)

The results are as follows:

1. *Ascending motion from initial velocity v_0:*

$$v = \frac{v_0\sqrt{\frac{c}{g}}\cos(t\sqrt{gc}) - \sin(t\sqrt{gc})}{v_0\frac{c}{g}\sin(t\sqrt{gc}) + \sqrt{\frac{c}{g}}\cos(t\sqrt{gc})}, \quad \text{............(10)}$$

$$y = \frac{1}{c}\log\left[\cos(t\sqrt{gc}) + v_0\sqrt{\frac{c}{g}}\sin(t\sqrt{gc})\right], \quad \text{.........(11)}$$

$$\tan(t_1\sqrt{gc}) = v_0\sqrt{\frac{c}{g}}, \quad \text{....................(12)}$$

$$Y = \frac{1}{2c}\log\left(1 + \frac{cv_0^2}{g}\right), \quad \text{..................(13)}$$

for the maximum height Y ascended.

2. *Descending motion, starting from rest:*

$$t = \frac{1}{2\sqrt{gc}}\log\frac{\sqrt{\frac{g}{c}} + v}{\sqrt{\frac{g}{c}} - v}, \quad \text{........................(14)}$$

or

$$v = \sqrt{\frac{g}{c}}\,\frac{e^{2t\sqrt{gc}} - 1}{e^{2t\sqrt{gc}} + 1} = \sqrt{\frac{g}{c}}\,\text{th}(t\sqrt{gc}), \quad \text{......(14a)}$$

$$y = \frac{1}{c}\log\frac{e^{t\sqrt{gc}} + e^{-t\sqrt{gc}}}{2} = \frac{1}{0{\cdot}4343\,c}\log\text{ch}(t\sqrt{gc}), \quad \text{......(15)}$$

giving the downward descent y in time t from rest.

From (14) and (15),

$$v = \sqrt{\frac{g}{c}}\sqrt{(1 - e^{-2cy})}, \quad \text{..................(16)}$$

the velocity v acquired in falling through y;

$$Y = \frac{1}{c}\log\frac{e^{t_2\sqrt{gc}} + e^{-t_2\sqrt{gc}}}{2} = \frac{1}{0{\cdot}4343\,c}\log\text{ch}(t_2\sqrt{gc}), \quad \text{...(17)}$$

which determines the total time t_2 of falling, when Y has been given by (13); and

$$v_e = \sqrt{\frac{g}{c}}\,\text{th}\,(t_2\sqrt{gc}), \quad \text{.......................(18)}$$

for the velocity of fall v_e.

3. *Descending motion with initial velocity of projection* v_0:

$$y = \frac{1}{0{\cdot}4343\,c}\log\left[v_0\sqrt{\frac{c}{g}}\,\text{sh}\,(t\sqrt{gc}) + \text{ch}\,(t\sqrt{gc})\right], \quad \text{...(19)}$$

for downward descent y after time t.

Tables for the hyperbolic functions, sh, ch, th, are given in: W. Ligowski, *Tables of hyperbolic and circular functions*, Berlin, Ernst and Korn, 1890; also in E. Jahnke and F. Emde, *Functionen-tafeln*, Leipzig, Teubner, 1909.

Consult § 10, for the value of c for air.

1. *Example.* A bullet with sectional area $0{\cdot}52 \times 10^{-4}\,\text{m}^2$, weight 0·01 kg, is dropped in a vertical line from a height $Y = 2600$ m, from rest, in air of mean density $\delta = 1{\cdot}08\,\text{kg/m}^3$; and $c = 0{\cdot}0039$, $g = 9{\cdot}81$.

The time of falling $t_2 = 56$ sec, and the striking velocity $v_e = 41$ m/sec, from formulae (17) and (18).

2. *Example.* Shooting vertically downwards in water. According to § 9, $c = \frac{\lambda\,.\,R^2\pi\,.\,\delta}{P}$, where $R^2\pi$ is the cross-section in m^2; P is the weight of the shell in kg, δ the weight in kg of one m^3 of water; λ is a numerical factor to be determined experimentally, depending on the form of the shell, the state of the water, and the velocity.

Take $P = 14$ kg, $R^2\pi = 0{\cdot}025\,\text{m}^2$, $\delta = 1{\cdot}050\,\text{kg/m}^3$, $\lambda = 0{\cdot}3$, and let the shell start from the surface of the water with the downward initial velocity of $v_0 = 150$ m/sec. What is the depth reached by the shell in $t = 0{\cdot}1$ sec?

By formula (19) the depth reached will be about 7 m.

Explanation of the preceding equations.

The differential equation $-dt = \frac{dv}{g + cv^2}$ leads at once to

$$-t\sqrt{gc} = \tan^{-1}\frac{v\sqrt{\frac{c}{g}} - v_0\sqrt{\frac{c}{g}}}{1 + \frac{c}{g}v_0 v};$$

or, since

$$\tan^{-1}\frac{\alpha \pm \beta}{1 \mp \alpha\beta} = \tan^{-1}\alpha \pm \tan^{-1}\beta,$$

equation (10) is obtained by solution for v.

Then if $v = \frac{dy}{dt}$ is integrated again, the relation (11) follows. Put $v = 0$ in (10), and then $t = t_1$, as in (12); and then $y = Y$ in (13).

Moreover, the equation $\frac{dv}{dt}=g-cv^2$ is easily integrated by a resolution into partial fractions, and then (14) is obtained, or (14a) when solved for v. This last equation, with $v=\frac{dy}{dt}$, integrated again with respect to t, leads to (15); and to (17) and (18) for the point of fall.

In a vacuum, where $c=0$, $Y=\frac{0}{0}=\frac{v_0^2}{2g}$, $v_e=\frac{0}{0}=gt_2$, and so on.

Remark. The purely academic question is often proposed, as to whether it is possible for an ordinary shell, fired vertically upwards, ultimately to leave the Earth, provided the initial velocity is made excessively high.

This question cannot be answered without further examination, considering that with the velocity of the shell the resistance of the air also increases.

Consider, as an extreme case, the problem of a small sphere of elder pith or of down, projected vertically with enormous initial velocity; here anyone would find it reasonable to suppose that such a body would not ascend to an infinite height, because the air opposes a very powerful resistance to the motion.

Assuming the cubic law of resistance, $cf(v)=cv^3$, the time t_1 at which the body is for a moment at rest, at its maximum height Y of ascent, is given by

$$t_1=\int_0^{v_0}\frac{dv}{g+cv^3},\quad Y=\int_0^{v_0}\frac{v\,dv}{g+cv^3}.$$

When $t_1=\tau$ and $Y=\xi$, for the case of $v_0=\infty$,

$$\tau=\int_0^{\infty}\frac{dv}{g+cv^3},\quad \xi=\int_0^{\infty}\frac{v\,dv}{g+cv^3};$$

and the question is whether these values are finite or infinite.

To calculate τ, divide the integral into two parts: the first reaching from $v=0$ to any assumed finite velocity V, and the second from $v=V$ to $v=u$, where u afterwards is made to tend to a limit ∞. Then

$$\tau=\int_0^{V}\frac{dv}{g+cv^3}+\left[\int_V^{u}\frac{dv}{g+cv^3}\right]_{u=\infty}.$$

Here the first integral is always finite, because V has been assumed finite.

In the second integral, divide numerator and denominator by v^3, and employ the first law of the mean, as is allowable, since $\frac{g}{v^3}+c$ and $\frac{1}{v^3}$ are finite, and $\frac{1}{v^3}$ does not alter in sign.

Denoting by M the mean value of $\frac{1}{c+\frac{g}{v^3}}$ between $v=V$ and $v=\infty$, then

$$\tau=\int_0^{V}\frac{dv}{g+cv^3}+\int_V^{u}\frac{\frac{dv}{v^3}}{\frac{g}{v^3}+c}=\int_0^{V}\frac{dv}{g+cv^3}-\tfrac{1}{2}M\left(\frac{1}{v^2}\right)_V^{\infty}$$

$$=\int_0^{V}\frac{dv}{g+cv^3}+\frac{1}{2}\frac{M}{V^2}.$$

Here M is finite, since in $\frac{g}{v^3}+c$ the velocity v varies between V and ∞, and so M lies between the two finite values $\frac{1}{\frac{g}{V^3}+c}$ and $\frac{1}{c}$: thus τ is finite.

In a corresponding way,

$$\xi=\int_0^\infty \frac{v\,dv}{g+cv^3}=\int_0^V \frac{v\,dv}{g+cv^3}+\left[\int_V^u \frac{\frac{dv}{v^2}}{\frac{g}{v^3}+c}\right]_{u=\infty}$$

$$=\int_0^V \frac{v\,dv}{g+cv^3}-M\left(\frac{1}{v}\right)_V^\infty=\int_0^V \frac{v\,dv}{g+cv^3}+\frac{M}{V}.$$

It can be proved, with the monomial law cv^n, that if $n>1$, τ has a finite value; and if $n>2$, ξ also has a finite value.

The maximum height ξ, that a shell can reach, and the corresponding time τ of the ascent, have been calculated by St Robert for the law of retardation

$$cf(v)=cv^2\left(1+\frac{b}{c}v\right);$$

and he finds for ξ,

$$\frac{2}{\lambda^2}(3g+c\lambda^2)\,\xi=\log\frac{b\lambda^3}{g}+\frac{6g}{\sqrt{[g\,(3g+4c\lambda^2)]}}\left[\frac{1}{2}\pi+\sin^{-1}\frac{1}{2}\sqrt{\frac{g}{b\lambda^3}}\right],$$

where λ is given by the cubic equation $b\lambda^3-c\lambda^2-g=0$.

For an iron sphere of weight 12 kg, in air of density $\delta=1{\cdot}208$ kg/m³, and with $g=9{\cdot}81$, $c=0{\cdot}000374$, $b=0{\cdot}00000086$, it was found that $\lambda=483{\cdot}63$, $\tau=19{\cdot}24$ sec, $\xi=3966$ m, so that the shell would not reach a height beyond that of Mont Blanc.

St Robert then took into account further the diminution of air density and gravity with the height. As the calculation led to a somewhat complicated differential equation when both influences were taken into account together, St Robert calculated an upper limit H for ξ, a value H that must always be greater than ξ, in the following way:

The motion is supposed to be divided into two parts, and the assumption is made that in the first part the resistance of the air alone is at work, without gravity: in the second part, gravity alone is at work, but not the air resistance.

The first part reaches from $v_0=\infty$ to some arbitrary finite velocity v_1, and the corresponding height of ascent is denoted by h_1. The second part reaches from v_1 to $v=0$, and the corresponding further ascent is h_2. Thus $H=h_1+h_2$ is always greater than the true value of ξ, the height that would be reached when air resistance and gravity are acting simultaneously.

The motion in the first part requires the differential equation,

$$\frac{dv}{dt}=v\frac{dv}{dy}=-cf(v)\,e^{-\frac{y}{8440}},\quad f(v)=cv^2\left(1+\frac{b}{c}v\right),$$

as the air resistance alone is assumed at work, and at the same time the barometric influence is taken into account.

The integration from $v=\infty$ to $v=v_1$, and $y=0$ to $y=h_1$, gives

$$h_1=-8440\log\left[1-\frac{1}{8440\,c}\log\left(1+\frac{c}{bv_1}\right)\right].$$

In the second part, where gravity alone, on Newton's law of gravitation, is acting, the differential equation is

$$\frac{dv}{dt} = v\frac{dv}{dy} = -g\left(\frac{r}{r+y}\right)^2,$$

where $r = 6370000$ m, the radius of the Earth: and this gives (since for $v = v_1$, $y = h_1$ and $v = 0$, $y = h_1 + h_2$),

$$h_2 = \frac{v_1^2(r+h_1)^2}{2gr^2 - v_1^2(r+h_1)}.$$

On the same assumptions as before, and taking the given value $v_1 = 175$ m/sec, it is found that $h_1 = 4248$ m, $h_2 = 1564$ m; thus the height ξ is always less than $H = h_1 + h_2 = 5812$ m.

Higher than this the shell can never fly, however great the initial velocity may be made.

The author is of the opinion that calculations of this nature are inconclusive; because the law of resistance employed by St Robert is purely empirical, and is a law based on experimental velocities up to about 600 m/sec; and an extrapolation from $v = 600$ to $v = \infty$ is not permissible.

In fact we may say that nothing is known of the resistance of the air at inordinately high velocities of the moving body.

§ 36. Shooting nearly vertical. Use of the auxiliary functions.

The following procedure relates to the case, where the angle ψ of the tangent of the trajectory with the vertical remains so small that in the expansion of the series for $\cos\psi$ and $\sin\psi$,

$$\cos\psi = 1 - \frac{\psi^2}{2!} + \ldots, \qquad \sin\psi = \psi - \frac{\psi^3}{3!} + \ldots,$$

only the first term need be retained; the whole of the ascending branch does not come into consideration, but only the nearly straight part.

A. *High angle firing.*

Let $P\,(xy)$ be any point of the path, reached after a time t; v the velocity, $cf(v)$ retardation of the air resistance, $\psi = \frac{1}{2}\pi - \theta$ the inclination to the vertical of the tangent of the path, ψ_0 the initial angle, and v_0 the initial velocity.

Since $\cos\psi = \sin\theta$, $\sin\psi = \cos\theta$, $d\psi = -d\theta$, the general equations of § 17 become

$$d(v\cos\psi) = -g\,dt - cf(v)\cos\psi\,dt, \quad \ldots\ldots\ldots\ldots\ldots(1)$$

$$d(v\sin\psi) = -cf(v)\sin\psi\,dt, \quad \ldots\ldots\ldots\ldots\ldots(2)$$

$$g\,dx = +v^2 d\psi, \quad \ldots\ldots\ldots\ldots\ldots(3)$$

$$g\,dt = +v \operatorname{cosec} \psi\, d\psi, \quad \text{.................(4)}$$

$$g\,dy = +v^2 \cot \psi\, d\psi, \quad \text{.....................(5)}$$

$$g\,d(v \sin \psi) = -cf(v)\, v\,d\psi, \quad \text{.......................(6)}$$

and thence, on the assumption above,

$$dt = -\frac{dv}{g + cf(v)}.$$

Integrating from $t = 0$ to t, and from $v_0 = 0$ to v,

$$t = -\int_{v_0}^{v} \frac{dv}{g + cf(v)} = M(v) - M(v_0), \quad \text{..............(7)}$$

where

$$M(v) = \int_v^{1200} \frac{dv}{g + cf(v)}, \quad M(v_0) = \int_v^{1200} \frac{dv}{g + cf(v)}.$$

Moreover, from (4)

$$\frac{d\psi}{\psi} = g\frac{dt}{v} = -\frac{g\,dv}{v[g + cf(v)]}.$$

Integrating from ψ_0 to ψ, and v_0 to v, the instantaneous slope of the tangent is obtained

$$\psi = \frac{\psi_0\, G(v)}{G(v_0)} \quad \text{.........................(8)}$$

where, in our notation,

$$G(v) = e^{N(v)}, \quad G(v_0) = e^{N(v_0)},$$

$$N(v) = \int_v^{1200} \frac{g\,dv}{v[g + cf(v)]}, \quad N(v_0) = \int_{v_0}^{1200} \frac{g\,dv}{v[g + cf(v)]}.$$

Equation (3) gives

$$dx = \frac{v^2}{g}\, d\psi = -\frac{\psi_0}{G(v_0)} \frac{G(v)\, v\, dv}{g + cf(v)}, \quad x = \frac{\psi_0}{G(v_0)}[P(v) - P(v_0)], \quad (9)$$

where

$$P(v) = \int_v^{1200} \frac{G(v)\, v\, dv}{g + cf(v)}, \quad P(v_0) = \int_{v_0}^{1200} \frac{G(v)\, v\, dv}{g + cf(v)}.$$

Finally the altitude y of the shell above the horizontal through the muzzle is given from (5),

$$g\,dy = \frac{v^2\, d\psi}{\psi} = -\frac{g\, v\, dv}{g + cf(v)}, \quad y = Q(v) - Q(v_0),$$

where

$$Q(v) = \int_v^{1200} \frac{v\,dv}{g + cf(v)}, \quad Q(v_0) = \int_{v_0}^{1200} \frac{v\,dv}{g + cf(v)}.$$

Statement of results.

Time of flight

$$t = M(v) - M(v_0),$$

Slope of tangent

$$\psi = \frac{\psi_0 G(v)}{G(v_0)},$$

Abscissa of trajectory

$$x = \frac{\psi_0}{G(v_0)}[P(v) - P(v_0)],$$

Ordinate of trajectory

$$y = Q(v) - Q(v_0).$$

Notation.

$$M(v) = \int_v^{1200} \frac{dv}{g + cf(v)},$$

$$G(v) = e^{N(v)},$$

$$N(v) = \int_v^{1200} \frac{g\,dv}{v[g + cf(v)]},$$

$$P(v) = \int_v^{1200} \frac{G(v)\,v\,dv}{g + cf(v)},$$

$$Q(v) = \int_v^{1200} \frac{v\,dv}{g + cf(v)}.$$

Meaning of the notation: x, y the coordinates in m of a point on the trajectory, reached after a time t sec; v m/sec the velocity; ψ the inclination to the vertical of the tangent of the trajectory at any point, ψ_0 that of the initial tangent; $cf(v)$ the retardation due to air resistance; for which the formula is assumed as in Table 12,

$$f(v) = 0{\cdot}2002\,.\,v - 48{\cdot}05 + \sqrt{[(0{\cdot}1648\,.\,v - 47{\cdot}95)^2 + 9{\cdot}6]} + \frac{0{\cdot}0442\,(v - 300)}{371 + \left(\frac{v}{200}\right)^{10}};$$

$$c = \frac{\delta\,.\,1000\,(2R)^2\,.\,0{\cdot}896\,.\,i}{1{\cdot}206P};$$

δ the air density in kg/m³; $2R$ the calibre in m; P the weight of the shell in kg; and i the coefficient of form is $=1$ according to Siacci, for the original Krupp normal shell with ogival head rounded to a radius of 2 calibres.

The curves of Table No. 1 (Vol. IV) give the values of M, G, P, Q, for $c = 6, 3, 1, 0{\cdot}5, 0{\cdot}2, 0{\cdot}1$; and M, and Q for the values $c = 5, 4, 2$.

Interpolation is not possible in all circumstances for intermediate values of c; but at least the values of c given in the Table will assign two limits for the trajectory.

When the value of i is not yet known, it must be determined by firing along a flat trajectory. A first tentative calculation will give an approximate range $\sqrt{(x^2 + y^2)}$; then if the trajectory is swung down according to the method of Burgsdorff, the corresponding value of i is taken out of a Range Table for direct fire.

The special case of vertical fire is given by $\psi_0 = 0$.

The total height reached at y_1, when $v = 0$, and the corresponding time t_1 for the complete upward motion, are then given by

$$y_1 = Q(0) - Q(v_0), \quad t_1 = M(0) - M(v_0).$$

The diminution of air density with the height can be taken into account, by calculating the path in several parts: but in most cases it is sufficient to take the mean density of the air through which the shell passes.

B. *Shooting vertically downwards.*

The origin O of coordinates is taken again at the point of departure, and the positive axis of y is drawn vertically downward. The only alteration from A is to replace $+g$ by $-g$.

Time of flight

$$t = M_1(v) - M_1(v_0).$$

Slope of tangent

$$\psi = \psi_0 \frac{G_1(v)}{G_1(v_0)}.$$

Abscissa of trajectory

$$x = \frac{\psi_0}{G_1(v_0)}[P_1(v) - P_1(v_0)].$$

Ordinate of trajectory

$$y = Q_1(v) - Q_1(v_0).$$

Where

$$M_1(v) = \int_v^{1200} \frac{dv}{-g + cf(v)},$$

$$G_1(v) = e^{N_1(v)},$$

$$N_1(v) = \int_v^{1200} \frac{-g\,dv}{v[-g + cf(v)]},$$

$$P_1(v) = \int_v^{1200} \frac{G_1(v)\,v\,dv}{-g + cf(v)},$$

$$Q_1(v) = \int_v^{1200} \frac{v\,dv}{-g + cf(v)}.$$

The curves of Table II (Vol. IV), for M_1, G_1, P_1, Q_1 should be consulted.

In the calculation of this Table it is assumed that the initial velocity of departure v_0 of the shell is greater than the terminal velocity v', for which the weight and the air resistance are equal, when $g = cf(v')$.

The velocity will then diminish from the initial value v_0, and tend more and more to the terminal velocity v'.

In the case where $v_0 = v'$, the velocity remains very nearly constant.

The case of $v_0 < v'$ is not taken into account, as in this case the quadratic law may be employed.

Remarks.

1. With ordinary small shot the previous laws of air resistance should not be employed without further examination, because of the very small cross-section and the small velocity which are to be considered, for which the usual constants in the laws of air resistance are possibly inaccurate.

2. These auxiliary functions, which in the form above were first taken to hold for very small values of ψ, may receive a somewhat more extended application, if in equation (1) above,

$$d(v \cos \psi) = -g\,dt - cf(v) \cos \psi\, dt,$$

the value of $\cos \psi$ is replaced by some constant mean value.

Denoting this by either σ_1, or γ_1, then

$$dt = -\frac{\sigma_1 dv}{g + c\gamma_1 f(v)}.$$

This generalisation is analogous to the methods of §§ 23—31, where σ_1 and γ_1 were determined by an appropriate choice; the consideration of this question will be resumed later.

For a numerical example (of vertical fire with the *S* bullet), and the researches of Preuss on time of flight and striking velocity in vertical or nearly vertical rifle fire, consult notes to §§ 35 to 37.

§ 37. General calculation of high angle trajectories. Allowance for the diminution of air density.

Undoubtedly the most accurate procedure for the plotting of a high angle trajectory is the experimental method, making use of two photo-theodolites, described in Vol. III, § 184. Here, however, we are concerned exclusively with methods of calculation.

1. The several processes of § 23 show how in the calculation of a steep trajectory any one of the methods of the second group of approximations can be employed, if the trajectory is divided up into several arcs.

The extent of any individual arc must be chosen smaller in proportion as the trajectory is more curved in the neighbourhood. This procedure, proposed by J. Didion, 1848, is in principle the following:

The system of equations is

$$x = \frac{1}{\alpha c}(D_u - D_{u_0}), \quad t = \frac{1}{c}(T_u - T_{u_0}),$$

$$\tan \theta = \tan \phi - \frac{\alpha}{2c}(J_u - J_{u_0}),$$

$$y = x \tan \phi - \frac{1}{2c^2}[A_u - A_{u_0} - J_{u_0}(D_u - D_{u_0})],$$

$$u = \alpha v \cos \theta, \qquad u_0 = \alpha v_0 \cos \phi.$$

An arbitrary choice is made of the angle θ of slope corresponding to the end of the first arc; and then given $\theta, \phi, \alpha, c, v_0$, the third equation will serve to calculate u.

The first equation then determines x, the second t, the fourth y, the fifth v for the end of the first arc. The factor α is here a mean value between $\sec\phi$ and $\sec\theta$ at the beginning and end of the arc.

According to Didion, $\alpha = \dfrac{\xi(\phi) - \xi(\theta)}{\tan\phi - \tan\theta}$,

according to v. Wuich $\alpha = \dfrac{\xi\left(\dfrac{\phi+\theta}{2}\right)}{\tan\dfrac{\phi+\theta}{2}}$;

or to St Robert $\alpha = \frac{1}{2}(\sec\phi + \sec\theta)$,

or to Hélie, $\alpha = \sqrt{(\sec\phi \sec\theta)}$.

2. The choice of the number and length of the arcs, of which the trajectory is built up, is then of especial importance.

In order to know whether the desired degree of accuracy in the calculation is actually reached with the arbitrary assumed system of division into arcs, we shall calculate some trajectories (for angles of departure, for instance, of $\phi = 50°$, $65°$, $80°$) according to a method, which has the advantage from a mathematical point of view of known accuracy.

For this purpose the 1909 method of the author is proposed, that was employed in § 32 in testing the different methods of solution and their accuracy; this may be called the "planimetric" method.

An integrable law is chosen for the representation of the air resistance as a function of the velocity (§ 17), for instance a monomial law, giving retardation cv^n. The corresponding Tables, A_u, D_u, J_u, T_u, will then be taken as the basis of the calculations.

The relation between the instantaneous velocity v of the shell in its path and the angle θ of slope with the horizon is then given by

$$\frac{1}{(v\cos\theta)^n} = -\frac{nc}{g}\int\frac{d\theta}{(\cos\theta)^{n+1}} + C.$$

The integration constant C is then to be calculated, so as to connect the zones, taking the pair of values (θ, v) at the end of one zone to be the same as at the beginning of the following zone.

The constant c is given by the mass of the shell, its dimensions and the shape, as well as the air density; and contains moreover a factor, which takes different values in going from one zone into another.

Taking the zone laws of Mayevski-Sabudski:

v m/sec	n	c	$\int \frac{d\theta}{(\cos\theta)^{n+1}}$
0 to 240	2	$\delta_y \,.\, a \,.\, 0{\cdot}0140$	$\int \frac{d\theta}{\cos^3\theta} = \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2}\log\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$ (Vol. IV, Table 10)
240 „ 295	3	$\delta_y \,.\, a : 0{\cdot}\overset{(4)}{0}5834$	$\int \frac{d\theta}{\cos^4\theta} = \tan\theta + \frac{1}{3}\tan^3\theta$ „ „
295 „ 375	5	$\delta_y \,.\, a \,.\, 0{\cdot}\overset{(9)}{0}6709$	$\int \frac{d\theta}{\cos^6\theta} = \tan\theta + \frac{2}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta$ (Vol. IV, Table 10)
375 „ 419	3	$\delta_y \,.\, a \,.\, 0{\cdot}\overset{(4)}{0}9404$	$\int \frac{d\theta}{\cos^4\theta} = \tan\theta + \frac{1}{3}\tan^3\theta$ „ „
419 „ 550	2	$\delta_y \,.\, a \,.\, 0{\cdot}0394$	$\int \frac{d\theta}{\cos^3\theta} = \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2}\log\tan(\frac{1}{4}\pi + \frac{1}{2}\theta)$ (Vol. IV, Table 10)
550 „ 800	1·70	$\delta_y \,.\, a \,.\, 0{\cdot}2616$	$\int \frac{d\theta}{(\cos\theta)^{2{\cdot}70}}$ „ „
800 „ 1000	1·55	$\delta_y \,.\, a \,.\, 0{\cdot}7130$	$\int \frac{d\theta}{(\cos\theta)^{2{\cdot}55}}$ „ „

Here $\frac{R^2\pi g i}{1{\cdot}206\,P} = a$, for brevity; $2R$ the calibre in m, P the weight of the shell in kg, i the form coefficient, taken $= 1$ for ogival shell of 2 calibre radius of rounding; δ_y the air density at the corresponding height y m above the ground, measured in kg/m³.

To allow for the diminution of air density δ_y with the height, the calculation is made with a mean density within the corresponding zone.

After v has been obtained as a function of θ, the values of $\frac{v^2}{g}$, $\frac{v^2}{g}\tan\theta$, $\frac{v}{g\cos\theta}$ are calculated for a large number of values of θ and the summation is carried out of

$$x = -\Sigma \frac{v^2}{g}\,d\theta, \quad y = -\Sigma \frac{v^2}{g}\tan\theta\,d\theta, \quad t = -\Sigma \frac{v\,d\theta}{g\cos\theta},$$

with the help of the planimeter, or of the integraph.

The details of the procedure will be made more clear by an example.

Example. (*a*) Angle of departure $\phi=80°$, calibre $2R=0{\cdot}037$ m, weight of shell $P=0{\cdot}680$ kg, initial velocity $v_0=500$ m/sec, $i=1$ (for the choice of i, see below), air density at the muzzle level $\delta=1{\cdot}206$ kg/m³.

1st Zone (500—419 m/sec). The air density is assumed at first throughout the whole zone to be the same as at the beginning of the zone, and so $\delta=1{\cdot}206$.

Then $$\frac{2c}{g}=\frac{R^2\pi\,.\,2\,.\,0{\cdot}0394}{P}=(0{\cdot}0185)^2\pi\times 2\times 0{\cdot}0394\,(0{\cdot}68)^{-1},$$

$$\log\frac{2c}{g}=\bar{4}{\cdot}09551.$$

The relation in this zone between v and θ is

$$\frac{1}{(v\cos\theta)^2}=-\frac{2c}{g}\int\frac{d\theta}{\cos^3\theta}+A=-\frac{2c}{g}\left[\frac{\sin\theta}{2\cos^2\theta}+\tfrac{1}{2}\log\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta\right)\right]+A.$$

The integration constant A is given at the beginning of the zone, where $\theta=\phi=80°$, and $v=v_0=500$; thence $A=0{\cdot}0023191$. At the end of the zone, $v=419$, and thence $\theta=+79°\,52'{\cdot}4$.

A provisional planimetric summation gives $y=-\Sigma\dfrac{v^2}{g}\tan\theta\,d\theta$ up to the end of the zone, and the result is $y_1=264{\cdot}8$ m, for the ordinate y_1 of the end point.

This makes the air density at the end of the zone

$$\delta_y=\delta\,(1-0{\cdot}00011y_1)=1{\cdot}206\times 0{\cdot}9709;$$

and the mean density (arithmetic mean of values at the muzzle level and at the end of the zone), is

$$\delta_1=1{\cdot}206\times 0{\cdot}9854.$$

Thence, more accurately

$$\frac{2c}{g}=(0{\cdot}0185)^2\pi\times 2\times 0{\cdot}0394\times 0{\cdot}9854\times(0{\cdot}68)^{-1},$$

$$\log\frac{2c}{g}=\bar{4}{\cdot}08914.$$

Repeating the calculation, the integration constant is now $A=0{\cdot}0022872$.

At the end of the zone, where $v=419$, the corrected value $\theta=+79°\,52'{\cdot}3$.

By the use of the planimeter $y_1=268{\cdot}41$ m (to a scale of 1 square cm of the drawing sheet to $0{\cdot}097$ m).

And the air density δ_{y_1} at the end of the first zone is given by

$$\delta_{y_1}=\delta\,(1-0{\cdot}00011\times 268{\cdot}4)=0{\cdot}9705\times 1{\cdot}206.$$

2nd Zone (419—375 m/sec). Here

$$\frac{1}{(v\cos\theta)^3}=-\frac{3c}{g}\left(\tan\theta+\tfrac{1}{3}\tan^3\theta\right)+B.$$

Assuming the air density inside the second zone as constant and equal to the amount, $0{\cdot}9705$, at the end of the first zone, then in the second zone

$$\frac{3c}{g}=3\times(0{\cdot}0185)^2\times\pi\times 0{\cdot}00009404\times 0{\cdot}9705\times(0{\cdot}68)^{-1},$$

$$\log\frac{3c}{g}=\bar{7}{\cdot}63640.$$

The integration constant B is derived from the (θ, v) pair of values at the end of the first zone; that is, $v=419$, $\theta=79°\ 52'{\cdot}3$, and thence $B=0{\cdot}000030237$.

The value at the end of the zone, where $v=375$, is therefore

$$\theta=+79°\ 45'{\cdot}53.$$

By use of the planimeter up to the end of the second zone, a first value is

$$y_2=175{\cdot}06 \text{ m}.$$

The air density at the end of the second zone is therefore

$$\delta_{y_2}=\delta\,[1-0{\cdot}00011\,(y_1+y_2)]=1{\cdot}206\times0{\cdot}95122;$$

and so the mean density inside the second zone is

$$\tfrac{1}{2}\,(\delta_{y_1}+\delta_{y_2})=1{\cdot}206\times0{\cdot}96085.$$

The calculation is repeated with this mean air density; and a closer value is found of

$$\log\frac{3c}{g}=\bar{7}{\cdot}63206,$$

and of $$B=0{\cdot}0000299618.$$

Using the planimeter, in four steps, to a scale of 1 square cm of the drawing sheet to 0·01164 m, the end value of the ordinate of the second zone is

$$y_2=176{\cdot}8 \text{ m}.$$

3rd Zone (375—295 m/sec). In a first approximation, the end values of the third zone, $v=295$, $\theta=+79°\ 15'{\cdot}09$, and thence $y_3=522{\cdot}29$ m.

Hence the mean relative air density in the third zone is 0·92230; thence closer values are $v=295$, $\theta=79°\ 14'{\cdot}33$, and $y_3=536{\cdot}20$ m.

4th Zone (295—240 m/sec). With the air density prevailing at the end of the third zone, namely $0{\cdot}8920\times1{\cdot}206$, the value obtained for the end of the zone is at first, for $v=240$, $\theta=+78°\ 19'{\cdot}7$; and thence $y_4=575{\cdot}15$ m. At the end of the zone this makes

$$\delta_{y_4}=\delta\,[1-0{\cdot}00011\,(y_1+y_2+y_3+y_4)]=\delta\times0{\cdot}82878,$$

or a mean density in the zone $\delta\times0{\cdot}86041$.

Repeating the calculation with this mean density, a closer value is obtained, $\theta=78°\ 18'{\cdot}4$, and $y_4=588{\cdot}85$ m.

5th Zone (240—0 m/sec). At $\theta=0$, the vertex of the trajectory,

$$v=v_s=33{\cdot}3 \text{ m/sec}.$$

The planimeter gave (in six parts) a first value $y=1925{\cdot}8$ m at the end of the zone; and so a first value of the air density at the end of the zone 0·61543; thus the mean density in the fifth zone is taken as 0·72135.

Repeating the calculation, a closer value is found at $\theta=0$ for the vertex velocity $v_s=34{\cdot}51$ m/sec.

The planimeter is employed again in four groups; first to $\theta=+78°\ 18'{\cdot}4$, secondly to $\theta=45°$, and thirdly to $\theta=0$ (in 3, 5, 9 steps respectively).

Thence the value found at the end of the fifth zone was $y=2000{\cdot}96$ m. The separate values of y, as well as of x and t, are then added for the ascending branch; and the results are shown in the table on the next page.

Slope of the tangent to the horizon	Altitude of the shell above the muzzle level	Horizontal advance of the shell from the point of departure	Time of flight from the start	Velocity of the shell in the trajectory
$\theta=80°$	$y=0$ m	$x=0$ m	$t=0$ sec	$v=500{\cdot}0$ m/sec
79° 55′	185·0	32·78	0·3984	442·8
79° 52′·3	268·4	47·66	0·5949	419·0
79° 45′·5	445·2	79·49	1·0486	375·0
79° 30′	745·9	134·5	1·9298	323·5
79° 14′·3	981·4	178·6	2·7056	295·0
78° 50′	1274·0	235·2	3·7682	266·5
78° 18′·4	1570·3	295·0	4·9631	240·0
77° 30′	1910·8	368·3	6·5119	212·7
76° 30′	2218·8	439·4	8·0917	187·7
76°	2340·1	469·1	8·7748	177·6
75°	2537·2	519·9	9·9780	160·8
74°	2689·2	562·0	11·0027	147·3
73°	2810·0	597·6	11·8895	136·2
72°	2900·4	628·3	12·6663	126·8
70°	3054·9	678·9	13·9667	111·8
68°	3158·4	718·7	15·0180	100·3
66°	3235·5	751·3	15·8887	91·2
64°	3294·1	778·5	16·6239	83·8
61°	3359·7	811·9	17·5384	74·9
58°	3406·0	839·0	18·2896	67·9
54°	3449·9	868·4	19·1096	60·7
50°	3480·5	892·3	19·7812	55·2
45°	3507·3	916·8	20·4694	49·9
40°	3526·3	937·2	21·0474	45·9
30°	3550·1	970·3	21·9845	40·50
20°	3562·6	997·0	22·7433	36·98
10°	3569·1	1020·0	23·4040	35·16
0°	$y_s=3571{\cdot}2$	$x_s=1041{\cdot}5$	$t_s=24{\cdot}0302$	$v_s=34{\cdot}51$

Thence the following range table can be obtained. The slope E of the ground is given, on which the target is seen; moreover the coordinates of the target, the time of flight t, and the striking velocity, measured along the tangent.

E	a	y	x	t	v
80°	0°	0 m	0 m	0 sec	500 m/sec
78	2	27·85	592	11·717	139
76	4	3430	855	18·719	64·1
74	6	3569·5	1024	23·524	35·0

In the table a is the sighting angle above the sloping ground.

Remarks.

1. In most cases in practice the calculation would be made to a fewer number of decimal places, and the variation of air density would not be found by a double calculation, but simply by assuming a mean value.

But in our case the question was to show that if the law of the air resistance and the relation between air density and height above the ground are known accurately, a steep high angle trajectory can be calculated by this method with all the desired accuracy.

The planimeter employed, in the measurement of a square of 16·00 cm, gave a probable error of 0·060 °/₀ for a single measurement. Thence it was seen that the vertex height y_s, abscissa x_s, and time of flight t_s determined on this method would have a probable error, for a single planimeter measurement, of the following magnitudes:

$$y_s = 3571 \cdot 2 \pm 6 \cdot 1 \text{ m},$$
$$x_s = 1041 \cdot 5 \pm 0 \cdot 89 \text{ m},$$
$$t_s = 24 \cdot 0302 \pm 0 \cdot 022 \text{ sec}.$$

Actually the work with the planimeter was done ten times.

2. The descending branch of the trajectory can be calculated on the same plan.

However the rotation of the axis of the shell has greater influence on the results of the calculation, than in the ascending branch.

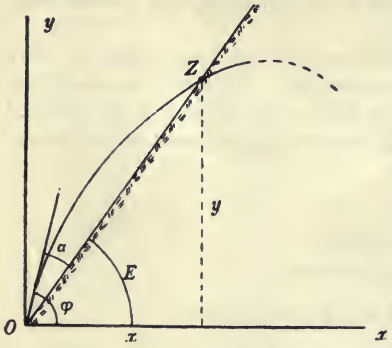
But the descending branch scarcely comes into consideration in fire against an aeroplane. On this account the continuation of this example is unnecessary.

(*b*) With the same P, $2R$, δ, i, v_0 as before, but with an angle of departure $\phi = 75°$, the results are as follows:

θ	y	x	t	v
75°	0 m	0 m	0 sec	500·0 m/sec
74° 45′	327·6	88·4	0·759	401·3
74° 38′·3	436·3	118·1	1·050	375·0
74° 30′	551·9	150·5	1·385	351·5₅
74°	885·0	244·3	2·449	303·6
73° 30′	1137·7	318·8	3·369	276·9
73°	1347·8	382·0	4·191	256·9₅
72° 28′	1538·9	440·4	4·984	240·0
72°	1681·0	485·9	5·622	228·5
71°	1933·7	579·1	6·843	207·9
70°	2133·4	640·8	7·903	191·4
68°	2429·1	753·7	9·678	166·2
66°	2636·6	841·8	11·110	147·7
64°	2788·4	912·3	12·296	133·5
61°	2949·4	996·1	13·744	117·3
57°	3091·5	1081·1	15·196	101·7
52°	3202·8	1160·5	16·706	88·0
46°	3286·3	1232·3	18·068	76·7
40°	3339·8	1288·0	19·096	68·7
33°	3379·4	1341·1	20·109	62·1
20°	3420·2	1419·8	21·612	54·6
0°	$y_s = 3439 \cdot 5$	$x_s = 1518 \cdot 1$	$t_s = 23 \cdot 511$	$v_s = 50 \cdot 5_5$

The range table is therefore as follows:

Angle of slope of ground $E=74°$, 72, 70, 68, or angle of sight relative to the ground $a=1°$, 3, 5, 7;



$x=419$, 870, 1170, 1373 m;

$y=1470$, 2697, 3217, 3398 m;

$OZ=r=1525$, 2827, 3416, 3656 m;

$t=4·72$, 10·89, 16·06, 19·85 sec;

$\theta=72° 30'$, 65°, 51° 15', 31° 45'.

The tangential velocity on striking the target is

$v=248$, 141, 88, 58 m/sec.

At the vertex

$y_s=3439·5 \pm 2·37$ m,

$x_s=1518·1 \pm 0·49$ m,

$t_s=23·511 \pm 0·006$ sec.

(*c*) The same P, $2R$, δ, i, v_0 as before, but angle of departure $\phi=70°$ with the following range table:

Angle of slope $E=70°$, 68, 66, 64, 62; angle of relative elevation $a=0°$, 2, 4, 6, 8; $x=0$, 782·6, 1183·8, 1469·4, 1685·5 m; $y=0$, 1937·0, 2658·0, 3012·4, 3174 m; $t=0$, 7·215, 12·080, 15·797, 18·615 sec; $v=500$, 202·4, 137·4, 96·85, 78·7 m/sec; direct distance of mark $OZ=r=\sqrt{(x^2+y^2)}=0$, 2089, 2909·5, 3351·8, 3590·3 m.

θ	y	x	t	v
70°	0 m	0 m	0 sec	500·0 m/sec
69° 50′	176·4	65·43	0·4055	442·3
69° 40′	319·2	117·9	0·766	401·1
69° 30′	437·8	162·0	1·094	371·8
69° 20′	540·4	200·1	1·398	350·2
69° 10′	631·9	235·3	1·684	334·2
69° 0′	715·2	267·1	1·956	321·4
68° 30′	930·7	350·7	2·709	294·2
68° 0′	1111·3	422·7	3·392	275·5
67° 30′	1267·2	486·5	4·021	260·3
67°	1403·8	443·8	4·604	247·5
66°	1634·4	643·8	5·662	227·6
65°	1823·5	728·8	6·605	211·8
63°	2114·0	869·7	8·221	187·1
61°	2326	982	9·556	168·5
59°	2486	1074	10·697	153·9
56°	2663	1186	12·108	137·0
52°	2826	1304	13·594	120·4
48°	2937	1397	14·810	108·2
40°	3074	1537	16·724	91·6
31°	3159	1654	18·346	79·9
19°	3216	1774	20·05	70·9
10°	3236	1850	21·17	67·2
4°	3241	1897	21·88	65·8
0°	$y_s=3243$	$x_s=1928$	$t_s=22·29$	$v_s=65·4$

(*d*) Same $2R$, P, δ, i, v_0 as before, but angle of departure $\phi = 65°$.

The first zone reached from $\theta = 65°$ to $\theta = 64° 46'$, the second from 64° 46′ to 64° 30′, the third from 64° 30′ to 63° 25′, the fourth from 63° 25′ to 61° 10′, the fifth from 61° 10′ to 39°.

As to the probable error of the results as derived from the arithmetic mean of a repeated operation with the planimeter,

$$y_s = 3026{\cdot}0 \pm 0{\cdot}1 \text{ m};$$
$$x_s = 2307{\cdot}83 \pm 0{\cdot}056 \text{ m};$$
$$t_s = 21{\cdot}919 \pm 0{\cdot}00055 \text{ sec};$$
$$v_s = 80{\cdot}40 \pm 0{\cdot}0.$$

Over the slope $E = 65°$, 64, 62, 60, 58, 56, 54, corresponding to $a = 0°$, 1, 3, 5, 7, 9, 11, the direct distances of the points struck worked out to

$$OZ = 0,\ 1140,\ 2300,\ 2918,\ 3310,\ 3565,\ 3724 \text{ m}.$$

θ	y	x	t	v
65°	0 m	0 m	0 sec	500·0 m/sec
64° 52′	116·47	54·48	0·271	459·7
64° 40′	260·36	112·29	0·635	415·2
64° 30′	360·56	169·9	0·912	387·3
64° 10′	527·5	249·9	1·415	349·7
63° 55′	632·7	301·3	1·760	331·2
63° 40′	727·5	348·0	2·087	317·0
63° 25′	813·9	391·1	2·397	305·6
63° 10′	894·2	431·4	2·695	295·1
62° 50′	992·5	481·5	3·075	285·2
62° 30′	1082·7	528·1	3·437	275·6
62° 10′	1166·0	571·7	3·783	267·1
61° 40′	1243·3	612·8	4·116	259·4
61° 10′	1382·8	688·5	4·744	246·1
60°	1590·2	804·8	5·761	227·5
59°	1736·6	891·4	6·537	214·1
58°	1862·7	975·9	7·247	202·7
56°	2067·4	1109·4	8·513	184·2
53°	2294·8	1269·9	10·110	163·4
48°	2541·4	1472·3	12·184	139·5
39°	2784·9	1725·3	14·971	113·7
25°	2951·5	1983·6	17·991	93·1
$\theta = 0°$	$y_s = 3026{\cdot}0$	$x_s = 2307{\cdot}8$	$t_s = 21{\cdot}919$	$v_s = 80{\cdot}40$

The preceding results from four steep high angle trajectories of the same gun are given here in detail because they give an opportunity of settling the value of the method, employed far too frequently in practical gunnery, of tilting the trajectory.

3. The foregoing planimetric method was simplified to a considerable extent by Freiherr von Zedlitz in 1913.

He proceeds equally from the division of the trajectory into a number of arcs, on the assumption of a monomial law cv^n for the retardation due to air resistance; and the relation between v and θ is first obtained.

So too the value of v at the end of an arc may be calculated, corresponding to arbitrary angles of slope of the tangent.

Thus for instance, as in § 20, on the quadratic law the relation is

$$\frac{1}{(v\cos\theta)^2} - \frac{1}{(v_0\cos\theta_0)^2} = \frac{2c}{g}[\xi(\theta_0) - \xi(\theta)].$$

Freiherr von Zedlitz next applies to such an arc the method of expansion in a series, described already in § 22a, where y, θ, $v\cos\theta$, t are given as functions of x.

Eliminate between the four equations the term involving c, and three equations are obtained between the three variables x, y, t; and in this way v. Zedlitz derives two systems of equations of different degrees of accuracy. For details reference must be made to the work of v. Zedlitz (see Note).

Collection of formulae.

On the preceding notation, and with

$$\frac{v_0\cos\theta_0}{v\cos\theta} = p, \qquad \frac{p^4-1}{\tan\theta_0 - \tan\theta} = q,$$

$$(1) \qquad x = \frac{2(v_0\cos\theta_0)^2(\tan\theta_0 - \tan\theta)}{g(1+p^2)},$$

$$(2) \qquad y = \frac{x}{q}\left(1 + q\tan\theta_0 - \frac{1}{3}\frac{p^6-1}{p^2-1}\right),$$

$$(3) \qquad t = \frac{2}{3}\frac{x}{v_0\cos\theta_0}\frac{p^3-1}{p^2-1};$$

and thence the end point of the first arc is determined.

Similarly for the end point of the second arc, and so finally for the complete trajectory.

The value of c differs in proceeding from one arc to the next, taking into account the diminution of air density with the height.

It may be mentioned that in this solution there is no need to assume a monomial law, cv^n. As stated already in § 17, the integral relation between v and θ or the so-called hodograph equation can be obtained with accuracy for any given law of air resistance.

Example (by Freiherr v. Zedlitz).

Calibre 10 cm, weight of shell 11·6 kg, initial velocity $v_0 = 353{\cdot}6$ m/sec, angle of departure $\phi = \theta_0 = 27°\ 15'$, air density 1·20 kg/m³. Results as follows:

θ	v m/sec	Calculated by (1) x m	Calculated by (2) y m	θ	v m/sec	x m	y m
27° 15′	353·6	0	0	0	229·6	3573	1010
25	325·4	460	226	− 8	220·9	4292	961
23	307·9	816	384	−16	217·2	4971	816
20	288·8	1290	572	−24	217·9	5642	572
16	269·8	1843	752	−32	222·7	6329	205
12	255·5	2333	874	−36	226·8	6688	− 38
7	242·3	2883	967				
4	236·1	3188	997				
0	229·6	3573	1010				

4. A first rough approximation to the calculation of a steep high angle trajectory is obtained by treating the path as a parabola of the third or fourth order.

Thus the formulae of Piton-Bressant and Hélie of § 22a may be employed:

$$y = x \tan \phi - \frac{gx^2}{2v_0^2 \cos^2 \phi}(1 + Kx),$$

$$\tan \theta = \tan \phi - \frac{gx}{2v_0^2 \cos^2 \phi}(2 + 3Kx),$$

$$\frac{v \cos \theta}{v_0 \cos \phi} = \frac{1}{\sqrt{(1 + 3Kx)}},$$

$$t = \frac{2}{9v_0 \cos \phi} \frac{(1 + 3Kx)^{\frac{3}{2}} - 1}{K}.$$

Thence K can be determined, given v_0 and ϕ, and the measured range X, by

$$1 + KX = \frac{v_0^2 \sin 2\phi}{gX}.$$

But we must not expect to obtain very exact results in this way, as it is manifest that on the same trajectory, and with same v_0 and ϕ, the factor K along the path cannot be taken as constant.

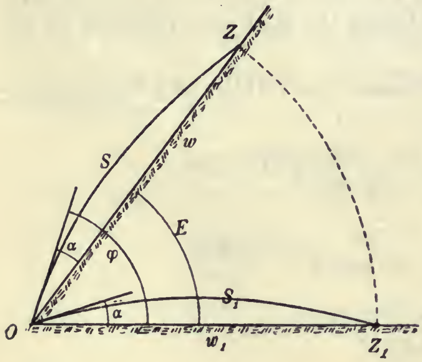
The Tables of Otto (Vol. IV) can serve the purpose, in so far as they go, of obtaining corrections with respect to the height of the vertex.

II. On the errors which can arise in the tilting of a trajectory.

A general consideration of these errors can only be made on the conditions of a vacuum (§ 4); in practice, the errors must be investigated for each particular case.

§ 38. Ordinary procedure of tilting or swinging a trajectory.

A trajectory OS_1Z_1, as in the figure, is treated as a rigid curve, and rotated through the angle E of the slope of the ground into the steep position OSZ. Conversely, if a mark Z on the line of sight OZ at the angle, E, of the slope of the ground, is struck when the actual angle of departure with the horizon is $\phi = E + \alpha$, it is assumed that the trajectory OSZ can be calculated, as if it were treated as a flat trajectory OS_1Z_1 with the angle of departure $\phi - E$ or α, and with the same initial velocity.



From the preceding examples the ranges on the steep slope, w or OZ, are known for four high angle trajectories OSZ, and also the angles of departure of the same shell, relating to several angles of slope E of the ground.

Then if the flat trajectory range w_1 is calculated by Vallier's method for the angle of tangent sighting $\phi - E$, and compared with w, the error ϵ is obtained, due to the employment of the method of swinging the trajectory.

The adjoining table enables us to see that under these conditions, with the employment of the same tangent elevation, the range is greater over the rising ground than over the horizontal; or in other words if, as usual, the tangent graduations of the weapon are marked for aiming at a target on the horizontal through the muzzle, then for aiming at a target at the same distance but at a high angle of sight, the tangent elevation must be smaller.

As stated in § 4, we are concerned chiefly with long range fire; and sometimes the corresponding relations may be found to be reversed completely.

	Slope of ground $E=$	Tangent elevation from ground slope $a=\phi-E=$	Flat-trajectory range w_1 with angle of departure a and slope zero	Actual range w on the slope for incline E and tangent elevation a	Difference between w and w_1 $\epsilon=w-w_1=$	Error as percentage of w_1 %
$\phi=80°$	78°	2°	$w_1=1111$ m	$w=2847$ m	1736	156
	76	4	1791	3535	1744	91
	74	6	2325	3713	1388	60
$\phi=75°$	74	1	667	1525	858	129
	72	3	1480	2827	1347	91
	70	5	2069	3416	1347	65
	68	7	2554	3656	1102	43
$\phi=70°$	68	2	1117	2094	977	88
	66	4	1791	2909·5	1118·5	62
	64	6	2326	3352	1026	44
	62	8	2770	3590	820	30
$\phi=65°$	64	1	668	1140	472	71
	62	3	1480	2300	820	55
	60	5	2069	2918	849	41
	58	7	2554	3310	756	30
	56	9	2966	3565	599	20
	54	11	3319	3724	405	12

At all events the errors that arise in the simple tilting of a trajectory may be serious.

§ 39. Burgsdorff and Gouin's method.

The principle is the following: Given a flat trajectory OZ_1 with the angle of departure α_1, as in the figure on the next page.

Here the height fallen $A_1Z_1=f$ is known, being the vertical distance from A_1 to the muzzle horizontal in Z_1, and also the distance $OA_1=a$; and OA_1Z_1 is considered as if it were a bar with hinges at O and A_1, or as a fishing-rod OA_1, with line A_1Z_1; the system is rotated about O, the point of departure, as a fixed centre of rotation into the position OAZ, with

$$OA=OA_1=a, \text{ and } AZ=A_1Z_1=f.$$

The rotation is carried out till Z_1 reaches the line OZ, sloping up at the given angle $ZOZ_1=E$.

Then let OZ or w be the range on the slope for the high angle fire, and let the angle of departure be reckoned from the horizontal $AOZ_1 = E + \alpha$, so that α is the tangent elevation with respect to the ground line of sight.

This angle α and the range w are given for this rotation; and they are found either by a graphical construction or else from the formulae

$$\sin\alpha = \sin\alpha_1 \cos E,$$

$$w = w_1 \tan\alpha_1 \cos(E + \alpha) \operatorname{cosec}\alpha.$$

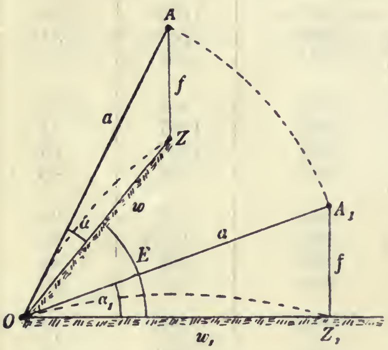
In the graphical construction we notice that in the motion described, Z_1 moves on a circle. If a point O_1 is placed vertically below O at a depth

$$OO_1 = A_1Z_1 = AZ = f,$$

a hinged parallelogram

$$OO_1Z_1A_1$$

can be changed into OO_1ZA in the second position, the fixed points of rotation being O and O_1. The point Z_1 moves along a circular arc Z_1Z with centre at O_1, and radius a.



This shifting movement can thus be carried out by means of a mechanism, or a parallelogram linkage, for any distance OZ of the target, or any angle of slope E, in order to determine the tangent elevation AOZ or α (the apparatus is constructed by the firm of Haker and Heidorn in Hamburg).

In the application of this principle the following mechanical assumptions have been made; the flat trajectory OZ_1 is described in time of flight T sec, and under the simultaneous influence of air resistance and gravity.

It is assumed that the same mark Z_1 will be reached, if these two forces come into action in succession for a time T.

The shell projected in the direction OA_1, subject to air resistance alone, but with gravity left out of account, will move in the straight line OA_1 to A_1; and afterwards under gravity, it will fall from A_1 to Z_1.

In the high angle trajectory OZ, on the same assumptions, these two steps remain the same, viz., $OA = OA_1 = a$, $AZ = A_1Z_1 = f$.

The difference between the flat and steep trajectories can only be due to the diminution of air density with the height.

But it is clear that this mechanical assumption is unsound in principle, even independently of the variation in air density. Because the movements along a and f are not independent of each other; just as the components of the motion of the shell along the x and y axes are mutually dependent (except for the special case of the law of air resistance, where the resistance is taken as proportional to the first power of the velocity).

If it were the case that this independence was true for all finite arcs, the solution of the problem would then become very simple.

The parallelogram law holds for a finite arc only when forces are constant in magnitude and direction.

But here this is not the case: the air resistance is always a variable quantity.

This swinging or tilting provides, therefore, only an approximate method, and an error is present. To determine this error the four steep high angle trajectories can be used; a, α, w are known, and then α_1 and w_1 can be plotted or calculated.

If then, corresponding to an angle of departure α_1 the correct flat trajectory range w_r is observed, or calculated on Vallier's method, it becomes a matter of comparing the values of w_1 and w_r.

In a previous example, with $\phi = 80°$, the value $E = 78°$ was chosen, and so $\alpha = 2°$. Thence we find $\alpha_1 = 9° 39'{\cdot}8$, $f = 572{\cdot}40$ m; and so $w_1 = f \cot \alpha_1 = 3361$ m.

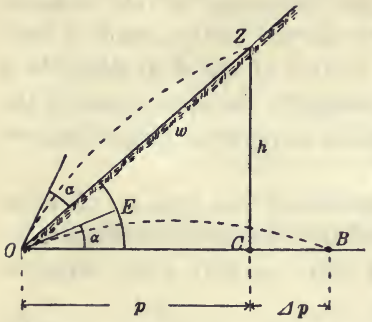
On the other hand, for the same angle of departure $\alpha_1 = 9° 39'{\cdot}8$, the true flat trajectory range $w_r = 3088$ m; and it is immaterial on which of the two ranges the percentage of error is to be estimated. With w_1 as the basis, the error works out to $8{\cdot}1\,^0/_0$.

With $E = 76°$, $\alpha = 4°$, $w_1 = 4712$ m, $w_r = 4169$ m; an error of 543 m, or $11{\cdot}5\,^0/_0$. For $E = 74°$, $\alpha = 6°$, $w_1 = 5459$, $w_r = 4722$; error 737 m, or $13{\cdot}5\,^0/_0$.

As in the other examples, it is seen that the percentage error increases with α; still it is smaller by the other processes than that which arises in the ordinary method of tilting.

§ 40. Percin's Method.

When a target Z is to be struck, situated on sloping ground OZ, at a slope $ZOB = E$, at a distance w m, the angle of departure α to be employed with respect to the ground OZ is that corresponding to a flat trajectory OB of range $p \pm \Delta p = OB$; where p denotes the horizontal projection $OC = w \cos E$ and $\Delta p = \frac{\alpha h}{100}$, where h is the vertical height CZ of the mark above O; the + or − sign to be employed, according as E is greater or less than 2α; so that $\Delta p = 0$ when $E = 2\alpha$. Under certain circumstances, Δp must be multiplied by a factor k, to be determined experimentally.



1. Example, as above, $\phi = 80°$.
For $\alpha = 2°$, and $h = 2785$ m,

$$\Delta p = \frac{2 \times 2785}{100} = 55{\cdot}7 \text{ m, and } OB = 647{\cdot}7 \text{ m},$$

with a correct angle of departure $\alpha = 0° 57'{\cdot}45$. According to Percin, $\alpha = 2°$; difference $1° 2'{\cdot}55$, or 52 %.

For $\alpha = 4°$, $h = 3430$ m, $OB = 992{\cdot}2$ m; and the correct $\alpha = 1° 41'{\cdot}7$; difference of 58 %.

For $\alpha = 6°$, $h = 3569{\cdot}5$ m, $OB = 1238{\cdot}2$ m, and correct $\alpha = 2° 14'{\cdot}1$; difference of 63 %.

2. Example, as above, $\phi = 75°$.
For α=1, 3, 5, 7°, the differences were 42, 50, 53, 58 %, respectively.

3. Example, as above, $\phi = 70°$.
For α = 2, 4, 6, 8°, differences 26, 29, 30, 32 %.

Result: Thus we see that the ordinary procedure of tilting the trajectory is not applicable with very high angle fire.

The method of Percin is, moreover, inexact, unless an empirical factor k is introduced. However on slopes up to about 65° the percentage error on this method comes out nearly constant, so that good results may be obtained by the introduction of such a factor.

The smallest errors arise from the employment of Burgsdorff's method; in particular for the steep part of the ascending branch of a high angle trajectory it can often give good and fairly accurate results.

CHAPTER VIII

Solution of various trajectories. Employment of experimental results for the construction of Range Tables

§ 41. I. Solution of problems by means of Table 12, Vol. IV.

System of formulae.

(A) for any given point (xy) on the trajectory.

Primary Functions D, J, A, T (Table 12a, Vol. IV):

$$\frac{x}{c'} = D(u) - D(v_0), \qquad (1)$$

$$t = \frac{c'}{\cos\phi}[T(u) - T(v_0)], \qquad (2)$$

$$\tan\theta = \tan\phi - \frac{c'}{2\cos^2\phi}[J(u) - J(v_0)], \qquad (3)$$

$$y = x\tan\phi - \frac{c'x}{2\cos^2\phi}\left[\frac{A(u) - A(v_0)}{D(u) - D(v_0)} - J(v_0)\right], \qquad (4)$$

$$v = \frac{u\cos\phi}{\cos\theta}. \qquad (5)$$

Secondary Functions E, H, L, etc. (Tables 12b to 12f):

$$\xi = \frac{x}{c'} = D(u) - D(v_0), \qquad (6)$$

$$t = \frac{c'}{\cos\phi} H(v_0, \xi), \qquad (7)$$

$$\tan\theta = \tan\phi - \frac{c'}{2\cos^2\phi} L(v_0, \xi), \qquad (8)$$

$$\tan\theta = \tan\phi\left[1 - \frac{c'}{\sin 2\phi} L(v_0, \xi)\right], \qquad (9)$$

$$y = x\tan\phi - \frac{c'x}{2\cos^2\phi} E(v_0, \xi), \qquad (10)$$

$$y = \frac{xc'}{2\cos^2\phi}\left[\frac{\sin 2\phi}{c'} - E(v_0, \xi)\right]. \qquad (11)$$

(B) At the point of descent ($y=0$, $x=X$):

$$\xi_e=\frac{X}{c'}, \qquad\qquad (12)$$

$$\sin 2\phi = XN(v_0, \xi_e)=c'E(v_0, \xi_e), \qquad\qquad (13)$$

$$v_e=\frac{u_e\cos\phi}{\cos\omega}, \qquad\qquad (14)$$

$$\xi_e=D(u_e)-D(v_0), \qquad\qquad (15)$$

$$T=\frac{c'}{\cos\phi}H(v_0, \xi_e) \qquad\qquad (16)$$

$$-\tan\theta_e=\tan\omega=\frac{c'}{2\cos^2\phi}M(v_0, \xi_e)=\tan\phi\,\frac{M(v_0, \xi_e)}{E(v_0, \xi_e)}. \qquad\qquad (17)$$

(C) Vertex ($x=x_s$, $y=y_s$, $\theta=0$):

$$\frac{\sin 2\phi}{c'}=L(v_0, \xi_s), \qquad\qquad (18)$$

$$t_s=\frac{c'}{\cos\phi}H(v_0, \xi_s), \qquad\qquad (19)$$

$$\xi_s=\frac{x_s}{c'}=D(u_s)-D(v_0), \qquad\qquad (20)$$

$$y_s=\frac{c'x_s}{2\cos^2\phi}M(v_0, \xi_s)=x_s\tan\phi\,\frac{M(v_0, \xi_s)}{L(v_0, \xi_s)}, \qquad\qquad (21)$$

$$v_s=u_s\cos\phi. \qquad\qquad (22)$$

(D) Some empirical and semi-empirical approximations:

$$\left.\begin{aligned} y_s&=1{\cdot}226T^2,\\ y_s&=T^2(1{\cdot}226+0{\cdot}002T),\\ y_s&=\tfrac{1}{4}X\sqrt{(\tan\phi\tan\omega)},\\ y_s&=\tfrac{1}{8}X(\tan\phi+\tan\omega),\\ y_s&=\tfrac{1}{2}X\frac{\sin\phi\sin\omega}{\sin(\phi+\omega)},\\ y_s&=\frac{0{\cdot}55X}{\cot\phi+\cot\omega}, \end{aligned}\right\} \qquad\qquad (23)$$

$$\left.\begin{aligned} x_s&=\tfrac{1}{4}X+y_s\cot\phi,\\ x_s&=X\left(0{\cdot}5+\frac{v_0}{10000}\right). \end{aligned}\right\} \qquad\qquad (24)$$

Practical rule: The abscissa x_s of the vertex is equal to that range in the Range Table, for which the angles of descent and of elevation are together equal to the elevation required for the whole range. This is derived from tilting the trajectory, because the tangent at the vertex is horizontal; compare §§ 38 to 40.

(E) The notation is as follows:

$2R$ = calibre in cm, P weight of shell in kg, (xy) any given point, x horizontal axis, positive in the direction of fire, y vertical axis and positive upwards, $(x_s y_s)$ coordinates of the vertex, X the range in m on the horizontal plane through the muzzle; v_0 the muzzle velocity in m/sec; v = velocity in the trajectory at any given point (xy); v_s vertex velocity, v_e velocity at point of descent $(y = 0, x = X)$, θ inclination to the horizon of the tangent at any point (xy); $\theta_e = -\omega$ at the point of descent; ω the acute angle of descent; t the time of flight to reach any point (xy); t_s time to the vertex $(x_s y_s)$, T the time to the point of descent, or the total time of flight; $u = \dfrac{v \cos\theta}{\cos\phi}$; u_s and u_e the values of u at the vertex and point of descent respectively; δ the air density in kg/m³:

$$c' = \frac{1{\cdot}206P}{R^2 \delta i_0 \beta} \quad (P \text{ in kg, } R^2 \text{ in cm}^2). \quad \ldots\ldots\ldots(25)$$

$i_0 \beta$ depends on the shape of the shell and the curvature of the trajectory; according to Vallier,

$$\beta = 1, \quad \ldots\ldots\ldots(26)$$

or more accurately $$\beta = \cos \tfrac{2}{3}\phi, \quad \ldots\ldots\ldots(27)$$

or $$\beta i_0 = \frac{6 i_0 K'(v_0) \sec^3\phi + 5 i(v_s)(1 - 0{\cdot}00011 y_s) K'(v_s)}{[6K'(v_0) + 5K'(u_s)] \sec^2\phi}, \ldots(28)$$

and according to Vallier, i depends on the velocity of the shell and on the semi-vertical angle γ_1 of the point of the shell; that is

$$\text{for } v > 330 \text{ m/sec}, \quad i(v) = \gamma_1 \frac{v - (180 + 2\gamma_1)}{41{\cdot}5(v - 263)}; \quad \ldots\ldots(29)$$

for $v < 330$ m/sec, i depends only on γ_1,

that is

$i =$	0·67	0·72	0·78	1·00	1·10
for $\gamma_1 =$	31°	33°·6	36°·9	41°·5	48°·2

and i_0 denotes the value of i for $v = v_0$.

For the original Krupp normal shell with ogival head of 2 calibres of rounding, or 1·3 calibres of height of head, or $\gamma_1 = 41°{\cdot}5$, i is constant $= i_0 = 1$.

According to O. v. Eberhard the value of i for any given velocity v is given in the equations on p. 49.

$K'(v)$ is given in the following table:

v	$10^{12} K'(v)$	v	$10^{12} K'(v)$	v	$10^{12} K'(v)$	v	$10^{12} K'(v)$
150	1870	270	765	390	687	700	228
160	1680	280	722	400	669	750	194
170	1510	290	687	420	628	800	164
180	1401	300	651	440	590	850	140
190	1293	310	667	460	550	900	121
200	1195	320	687	480	511	950	106
210	1124	330	706	500	474	1000	92
220	1037	340	720	525	431	1050	81
230	971	350	724	550	393	1100	72
240	913	360	722	575	351	1150	64
250	856	370	714	600	325	1200	57
260	808	380	702	650	272	1250	51

The law of the retardation $cf(v)$ due to air resistance, given in kg, is as follows:

for $v > 330$ m/sec, $cf(v) = c \times 0{\cdot}125(v - 263)$ (Chapel-Vallier law),

$$\left.\begin{array}{ll} \text{for } 330 > v > 300, & cf(v) = c \times \overset{(11)}{0{\cdot}021692}\, v^5 \\ \text{for } 300 > v, & cf(v) = c \times \overset{(6)}{0{\cdot}033814}\, v^{2{\cdot}5} \end{array}\right\} \text{(Hojel law)},$$

$$c = \frac{R^2(\text{m}^2)\, 1000\, \delta_y i}{P \times 1{\cdot}206} = \frac{R^2(\text{cm}^2)\, \delta_y i}{P \times 1{\cdot}206};$$

$i = 1$ for ogival shell of 2 calibre radius of rounding, since the laws of air resistance were obtained by experiment with shell of this form; moreover i, as shown later, is slightly variable with the velocity, as well as with the shape of the shell; δ_y is the air density at the height y m above the ground; and according to St Robert and E. Vallier,

$$\delta_y = \delta(1 - 0{\cdot}00008y);$$

according to Charbonnier, $\delta_y = \delta(1 - 0{\cdot}00011y)$, where δ is the air density on the ground.

Some examples.

1. Given $2R$, P, δ, i_0, v_0, and ϕ. To determine the elements of the vertex, and of the point of descent. In the first approximation β is taken as in (26); or if $\phi > 10°$, from (27); thence c' from (25), then $L(v_0, \xi_s)$ from (18), and thence ξ_s, and consequently $x_s = c'\xi_s$; moreover y_s from (21), and $D(u_s)$ and consequently u_s from (20), as well as v_s from (22).

With these values of u_s, v_s, and thence of $K'(v_s)$ and $K'(u_s)$, calculate a closer value of βi_0 from (28); and then the new c' is obtained from (25), and at the same time a closer value of $(x_s y_s)$ at the vertex.

Then E follows from (13), and thence ξ_e; and from (12) the range X; further the angle of descent ω from (17); and from (16) the total time of flight T; from (15) $D(u_e)$, and thence u_e; and then the final velocity v_e from (14). Instead of i_0, γ_1 may be given.

Numerical example. $2R=15$ cm, $P=27{\cdot}44$ kg, $\delta=1{\cdot}27$ kg/m³, 2 calibre radius of rounding of the ogival head of the shell, and so $i=i(v_0)$ is constant $=1$, $v_0=460$ m/sec, $\phi=40°\,7'$.

Since $\phi>10°$, $\beta=\cos\frac{2}{3}\phi=0{\cdot}89301$, in a first approximation; and then in (25), $c'=0{\cdot}5187$, so that from (18)

$$L(v_0,\xi_s)=\frac{\sin 2\phi}{c'}=\frac{\sin 80°\,14'}{0{\cdot}5187}=1{\cdot}8998.$$

Then from Table 12 e, Vol. IV, interpolation gives the value $\xi_s=8915$.

Thence it follows in the first approximation

from equation (20), $x_s=c'\xi_s=8915\times 0{\cdot}5187=4625$ m,
" Table 12 f, $M(v_0,\xi_s)=1{\cdot}142$,
" equation (21), $y_s=2342$ m,
" equation (20), $u_s=237{\cdot}8$,
" equation (22), $v_s=181{\cdot}2$.

The Table of K' values gives

$$K'(v_0)=550.10^{-12},\quad K'(u_s)=925.10^{-12},\quad K'(v_s)=1379.10^{-12};$$

and with these values a closer value of $i_0\beta$ is obtained from (28), that is

$$\beta=\frac{6.1.550.10^{-12}\sec^3 40°\,7'+5.1(1-0{\cdot}00011.2342)\,1379.10^{-12}}{(6.550.10^{-12}+5.925.10^{-12})\sec^2 40°\,7'},$$
$$\beta=0{\cdot}946.$$

A recalculation with this value 0·946 leads to a third value, $\beta=0{\cdot}956$; further repetitions will not increase the accuracy of the solution.

2. Given $2R$, P, δ, i_0 (or the value of γ_1, instead of i_0), and further v_0 and X. To determine the angle of departure ϕ. Calculate β from (26) in a first approximation, c' from (25), ξ_e from (12), $\sin 2\phi$ and thence ϕ from (13).

Next in the second approximation, determine β from (27), and so on, as before.

3. Given $2R$, P, δ, i_0 (or γ_1), X and ϕ; to determine the initial velocity v_0.

Calculate β from (27), then c' from (25), ξ_e from (12), N from (13), whence v_0 is known. After this a closer determination of v_0; calculation of the vertex elements, as in example 1; thence a closer value of β from (28), and a closer c' from (25), with which to repeat the calculation.

4. Given v_0, ϕ, X, to determine elements of the point of descent, and of the vertex (for calculation of range tables and so forth).

From (13) N is found, and thence ξ_e; then c' from (12), ω from (17), u_e from (15), and v_e from (14), T from (16). Next for the vertex: L from (18), and with it ξ_s and x_s from (20), also u_s; v_s from (22), t_s from (19), y_s from (21).

5. Given X, ϕ, T, to determine the initial velocity, etc.

A first approximate value of v_0 is given by the formula $v_0=\frac{3}{2}\sqrt{\frac{gX}{\sin 2\phi}}$ (as for a vacuum, with the factor $\frac{3}{2}$), or better from a suitable range table; suppose it v_0^{I}. Calculate N from (13) with it, whence ξ_e is given; hence a first approximate value of T from (16); denote it by T^{I}.

Another value of v_0 is now assumed, smaller than v_0^{I} if the observed time of flight is smaller than the calculated T^{I}, or conversely; denote this value of v_0 by v_0^{II}. With it in the same way as before, another value of T is obtained; denote it by T^{II}.

Finally we interpolate; taking v_0^{I} and v_0^{II}, from which T^{I} and T^{II} were calculated, T is determined, and also v_0, by means of

$$\frac{v_0 - v_0^{II}}{v_0^{I} - v_0^{II}} = \frac{T - T^{II}}{T^{I} - T^{II}}.$$

§ 42. Solution of examples by means of other tables and formulae.

II. Solution with Siacci's Table 13, in Vol. IV.

System of formulae.

$$\frac{x}{c'} = D(u) - D(v_0), \quad \text{......(1)}$$

$$y = x \tan \phi - \frac{c'x}{2\cos^2\phi}\left[\frac{A(u) - A(v_0)}{D(u) - D(v_0)} - J(v_0)\right], \quad \text{......(2)}$$

$$\tan \theta = \tan \phi - \frac{c'}{2\cos^2\phi}[J(u) - J(v_0)], \quad \text{......(3)}$$

$$t = \frac{c'}{\cos \phi}[T(u) - T(v_0)], \quad \text{......(4)}$$

$$u = \frac{v \cos \theta}{\cos \phi}, \quad \text{......(5)}$$

$$c' = \frac{1}{c\beta}. \quad \text{......(6)}$$

Then in $cf(v)$, the retardation due to air resistance,

$$c = \frac{(2R)^2 \delta \,.\, 1000 \,.\, 0{\cdot}865\, i}{P \,.\, 1{\cdot}206} \quad \text{......(7)}$$

(calibre $2R$ in m, weight of shell P in kg, $i = 1$ for ogival shell of 2 calibre rounding), and

$$f(v) = 0{\cdot}2002v - 48{\cdot}05 + \sqrt{[(0{\cdot}1648v - 47{\cdot}95)^2 + 9{\cdot}6]}$$
$$+ \frac{0{\cdot}0442v\,(v - 300)}{371 + \left(\frac{v}{200}\right)^{10}} \text{ (see Table 7 in Vol. IV).} \quad \text{......(8)}$$

The value of β is taken from Table 13 of Siacci.

In those cases where the table does not go far enough, β is to be calculated from the expression

$$\beta\left[6\frac{f(v_0)}{v_0^4}+5\frac{f(u_s)}{u_s^4}\right]\sec^2\phi$$
$$=6\frac{f(v_0)}{v_0^4}\sec^3\phi+5(1-0{\cdot}00011y_s)\frac{f(v_s)}{v_s^4}, \quad \ldots(9)$$

in which u_s and v_s are determined in the following manner: Calculate y_s and x_s, as described later in III: then $\frac{x_s}{c}$, from (1), taking $\beta=1$; thence u_s and $v_s=u_s\cos\phi$.

The corresponding values in (8) of $D(u)$, $J(u)$, $T(u)$, $A(u)$, are to be found in Table 13, Vol. IV.

The secondary tables have been calculated by E. Fasella; see *Tavole balistiche secondarie,* Genoa, 1901.

The ballistic curves, which are described later, can be used instead of Fasella's tables.

III. Solution by means of the ballistic curve tables, Nos. IIIa to IIIg, in Vol. IV.

These tabulated curves, as mentioned already in § 30a, are employed for the elements of the point of descent and the vertex, and serve for the solution of the most important problems of trajectories with sufficient accuracy for practical purposes.

The tables contain, for a series of values of $c\beta X$, and for many values of v_0, the values of

$$\frac{v_0^2\sin 2\phi}{X},\quad \frac{\tan\omega}{\tan\phi},\quad \frac{v_e\cos\omega}{v_0\cos\phi},\quad \frac{T}{\sqrt{(X\tan\phi)}},\quad \frac{x_s}{X},\quad \frac{y_s}{X\tan\phi},\quad c\beta v_0^2\sin 2\phi.$$

Here c has the value given in (7); and $2R$ = calibre of shell in m, etc. β = value from Table 13, Vol. IV.

In the Tables III a to III g, the curves are drawn for different values of v_0; on the assumption of a quadratic or cubic law the curves would be coincident.

Examples.

1. Given v_0, ϕ, X: calculate $\frac{v_0^2\sin 2\phi}{X}$, and look out on the curve, corresponding to the value of v_0 in Table IIIa, the ordinate A_1; and then the abscissa $c\beta X$. Look out for the same abscissa $c\beta X$ all the

ordinate values in the other tables. Since v_0, ϕ, X are given, this determines ω from Table IIIb, $v_e \cos \omega$ from Table IIIc, and thence v_e is known; Table IIId gives T; Table IIIe the value of x_s, Table IIIf the value of y_s.

Finally the value of c is obtained from the given abscissa $c\beta X$, because β is given in Table 13 of Vol. IV. If $2R$, P, δ are known, then i is given.

2. Given c, ϕ, v_0, to determine X. Proceeding from $c\beta v_0^2 \sin 2\phi$ with $\beta = 1$, determine, as before, ω, $v_e \cos \omega$, v_e, T, x_s, y_s, and X; the last with the help of Table IIIa, because $\frac{v_0^2 \sin 2\phi}{X}$ is known, in which v_0 and ϕ are given; thence β is found more accurately by a repetition of the operations.

3. Given v_0, c, X, to determine ϕ and the remaining elements.

First take $\beta = 1$, and determine all the ordinate values; from Table IIIa the value of $\frac{v_0^2 \sin 2\phi}{X}$, and thence ϕ; thence a closer value of β from the β Table 13 in Vol. IV: then repeat the procedure.

The remaining tables then give ω, v_e, T, x_s, y_s.

4. Given v_0, ϕ, ω, to determine X and the remaining elements.

Start with Table IIIb; in the corresponding v_0 curve the ordinate $\frac{\tan \omega}{\tan \phi}$ is given; then all the rest are known. X is given by Table IIIa, and the value of c from the abscissa $c\beta X$, since β is given in the table.

Example. Given the initial velocity $v_0 = 550$ m/sec; range $X = 6841$ m, angle of departure $\phi = 20°$; weight of shell $P = 6{\cdot}9$ kg, calibre $2R = 0{\cdot}077$ m; mean air density $\delta = 1{\cdot}206$ kg/m³.

We find then $\frac{v_0^2 \sin 2\phi}{X} = A_1 = 28{\cdot}4$; thence $A_2 = 1{\cdot}74$, $A_3 = 0{\cdot}37$, $A_4 = 0{\cdot}512$, $A_5 = 0{\cdot}563$, $A_6 = 0{\cdot}3440$, $A_7 = 145000$. From A_2 we find $\tan \omega = 1{\cdot}74 \tan 20°$, angle of descent $\omega = 32° 22'$. From A_3, $v_e \cos \omega = 191$, and so velocity $v_e = 226$ m/sec. From A_4, time of flight $T = 25{\cdot}5$ sec: from A_5, vertex abscissa $x_s = 3860$ m: from A_6, vertex ordinate $y_s = 856$ m. From A_7, value of $c\beta = 0{\cdot}746$.

Now since $c\beta = \frac{(2R)^2 \delta i\beta . 1000 . 0{\cdot}865}{P . 1{\cdot}206}$ it follows that $i\beta = 1{\cdot}00_3$; whence the form coefficient i can be calculated by means of Table 13 for β.

IV. Solution by means of the formulae of Piton-Bressant.

System of formulae. See § 22a.

$$y = x \tan \phi - \frac{gx^2}{2v_0^2 \cos^2 \phi}(1 + Kx), \quad \text{.........(10)}$$

$$\tan \theta = \tan \phi - \frac{gx}{2v_0^2 \cos^2 \phi}(2 + 3Kx), \quad \text{.........(11)}$$

$$v \cos \theta = \frac{v_0 \cos \phi}{\sqrt{(1 + 3Kx)}}, \quad \text{.........(12)}$$

$$t = \frac{2}{9Kv_0 \cos \phi}[(1 + 3Kx)^{\frac{3}{2}} - 1]. \quad \text{.........(13)}$$

For the point of descent on the muzzle horizon ($y = 0$, $x = X$):

$$\frac{v_0^2 \sin 2\phi}{gX} = Z, \quad \text{.........(14)}$$

$$\frac{\tan \omega}{\tan \phi} = 2 - \frac{1}{Z}, \quad \text{.........(15)}$$

$$\frac{v_e \cos \omega}{v_0 \cos \phi} = \frac{1}{\sqrt{(3Z - 2)}}, \quad \text{.........(16)}$$

$$T = \frac{2X}{9v_0 \cos \phi} \frac{(3Z - 2)^{\frac{3}{2}} - 1}{Z - 1}, \quad \text{.........(17)}$$

$$1 + KX = Z. \quad \text{.........(18)}$$

For the vertex ($x_s y_s$):

$$x_s = \frac{\sqrt{[1 + 3KX(1 + KX)]} - 1}{3K}, \quad \text{.........(19)}$$

$$y_s = X \tan \phi \frac{1 + 2KX}{2 + 3KX}. \quad \text{.........(20)}$$

The factor K depends on v_0 and ϕ, and is assumed as practically constant along a flat trajectory.

On the other hand K alters rapidly from one trajectory to another on the same range table.

Putting $K = \frac{\lambda v_0^2}{\cos \phi}$, then λ does not vary so rapidly. The value of K will generally be determined by (14) and (18), and so from

$$\frac{v_0^2 \sin 2\phi}{g} = X(1 + KX),$$

for a definite trajectory, for which v_0, ϕ, and X are known.

§ 43. Practical applications to the calculation of range tables.

For the preparation of range tables see Heydenreich, *Lehre vom Schuss*, 1908, Part I, pp. 70 etc.

Here it can be stated briefly that the following quantities are to be measured, in suitable weather: velocity of shell near the muzzle, angle of departure from angle of elevation and jump, moreover the relation between a number of angles of elevation and the mean range attained and the mean time of flight, as well as the vertical deflection of the mean point of impact from the plane of fire (the vertical plane through the axis of the gun).

With this practice for mean ranges, the measurement of the dispersion in range and direction will be possible at each distance.

In shells with time fuzes, the relations between elevation, fuze-setting, and mean time of burning will be determined, as well as the variations in any of these.

The experiments will be carried out with all charges for guns of variable loading, in such a way that interpolation may be permissible for intermediate charges.

See Vol. III for the determination of the velocity of the shell, and the aberration of angle of departure (jump).

For the calculation of a range table in general for a gun, with a given charge, and for a large number of angles of departure ϕ, the mean range X requires to be measured, the mean time of flight T, and for time fuzes, the time setting, and the corresponding range. Moreover the air density δ_t is to be measured (§ 15).

The procedure of the calculations is now in general as follows:

1. *Estimation of the initial velocity* $v_{0,t}$ *of the shell.*

In reality it is not possible, in range table experiments, to measure the initial velocity in the immediate neighbourhood of the muzzle over a very short range, as for instance by spark photography (Vol. III, §§ 126—130). Generally the mean velocity of the shell will have to be measured at a distance of 25 to 100 m from the muzzle by means of the Boulengé apparatus.

In this way, for instance, the first screen of wires is placed at 25 m from the muzzle, and the second screen at 75 m; the mean velocity of the shell is measured over a distance of 25 to 75 m in front of the muzzle, and this is called v_{50}; but more accurately, owing to the use of vertical screens of measurement, this is the mean horizontal component, $v \cos \theta$.

From a series of measurements, of 5 to 10 shells, the mean value of the horizontal component of the velocity at 50 m in front of the muzzle is obtained.

The mean value is first to be reduced to the value at the muzzle, and that can be carried out by equation (1), § 41.

If the measurements of velocity are carried out at a small angle of departure ϕ with vertical screens, then from (1), § 41,

$$\frac{50}{c'} = D(v_{50}) - D(v_0 \cos\phi),$$

from which $v_0 \cos\phi$ follows, and then v_0.

The factor c' is chosen as a first approximation from the value for a similar gun, when it is not known from previous experiments with the same gun and shell; then a first approximate value is obtained for the initial velocity.

Employing this value in the shortest range X and its angle of departure ϕ, a fresh calculation gives a closer value of c', and this will be used to redetermine the velocity at the muzzle.

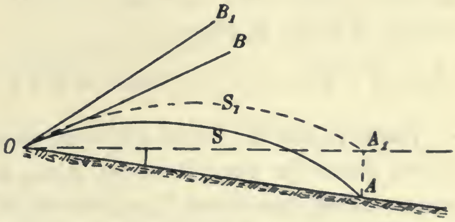
For these calculations the Krupp tables are most convenient, Vol. IV, Table 8. A numerical example is given in § 27.

Tables have been constructed lately by Krupp, after the tables of Fasella, by the use of which these calculations can be simplified.

The value so obtained of the initial velocity, denoted by $v_{0,t}$, called the initial velocity of the day, is to be employed in the further calculations.

2. *Repetition of the calculations on the observations over the muzzle horizon.*

When the angle of departure and of slope of the ground is so small, that the principle of swinging the trajectory may be employed, the trajectory OSA is supposed to be rotated as if it were a rigid curve, into the position OS_1A_1.



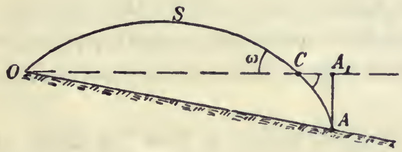
Suppose for instance the angle of elevation to be 5°, the error of departure (jump) to be + 10′, so that the angle of departure is 5° 10′; and further the difference of height AA_1 between the point of descent, A, and of departure, O, to be equal to 5 m, at a range of 2500 m along the sloping ground, so that the slope of the ground is $-7'$.

In the tilted trajectory OS_1A_1 the corrected angle of departure above the muzzle horizon is then

$$B_1OA_1 = \phi = 5° + 10' + 7' = 5° \, 17'.$$

These values, $X = 2500$ m and $\phi = 5° \, 17'$, will serve as a basis for the calculations.

If the tilting of the trajectory is not allowable (see above, § 38), determine the point of intersection of the trajectory with the horizontal OA_1, and suppose for example $AA_1 = 4$ m; then calculate as a first approximation the acute angle of descent $A_1CA = \omega$ (and also the time of flight T for subsequent purposes: this is easily done with the ballistic curves, III b and III d, of the curve tables of Vol. IV).



Suppose it is found that $\omega = 30°$; then $A_1C = \sim 4 \cot 30° = 7$ m. Thence the reduced range over the muzzle horizon OC or

$$X = OA_1 - A_1C = \sim OA - A_1C = 2300 - 7 = 2293 \text{ m}.$$

This procedure is not permissible except under certain assumptions on the smallness of the land slope AOA_1. Whether it is to be allowed must be investigated in each special case.

3. *The effect of the wind.*

The corresponding values of X, ϕ, v_0 may require to be corrected for the effect of a wind, having a component $+w$ in the direction of the line of fire (but w is to be taken negative when the wind is in the opposite direction, and follows the direction of motion). To make the necessary correction, calculate with the reduced values X_r, v_r, ϕ_r, instead of X, v_0, ϕ, where

$$X_r = X - Tw, \quad v_r^2 = v_0^2 - 2v_0 w \cos\phi + w^2, \quad \tan\phi_r = \frac{v_0 \sin\phi}{v_0 \cos\phi - w} \quad (\S\ 47).$$

These values X_r, v_r, ϕ_r serve as data for the further calculations: denote them for simplicity by X, v_0, ϕ. A calculation can also be made, slightly less exact, by equation (10), § 47. The equations there given can be used for the lateral deviation of the point of mean impact from the plane of fire.

4. We therefore know how to determine the mean range X_t, mean lateral deviation Z_t from the plane of fire, and also the mean time of flight T_t, under these conditions.

The uses, to which the results are put, depend on the object of the calculations.

In a complete calculation of a range table the values X_t, ϕ corresponding to the initial velocity $v_{0,t}$ of the day will be employed to determine the ballistic coefficient c'_t of the day by calculation.

If for instance the table of § 41 (Vol. IV, Table 12) is employed for the purpose, the calculations will proceed as follows:

Calculate for every group of observed $v_{0,t}$, ϕ, X_t by equation (13), § 41, $\sin 2\phi = XN(v_0, \xi_e)$, the value of ξ_e; and then from equation (13) and the observed air density δ_t of the day, and the range X_t obtained on the day, the ballistic coefficient c'_t which holds good.

This value is to be corrected for a normal air density δ_n, as derived from the equation $c' = \dfrac{1{\cdot}206\, P}{R^2 \delta i_0 \beta}$, and $c'_n = c'_t \dfrac{\delta_t}{\delta_n}$.

The series of c'_n values so obtained are plotted graphically as a function of X.

If the experiments have been made on several days, and repeated several times, then the results are plotted out. A curve is then drawn through the plotted points, and smoothed if necessary.

From the curve so obtained of $c'_n = f(X)$, the corresponding values of c'_n can be read off for the separate values of X, proceeding at given definite intervals; for instance for $X = 1000, 3000, 5000, \dots$ m; and then for any such pair of values of c'_n, X, the value is found of $\xi_e = \dfrac{X}{c'_n}$.

The angle of departure ϕ is then given for this value of ξ_e by $\sin 2\phi = XN(v_0, \xi_e)$, and ω the angle of descent by

$$\tan \omega = \frac{c'_n}{2 \cos^2 \phi} M(v_0, \xi_e);$$

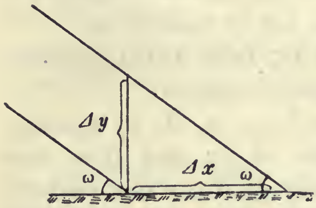
the time of flight $T = \dfrac{c'}{\cos \phi} H(v_0, \xi_e)$, and u_e from $D(u_e) = \xi_e + D(v_0)$; thence the final velocity $v_e = u_e \cos \phi \sec \omega$.

For a change $\Delta\phi$ of the angle of departure, in circular measure, the corresponding change of range is $\dfrac{\Delta X}{X} = \dfrac{2 \tan \phi}{\tan \omega} \dfrac{\Delta \phi}{\tan 2\phi}$.

The corresponding change Δy for given X is given on the figure, $\Delta y = \Delta X \tan \omega$.

Choosing $\Delta\phi = \frac{1}{16}$ of a degree, this formula gives the measure of the change of height in the striking point due to a change of $\frac{1}{16}$ degree, and the change in range.

In many cases it is desirable in these calculations to start with the angle of departure, and not from the range.



The procedure is then as follows, again using Table 12, Vol. IV: the former values obtained for c'_n are not to be plotted as functions of the range, but of the departure angle.

From the curve $c'_n = f(\phi)$ for definite values of ϕ, for instance $\phi = 2°, 5°, 10°, \ldots$ the value of c'_n is determined.

Calculate for every such pair of values, ϕ, c'_n, from equation (13), §41,

$$E(v_0, \xi_e) = \frac{\sin 2\phi}{c'_n}.$$

From the value of $E(v_0, \xi_e)$ thus obtained and by the use of Table 12 b, Vol. IV, the value of ξ_e is determined corresponding to the value v_0 of the initial velocity in the Range Table; and thence $X = c'_n \xi_e$ is given. The calculation of the remaining elements of the range table follows as before.

The elements of the trajectory obtained in this way, ϕ, v_e, T, are plotted as functions of X; and then on the curve, thus obtained, the separate elements of the trajectory are read off for $X = 100$, 200, 300,....

The secondary tables of Fasella are convenient for use in these calculations.

5. *Application of the results of fire for initial velocity and air density as required in range tables.*

According to the state of the day, the nature of the weapons and the difference of the temperature of the powder and so forth, the initial velocities of the day will differ more or less from the mean initial velocity as given in the range table. In a recalculation from the observed initial velocity $v_{0,t}$ of the day, and in a reduction to the mean value of the initial velocity v_0 of the range table, the equations employed are those of § 45:

$$\frac{\Delta X}{X} = \frac{2 \tan \phi}{\tan \omega} \frac{\Delta v_0}{v_0} \text{ (for howitzers and mortars),}$$

$$\frac{\Delta X}{X} = \frac{3 \tan \phi - \tan \omega}{\tan \omega} \frac{\Delta v_0}{v_0} \text{ (for guns and rifles).}$$

The amount ΔX is then obtained, which serves for the determination of the difference Δv_0 between the mean value in the range table, and the value obtained on the day.

When the difference, $\Delta \delta$, between the air density of the day, and the mean density in the range table is taken into account, we have, as in § 45,

$$\frac{\Delta X}{X} = -\frac{\tan \omega - \tan \phi}{\tan \omega} \frac{\Delta \delta}{\delta},$$

and this will give the correction ΔX.

6. By the employment of the ballistic curves, Tables III a to III g of Vol. IV, the calculation can be made in a simple manner as follows.

Curves are interpolated corresponding to the mean values of v_0 as nearly as may be; and if required the curves are drawn in pencil. All reductions are assumed as already known for comparison with the mean range table values.

Calculate $\frac{v_0^2 \sin 2\phi}{X}$ for any observed X and its corresponding ϕ, and look out in Table III a the ordinate, and then in all the other tables the ordinates relating to the same abscissa $c\beta X$. Then Table III b gives $\tan \omega : \tan \phi$, and so the angle of descent ω: Table III c gives $\frac{v_e \cos \omega}{v_0 \cos \phi}$, and $v_e \cos \omega$ and v_e too, since ω is now known; Table III d gives $\frac{T}{\sqrt{(X \tan \phi)}}$ and thence T; Table III e gives $\frac{x_s}{X}$, and x_s; Table III f gives y_s.

All these elements of the trajectory ϕ, ω, v_e, T, x_s, y_s, are plotted graphically as functions of X; and then from these curves the values corresponding to $X = 100, 200, 300, \ldots$ of ϕ, ω, v_e, T, x_s, y_s are interpolated.

If the observations of X and ϕ are not numerous enough, we proceed as in 4: corresponding to the observed X, ϕ, v_0, the value of $\frac{v_0^2 \sin 2\phi}{X}$ is calculated: the value of the abscissa $c\beta X$ is ascertained, and this gives c, since β is given by the β Table 13, Vol. IV. The c values are plotted graphically, and then the values of c read off corresponding to $X = 100, 200, \ldots$.

Then to the values, so obtained, of $c\beta X$ the tables will give the corresponding values of ϕ, ω, v_e, T, x_s, y_s.

7. Finally, when the formulae of Piton-Bressant are employed, it is convenient to represent on a diagram to a sufficiently large scale the functions of Z arising in equations (15), (16), (17), § 42, such as $2 - \frac{1}{Z}$, $\frac{1}{\sqrt{(3Z-2)}}$, $\frac{(3Z-2)^{\frac{3}{2}} - 1}{Z-1}$ so as to read off the value of each as a function of Z; a table of these functions has been given already in § 22 a.

The process is as follows: Calculate $Z = \frac{v_0^2 \sin 2\phi}{gX}$ for the different reduced ranges X and corresponding ϕ, and look out the functions on the diagram for every value of Z.

By means of the equations (15), (16), (17), § 42, and without much calculation the values of ω, v_e, T are obtained; and the intermediate values for equidistant X can be read off by interpolation.

8. In the range tables the columns of the zones of probable scattering are obtained usually by shooting at some particular ranges.

The point of impact must always be measured exactly; the point of burst of time-fuze shell is observed by photography. Consult §§ 63 to 65 on the calculation of the individual points of impact or burst. The probable errors are again to be plotted graphically as functions of the range.

A curve is then drawn through the points so plotted, and the results entered on the range table.

9. The times of flight are of frequent value in firing practice; as, for instance, in shooting at a moving mark, aircraft, ships, cavalry, etc.

Moreover the times of flight give a guide in setting a time fuze, which with a mechanical fuze can be relied on to a certain extent.

The measurement of these mean fuze settings for the range table can be carried out in different ways.

In a small calibre, the expense of ammunition is not so important a detail of the range table experiments. At the present time, according to Heydenreich, fuze experiments should be carried out with weapons of different states of wear, as, for instance, with a new gun, a similar one that has fired an average number of rounds, and one that has fired a much greater number, and at different seasons of the year, in summer and winter. From a comprehensive series of such experiments a curve can be obtained which will give the fuze setting to be inserted in the range table.

If time does not allow of this experimental procedure, or if the cost is too high owing to the calibre of the gun, the following plan is recommended:

The photographs are taken to give the mean value of the time of burning, and the time of flight. The ratio of $\frac{B_t}{T_t}$, i.e., of the time of burning to the time of flight, is noted. This ratio alters only slightly with the range, but may be different from day to day.

The ratios are again plotted on a curve. The results are taken as the basis of a further determination.

Multiplying the mean range-table time of flight T_s by the ratio $\frac{B_t}{T_t}$ for the same range on the curve, the mean range-table fuze setting B_s is obtained for the corresponding range.

This method of calculation is only approximate, and assumes that at a given range the mean range-table fuze setting B_s bears to the setting of the day B_t the same ratio as the mean range-table time of flight to the time of flight of the day, that is

$$B_s : B_t = T_s : T_t.$$

10. The calculation of the mean range-table lateral deflection, which serves also for adjusting the amount of drift imparted by the rifling, is derived from the perpendicular distance from the plane of fire of the mean point of impact.

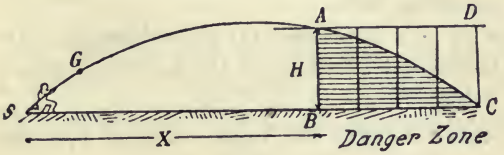
By means of a formula of Hélie, § 56, from these values of Z_t the factor A is calculated, and plotted graphically as a function of the range, taking into account the results on different days of fire.

On this curve the values of A are read off conversely for the individual ranges, and by means of the values of A and using the mean range-table v_0, the mean range-table lateral deviation Z from the plane of fire is obtained for each separate range.

The value of Z, calculated in angular measure (sixteenths of a degree, or thousandths of the radian), is plotted as a function of the range, and the curve so obtained is used for reading off the lateral deviation for the various ranges.

11. The danger zone R for a height H m of the target (for instance $H = 1$ m) is frequently given in the range table.

This is the horizontal stretch CB in the descending branch of the trajectory reaching from the point of descent C back to the point A with the ordinate $BA = H$, in the figure.



The calculation of this length $CB = R$ follows from equation (10), § 41, by finding the point A, for which $y = H$.

As this calculation is rather troublesome, an approximation can be made generally by assuming a parabolic element CA of the trajectory, when

$$R = \tfrac{1}{2}X\left[1 - \sqrt{\left(1 - \frac{4H}{X \tan \omega}\right)}\right]$$

so that in the case above, with $X = 1200$, $H = 1$ m, $R = 21$ m.

Examples of range table calculations.

1. For a rifle: Assume the mean velocity v_{25}, given by a rifle, between the muzzle and a point 50 m distant to be $v_{25} = 860$ m/sec.

The error of departure (jump) is found to be 0′·3; let it be assumed to be constant.

Next let targets be set up at distances of 1800, 1500, 1200, 900, 600, ... m; and suppose for example that on the target set up at 1200 m, the mean point of impact, when shooting with tangent elevation "1200," is at a height 105 cm above the muzzle level. Take the air density of the day at 1·255 kg/m³, and a head wind against the direction of firing of 1·53 m/sec. Take the height of the tangent sight for "1200" as 31·94 mm, the distance between the sights 500·0 mm, the height of the foresight 20·01 mm.

(*a*) Calculation of the muzzle velocity v_0: Here the equation (1), § 41, is employed, $D(v_0) = D(v_{25}) - \frac{25}{c'}$; c' is calculated from P, the weight of the bullet, $2R$ the calibre, the air density δ, and the form coefficient i. This last is not known, and so, in default of other data, it must be calculated approximately as in § 13 (Law of Lössl) from the form of the head of the bullet; and so it is found that $c' = \frac{P \times 1{\cdot}206}{R^2 \delta i}$; thence v_0 is found $= 880$ m/sec.

(*b*) Calculation of the angle of departure (§§ 146—149).

With the tangent elevation a, corresponding, for example, to a range of 1200 m, $\tan a = (31{\cdot}94 - 20{\cdot}01) : 500$, $a = 1°\ 22'{\cdot}0$.

The angle of correction ϵ is given by $\tan \epsilon = 1{\cdot}05 : 1200$, $\epsilon = 0°\ 3'\ 0''$; and so the angle of departure $\phi = 1°\ 22'{\cdot}0 + 0'{\cdot}3 - 3'{\cdot}0 = 1°\ 19'{\cdot}3$.

The value of ϕ for the other ranges is calculated in a corresponding way.

(*c*) Correction for the wind: A calculation of the time of flight T and the angle of descent ω (§ 47) gives $\Delta X = +4{\cdot}5$ m; so that, with $v_0 = 880$ m/sec, $\phi = 1°\ 19'{\cdot}3$, $X = 1204{\cdot}5$ m, $\delta = 1{\cdot}255$ the calculation is continued.

(*d*) Reduction to normal air density 1·225 kg/m³: From equation (13), § 41, $\sin 2\phi = XN(v_0, \xi_e)$, in which v_0, ϕ, X are known, $\xi_e = 12434$; then

$$c' = X : \xi_e = 1204{\cdot}5 : 12434.$$

The air density of the day $\delta = 1{\cdot}255$, and the normal air density is $\delta_n = 1{\cdot}225$; so that

$$c'_n = c' \frac{\delta}{\delta_n} = \frac{1204{\cdot}5}{12434} \cdot \frac{1{\cdot}255}{1{\cdot}225} = 0{\cdot}099.$$

Further reduction is not required, if $v_0 = 880$ m/sec is the initial velocity of the range table.

The corresponding values of c'_n are calculated in the manner described for all the ranges and are plotted graphically as a function of X; figures are then interpolated for values of the range, such as $X = 1800, 1700, 1600, \ldots$.

Thus, for example, for $X = 1200$, interpolation gives c'_n, or for brevity $c' = 0{\cdot}098_5$; thence $\xi_e = X : c' = 12180$.

Again the time of flight T is given by (16), § 41, and thus $T = c' \sec\phi\, H(v_0, \xi_e)$, and $T = 2{\cdot}78$ sec: the angle of descent ω from (17), $\tan\omega = \frac{1}{2} c' \sec^2\phi\, M(v_0, \xi_e)$, $\omega = 2°\ 46'{\cdot}6$. Moreover (18) gives $\xi_s = 7228$, and (20) $x_s = c'\xi_s = 712$ m; and then from (21) $y_s = 10{\cdot}1$ m.

Finally from (20) $u_s = \sim v_s = 390{\cdot}6$ m/sec.

The elements of the trajectory are calculated in a corresponding manner for the other ranges.

In the case of the infantry rifle, a table is usually added to the principal range table giving the ordinates of the trajectory: for this purpose, as for example for $X = 1200$ m, the corresponding angle of departure ϕ is calculated, and then from (10) or (11) § 41, the ordinate y, corresponding to a number of values of the abscissa x, as $x = 100, 200, 300, \ldots$ up to 1200.

2. For a gun: Let the mean value of the horizontal component of the velocity from five rounds at an elevation of 3° be measured at 75 m from the muzzle, and be found to be 693·5 m/sec.

The error in the angle of departure (jump), as determined by Siacci's method, with two paper screens (Vol. III, § 151, 3) is found to be 5′ 37″.

Next, suppose the mean ranges obtained with elevation angles 3°, $4\frac{1}{16}°$, $7\frac{9}{16}°$, $12\frac{1}{16}°$ to be 4152, 5046, 7409, 9724 m. The air density of the day on the ground is 1·193 kg/m³, and a wind is blowing from the right with a maximum strength of 1·5 m/sec.

(*a*) Calculation of $v_{0,t}$: Preliminary estimate of $c' = 0{\cdot}288$: thence

$$\frac{x}{c'} = 75 : 0{\cdot}288 = 260.$$

According to Krupp's Table 8, Vol. IV, $D(693{\cdot}5) = 10798$. Thence

$$D(v_0 \cos 3^\circ) = 10798 - 260 = 10538, \text{ and } v_0 \cos 3^\circ = 700 \text{ m/sec},$$

from Krupp's table. Since $\cos 3^\circ = 0{\cdot}9986$, we can take $v_0 = 700$ m/sec, in the further calculations.

(*b*) For example, at 9724 m, with angle of departure $12\tfrac{1}{16}^\circ$, the slope of ground, $-1' 52''$, and an error of departure $+5' 37''$, the true angle of departure for the furthest target is $\phi = 12^\circ 3' 45'' + 1' 52'' + 5' 37'' = 12^\circ 11' 14''$; and so $2\phi = 24^\circ 22' 28''$, $\sin 2\phi : X = 0{\cdot}00004241 = N(v_0, \xi_e)$.

In Table 12c (see Vol. IV), with $v_0 = 700$ m/sec, $\xi_e = 8602{\cdot}5$, then

$$c' = 9724 : 8602{\cdot}5 = 1{\cdot}132.$$

The calculation for the other ranges is carried out in a corresponding manner.

If the secondary tables of Fasella are employed, $\sin 2\phi : X$ is to be calculated.

Fasella denotes this quotient by f_1, and from Fasella's Table III, and $v_0 = 700$ m/sec, we get $f_0 = 2896$, $c'_t = 9724 : 2896 = 3{\cdot}358$; and then

$$c'_n = 3{\cdot}358 \, \frac{1{\cdot}193}{1{\cdot}220} = 3{\cdot}278.$$

It must be noticed that according to the method employed, the factor c' has a different meaning, so that it has not always the same value.

In a corresponding way, by the employment of Fasella's method, applied to the other data of shooting and their results, we find the values of c'_n to be 3·345, 3·158, 3·152, 3·278 (the last being as above).

In a second trial with the same gun the value $v_0 = 691$ m/sec was obtained; and with angles of elevation of 3°, $4\tfrac{1}{16}^\circ$, $7\tfrac{9}{16}^\circ$, 13°, the mean ranges were 4043, 4946, 7363, 10082 m. The air density was 1·256 kg/m³.

The corresponding values of c'_n determined on the procedure of Fasella were

3·343, 3·319, 3·442, 3·555.

The ballistic coefficients are thus seen to be variable, not only on the two days, but also during the same day; and in fact they stand on the second day on an average higher than on the first.

A curve is drawn of c'_n, for the values thus obtained on the two days of practice, to represent its value graphically as a function of X; and it is seen to rise somewhat with the range.

Based on the measurement of the initial velocities on the two days of shooting of 700 and 691 m/sec, with the temperature of the powder as +28° and +8°, the mean value for the range table is taken to be 692 m/sec, at a temperature +10°.

For example, at $X = 8000$ m, suppose the value of c'_n on the curve to be 3·275.

Calculating again with Fasella's method, $f_0 = X : c' = 8000 : 3{\cdot}275 = 2443$; and to this f_0 and $v_0 = 692$, Fasella's Table III gives $f_1 = 0{\cdot}00003848$; and then from $\sin 2\phi = X f_1$, $\phi = 8^\circ 58'$.

The error in the angle of departure (jump) is to be subtracted to obtain the angle of elevation. Similarly the calculations for $X = 1000, 2000, 4000, \ldots$ can be made.

Fasella's Table IV gives, for the above values of f_0 and v_0, the factor f_2, and thence from the equation $\tan\omega = f_2 \tan\phi$, the value of ω, the angle of descent; and so in our numerical example $\omega = 14° 47'$.

Fasella's Table V gives, for f_0 and v_0, a factor f_3, and then from the equation $t = \frac{c'}{\cos\phi} f_3$, the time of flight t; and in our numerical example $t = 18{\cdot}1$ sec.

Fasella's Table I for the same quantities gives a factor u, and thence from the equation $v_e \cos\omega = u \cos\phi$, the final velocity v_e; in our example $v_e = 326$ m/sec.

If the results of the firing are employed in other methods of calculation, then at 8000 m, under mean range-table conditions, Siacci II makes $\omega = 14° 37'$, Didion $\omega = 14° 58'$, against 14° 47′ above.

And for the final velocity, 329 according to Siacci II, 312 according to Didion, against 326 above.

So too, different results were found for the other ranges in angle of descent and final velocity; but the results are not recorded here. The calculation has the mere character of a process of interpolation.

(*c*) The calculation of the mean range-table fuze-setting scale proceeds as follows, selecting the previous example for illustration.

With the setting of 26 seconds a range was obtained of 9961 m, and a mean time of flight of 24·4 seconds was observed.

The ratio of the setting to the observed time of flight, $B_t : T_t$ works out to $26 : 24{\cdot}4 = 1{\cdot}07$. On the same day, and at ranges of 7576, 5024, 2487 m, the ratio obtained in the same way worked out to 1·16, 1·27, 1·36.

In another experiment the value of the ratios for ranges of 10160, 7125, 4718, 2532 m, worked out respectively to 1·05, 1·15, 1·32, 1·31.

These ratios, plotted graphically, make up a curve which with increasing range approaches the abscissa axis slowly.

On this curve, the value at 8000 m was 1·16, and in the mean range table the time of flight read off was 18·1 sec.

Consequently, on the mean range-table conditions, and at a range of 8000 m, the setting is $18{\cdot}1 \times 1{\cdot}16 = 21$ seconds.

At a range of 8000 m, and with a fuze setting of 21 seconds, the mean point of burst will be 8000 m from the muzzle, and at the same level; that is to say, it will give on the average 50 % of hits.

According to the construction of the gun, and the facts concerning target, ground, etc., the conditions of the height of mean burst above the muzzle horizon are determined.

In our example this may be put at 21 m at a range of 8000 m.

The trajectory of the mean range-table conditions must then be tilted to give this 21 m of extra height, in order to make the 21 seconds of time of burning give the best result.

But then, as evident from the tilting of the trajectory, the range will be increased from 8000 to 8082 m. So that according to mean range-table conditions, the fuse setting of a length of 21 seconds relates to a striking range of 8082 m; and corresponding relations can be worked out for the other ranges.

The relations thus obtained between the striking range and the fuze setting, plotted graphically, are found useful in practical work (*d*) The determination of the value of the mean lateral deflection is carried out in the following manner, on the same example as before.

Suppose for instance, with $v_0 = 700$ m/sec, and a departure angle 12° 11′ 14″ the mean range is 9724, and the lateral deflection of the mean point of impact from the plane of fire is $Z_t = 69$ m.

According to Hélie this gives the factor $A = 0{\cdot}00315$.

In the other ranges, mentioned in (*b*) above, the value of A was 0·00370, 0·00349, 0·00515, 0·00777, 0·00699, 0·00724, 0·01276.

The points so obtained, plotted as functions of the angle of departure, give for an angle of departure of 15° the value $A = 0{\cdot}00550$, and thence by Hélie's formula a lateral deviation of $\frac{15}{16}$ degree.

A calculation of this kind is recorded in the form of a curve and the intermediate values can then be read off.

Some examples of Range Tables.

1. Range table of the Mauser rifle M/71, according to Hebler's results.

Calibre 11·00 mm; bore across the grooves 11·60 mm; depth of grooves 0·30 mm; length of rifling 550 mm; number of grooves 4, breadth of the lands 4·3 mm, breadth of the grooves 4·3 mm, length of the bullet 27·5 mm, diameter of the bullet 11·00 mm, weight of bullet 25·0 g, material soft lead, with paper cover, charge 5·0 g, initial velocity 440 m/sec, weight of one m³ of air 1·226 kg, weight of rifle 4·5 kg, length of cartridge case 60 mm, weight of cartridge case 12·2 g, length of cartridge 78 mm, weight of complete loaded cartridge 42·8 g, sectional density of the bullet 0·263 g/mm² of the cross-section of the bore. The table is given on p. 266.

Table of the ordinate y at different distances x from the muzzle.

Range in m	Distance x in m									
	100	200	300	400	500	600	700	800	900	1000
200	0·38	0	−1·38							
300	0·84	0·93	0	−2·03						
400	1·35	1·95	1·54	0	−2·61					
500	1·87	2·99	3·10	2·09	0	−3·64				
600	2·48	4·20	4·92	4·52	3·03	0	−4·53			
700	3·12	5·50	6·86	7·11	6·27	3·89	0	−5·89		
800	3·86	6·97	9·07	10·05	9·95	8·30	5·15	0	−7 15	
900	4·65	8·56	11·45	13·23	13·92	13·07	10·71	6·36	0	−8·86
1000	5·54	10·33	14·11	16·77	18·35	18·38	16·91	13·44	7·97	0

2. Range table for the infantry rifles M/88 and M/98 S, according to v. Burgsdorff; see p. 267.

3. Example of an artillery range table.

Extract from an old range table of the German heavy field gun C/73, for shell and explosive shell.

Initial velocity 442 m/sec, error in the angle of departure (jump) $+\frac{3}{16}$ degree.

1 Range m	2 Angle of elevation degrees	3 Lateral deviation (see below)	4 Angle of descent degrees	5 Time of flight sec	6 $\frac{1}{16}$ degree displaces the height of the point of impact m	7 $\frac{1}{16}$ degree alters the range m	8 Final velocity m/sec	9 Danger zone for an object 1·7 m high m
100	0	30	$\frac{3}{16}$	0·2	0·1	45	424	—
200	0	30	$\frac{6}{16}$	0·5	0·2	42	407	—
300	$\frac{2}{16}$	30	$\frac{9}{16}$	0·7	0·3	39	392	—
1000	$1\frac{8}{16}$	31	$2\frac{5}{16}$	2·7	1·1	27	319	43
2000	$4\frac{4}{16}$	32	$6\frac{4}{16}$	6·2	2·2	20	268	16
3000	$7\frac{4}{16}$	34	12	10·3	3·2	15	233	8
4000	$12\frac{9}{16}$	37	$19\frac{12}{16}$	15·2	4·2	12	208	6
5000	$19\frac{2}{16}$	41	$30\frac{15}{16}$	21·4	4·8	8	190	3
6000	$29\frac{5}{16}$	49	$46\frac{2}{16}$	30·3	4·7	4	188	—

The angle of elevation in the second column is given in degrees and sixteenths of a degree, while the lateral deflection in column 3 is in units of an angle for which the arc is one-thousandth part of the radius, or 0·001 radian.

The zero of the graduation of the side deflection in the ordinary range table is numbered 30. The third column gives in a simple way the measure of the right-handed drift of the shell caused by the rifling.

Example. What is the drift at 4000 m?

Column 3 gives 37, that is 7 from zero; thence the drift at 4000 m is

$$\frac{7}{1000} \times 4000 = 28 \text{ m}.$$

The application of the results in columns 6 and 7 depends on the assumption that the trajectory can be raised or lowered like a rigid line, in accordance with the principle of tilting or swinging the trajectory; strictly speaking, this holds only for a small amount of elevation or depression.

Column 6 enables us to find the height of the shell at every range, and to find approximately the ordinate of the vertex.

The calculation is made as follows: to determine, for example, in a range of 6000 m the height of the shell above the ground at a distance of 5000 m. The elevation of 5000 m is $19\frac{2}{16}$ degrees, and for 6000 m is $29\frac{5}{16}$ degrees, which is $10\frac{3}{16}$ degrees greater. At 5000 m an extra elevation of $\frac{1}{16}$ degree will raise the height of the point struck by 4·8 m, and so $10\frac{3}{16}°$ will raise it $163 \times 4{\cdot}8 = 782$ m, in round numbers; and so in firing at a range of 6000 m, the shell at 5000 m will be 782 m above the ground.

The height of the vertex above the ground in a range of 6000 m can be found in the following empirical manner: the vertex is at that distance from the

Range	Final velocity	Time of flight	Distance fallen	Angle of departure	Angle of descent	Danger zone		Radius of 50 °/° zone	Striking energy of the shell	Penetration in pine wood	Deflection due to a moderate side wind $v=5$ m
						$H=1\cdot70$ m	$H=1\cdot80$ m				
m	m/sec	sec	m			m	m	cm	m-kg	cm	m
0	440	0	0	0°	0°	—	—	0	247	24·0	0
100	384	0·24	0·29	0° 10′ 1″	0° 11′ 25″	—	—	7	188	23·7	0·013
200	341	0·52	1·33	0° 22′ 50″	0° 28′ 43″	—	—	14	148	23·2	0·059
300	307	0·83	3·38	0° 38′ 44″	0° 50′ 9″	116·5	123·4	21	120	22·4	0·149
400	278	1·17	6·55	0° 56′ 16″	1° 11′ 26″	81·8	86·6	30	98·8	21·4	0·297
500	255	1·55	10·8	1° 14′ 27″	1° 36′ 21″	60·6	64·2	41	82·8	20·1	0·518
600	235	1·96	16·6	1° 34′ 49″	2° 8′ 57″	45·3	48·0	55	70·5	18·6	0·827
700	218	2·40	23·9	1° 57′ 27″	2° 46′ 43″	35·0	37·1	73	60·7	16·9	1·24
800	204	2·87	33·2	2° 22′ 29″	3° 29′ 57″	27·8	29·4	95	52·8	15·0	1·78
900	191	3·38	44·5	2° 50′ 0″	4° 18′ 31″	22·6	23·9	121	46·4	12·9	2·47
1000	180	3·92	58·3	3° 20′ 2″	5° 13′ 20″	18·6	19·7	158	41·1	11·1	3·32
1100	169	4·59	74·6	3° 52′ 49″	6° 13′ 50″	15·6	16·5	202	36·6	9·6	4·36
1200	160	5·10	93·8	4° 28′ 15″	7° 19′ 50″	13·2	14·0	256	32·8	8·5	5·62
1300	152	5·74	116	5° 6′ 30″	8° 31′ 20″	11·3	12·0	321	29·6	7·4	7·12
1400	145	6·41	142	5° 47′ 30″	9° 49′ 20″	9·8	10·4	403	26·8	6·6	8·89
1500	138	7·12	172	6° 31′ 30″	11° 12′ 40″	8·6	9·1	502	24·4	6·0	10·9
1600	132	7·86	205	7° 18′ 20″	12° 41′ 20″	7·5	8·0	624	22·4	5·5	13·3

Range	Angle of departure		Angle of descent		Distance of vertex		Height of vertex		Time of flight		Final velocity		Striking energy		Danger zone for object 1 m high		Height of 50 % zone		Height of 50 % zone (maximum)	
m					m		m		sec		m/sec		m-kg		m		m		m	
	S	88	S	88	S	88	S	88	S	88	S	88	S	88	S	88	S	88	S	88
0	—	—	—	—	—	—	—	—	—	—	875	640	390	307	—	—	—	—	—	—
100	0° 2′ 20″	0° 4′ 40″	0° 2′ 20″	0° 5′	50	50	0·02	0·04	0·12	0·17	810	565	335	239	100	100	0·03	0·03	0·14	0·21
200	0° 5′	0° 10′	0° 6′	0° 12′	102	105	0·08	0·15	0·25	0·36	742	498	281	186	200	200	0·06	0·08	0·30	0·43
300	0° 8′ 20″	0° 16′	0° 10′	0° 21′	157	162	0·20	0·40	0·39	0·57	673	440	231	145	300	300	0·09	0·12	0·46	0·66
400	0° 12′	0° 23′ 40″	0° 15′	0° 33′	215	220	0·39	0·82	0·54	0·81	605	389	187	113	400	400	0·12	0·18	0·65	0·88
500	0° 16′	0° 32′ 40″	0° 23′	0° 49′	274	278	0·70	1·44	0·72	1·08	538	351	148	92	500	95	0·16	0·25	0·85	1·11
600	0° 21′	0° 43′ 20″	0° 32′	1° 8′	338	336	1·13	2·40	0·92	1·38	470	323	113	78	170	55	0·22	0·32	1·05	1·34
700	0° 28′	0° 55′ 40″	0° 46′	1° 31′	405	394	1·87	3·70	1·15	1·69	403	302	83	68	85	42	0·28	0·41	1·28	1·58
800	0° 36′ 20″	1° 9′	1° 5′	1° 57′	468	450	2·95	5·38	1·41	2·04	355	285	64	61	50	33	0·35	0·51	1·52	1·82
900	0° 45′ 40″	1° 24′ 40″	1° 28′	2° 26′	530	504	4·36	7·57	1·71	2·39	324	269	54	54	40	25	0·43	0·63	1·76	2·08
1000	0° 56′ 40″	1° 41′ 40″	1° 54′	2° 59′	592	560	6·18	10·29	2·03	2·79	301	255	46	49	30	20	0·52	0·75	2·02	2·35
1100	1° 9′ 20″	2° 0′ 40″	2° 24′	3° 35′	650	620	8·52	13·51	2·37	3·18	282	241	41	44	25	16	0·63	0·90	2·30	2·63
1200	1° 24′ 40″	2° 21′	2° 57′	4° 16′	714	678	11·53	17·42	2·74	3·61	266	228	36	39	21	13	0·76	1·07	2·57	2·94
1300	1° 41′	2° 43′	3° 34′	5° 3′	778	740	15·10	22·05	3·13	4·06	250	217	32	35	17	11	0·91	1·26	2·87	3·27
1400	1° 59′ 20″	3° 7′ 20″	4° 16′	5° 55′	840	800	19·38	27·52	3·54	4·54	236	208	28	32	14	9	1·09	1·49	3·19	3·65
1500	2° 20′ 20″	3° 34′ 20″	5° 4′	6° 51′	900	862	24·75	34·10	3·97	5·04	222	198	25	29	12	8	1·31	1·76	3·54	4·08
1600	2° 43′	4° 2′ 40″	6°	7° 55′	965	922	31·08	41·50	4·44	5·57	211	188	23	26	10	7	1·56	2·07	3·96	4·54
1700	3° 9′ 20″	4° 34′ 40″	7° 2′	9° 7′	1030	984	38·66	50·47	4·92	6·10	199	181	20	24	8	6	1·85	2·43	4·48	5·09
1800	3° 38′	5° 10′ 20″	8° 14′	10° 30′	1096	1052	47·63	61·16	5·44	6·67	188	171	18	22	7	5	2·20	2·85	5·14	5·77
1900	4° 12′ 40″	5° 51′ 20″	9° 43′	12° 5′	1160	1130	59·00	74·35	5·98	7·27	177	164	16	20	6	5	2·60	3·35	5·93	6·71
2000	4° 56′ 20″	6° 41′ 20″	11° 35′	14° 5′	1230	1212	74·40	91·53	6·57	7·89	166	164	14	18	5	4	3·09	3·96	7·10	8·17

muzzle at which the sum of the corresponding angles of elevation and descent is equal to the elevation corresponding to the whole range.

Find the range at which the sum of the angles in columns 2 and 4 is equal to $29\frac{5}{16}°$, the elevation for 6000 m range. At 3000 m the sum is $7\frac{4}{16}+12=19\frac{4}{16}$; at 4000 m the sum is $12\frac{9}{16}+19\frac{12}{16}=32\frac{5}{16}$; the vertex must lie between the 3000 and the 4000 m range. A closer value is found at 3800 m; and at this range the height y of the shell above the ground is worked out.

Consult § 41, p. 244, on the calculation of the vertex ordinate by help of an approximate formula, employed in conjunction with the range table.

CHAPTER IX

Lateral deviations of a shell

§ 44. On lateral deviation of a shell in general.

When the same weapon, the same ammunition, and the same sighting are used for firing at the same mark Z, the striking points of the shells group themselves round a point of mean impact T.

In general, the point T will have a lateral deviation from Z, due to the following causes.

In the first place the sighting, and the angle of departure ϕ, may be wrong; this will be the case if the distance of the target has been wrongly estimated; or the weapon may not be sighted correctly, that is, the setting of the sight has not been measured correctly to correspond with the scale divisions of the range.

Secondly, the ballistic coefficient c may not have the normal value, as taken from the range table, inasmuch as on the day of the experiment the weight δ of a cubic metre of air, as determined by air temperature, height of barometer, humidity of the air, height above the sea, is not the same as that employed in the range table: in a few cases the shape of the point of the shell, its weight, and the calibre may differ from the normal.

Thirdly, the initial velocity v_0 of the shell may differ from the range table value, in consequence of the charge of powder being too small or too great, or not at the normal temperature, or heated too much in the bore of the gun.

Fourthly, the azimuth of the plane of fire may have been chosen incorrectly, since lateral deviations must be taken into account; such deviations may arise from a side wind, rotation of the shell, or rotation of the Earth, concerning which latter influence we do not at present know whether it is important enough to be taken into account. In rifles too a lateral deviation may arise from fixing the bayonet; moreover there is the effect of the slope of the axis of the wheels of a gun, or of the canting of the rifle.

These deviations are constant lateral deflections, since, in general, they occur to one side. They can be corrected, either by calculation or observation.

By means of the sighting the deviation due to the rotation of the shell, and the slant of the axle of the wheels may be allowed for; the old spherical shells showed deviations in an indeterminate direction; but by the introduction of eccentric shells, and placing the shell in the bore with the centre of gravity either up or down, these deviations were made regular, and so could be calculated beforehand.

§ 45. Influence of a small alteration of the angle of departure, or of the initial velocity, or of the ballistic coefficient on the range.

I. The retardation due to air resistance being $cf(v)$, take a given c, ϕ, v_0 to give a definite trajectory with range X.

When ϕ alters by $\Delta\phi$, and v_0 by Δv_0, and c by Δc, a neighbouring trajectory arises with the same point of departure O, and a range $X + \Delta X$; it is required to calculate ΔX.

The most accurate determination of ΔX follows evidently from a repetition of the trajectory calculation, for the new elements

$$\phi + \Delta\phi,\ c + \Delta c,\ v_0 + \Delta v_0.$$

An approximate calculation can be made by means of the differential formulae, and strictly speaking these must be considered separately for every solution; the assumption is made that the infinitely small alterations $d\phi$, dc, dv_0, dX can be replaced with sufficient accuracy by the finite small alterations $\Delta\phi$, Δc, Δv_0, ΔX, as explained above in § 43.

As examples the differential formulae are developed here for the quadratic and the cubic laws of air resistance.

In the first, $cf(v) = c_1 v^2$, $c_1 = \dfrac{R^2\pi\delta i g}{1{\cdot}206P}\lambda_1$, where λ_1 has a constant mean numerical value; for instance, $v < 240$ m/sec, $\lambda_1 = 0{\cdot}014$.

The range is calculated from the equation

$$\frac{v_0^2 \sin 2\phi}{gX} = B(Z), \quad Z = 2c_1\alpha X. \quad \text{..............(1)}$$

By logarithmic differentiation of (1), we have

$$2\frac{dv_0}{v_0} + \frac{d\sin 2\phi}{\sin 2\phi} - \frac{dX}{X} = \frac{dB}{B};$$

and B alters with Z, and therefore with c_1 and X, ignoring the alteration of the mean value factor α, and so

$$dB = \frac{\partial B}{\partial c_1} dc_1 + \frac{\partial B}{\partial X} dX,$$

and here

$$\frac{\partial B}{\partial c_1} = \frac{\partial B}{\partial Z} \cdot \frac{\partial Z}{\partial c_1} = \frac{\partial B}{\partial Z} \cdot 2\alpha X, \quad \frac{\partial B}{\partial X} = \frac{\partial B}{\partial Z} \cdot \frac{\partial Z}{\partial X} = \frac{\partial B}{\partial Z} \cdot 2c_1 \alpha.$$

Denoting $\frac{\partial B}{\partial Z}$ by B', we have

$$2\frac{dv_0}{v_0} + 2\frac{d\phi}{\tan 2\phi} - \frac{dX}{X} = \frac{B'}{B} \cdot 2c_1\alpha \,.\, dX + \frac{B'}{B} \cdot 2\alpha X \,.\, dc_1. \quad \ldots(2)$$

To obtain the fraction $\frac{B'}{B}$ in a convenient form, let us employ the equation of the trajectory

$$y = x \tan \phi - \frac{gx^2}{2v_0^2 \cos^2 \phi} \cdot B(z), \text{ where } z = 2c_1 \alpha x \text{ (§ 35)};$$

$$\frac{dy}{dx}, \text{ or } \tan \theta = \tan \phi - \frac{g}{2v_0^2 \cos^2 \phi}\left(2xB + x^2 \frac{\partial B}{\partial z} \cdot 2c_1 \alpha\right);$$

and at $\theta = -\omega$, $x = X$,

therefore $$-\tan \omega = \tan \phi - \frac{g}{2v_0^2 \cos^2 \phi}(2XB + X^2 B' \,.\, 2c_1 \alpha),$$

or $$\frac{\tan \phi + \tan \omega}{\tan \phi} = \frac{g}{v_0^2 \sin 2\phi}(2XB + X^2 B' \,.\, 2c_1\alpha),$$

$$= \frac{1}{BX}(2XB + X^2 B' \,.\, 2c_1 \alpha) = 2 + X\frac{B'}{B} \cdot 2c_1 \alpha,$$

from (1); then the expression is obtained

$$\frac{B'}{B} = \frac{1}{2c_1 \alpha X} \frac{\tan \omega - \tan \phi}{\tan \phi}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Substituting in (2) we have

$$2\frac{dv_0}{v_0} + \frac{2d\phi}{\tan 2\phi} - \frac{dX}{X} = \frac{\tan \omega - \tan \phi}{\tan \phi} \frac{dX}{X} + \frac{\tan \omega - \tan \phi}{\tan \phi} \frac{dc_1}{c_1},$$

$$\frac{dX}{X} \frac{\tan \omega}{\tan \phi} = 2\frac{dv_0}{v_0} + \frac{2d\phi}{\tan 2\phi} - \frac{\tan \omega - \tan \phi}{\tan \phi} \frac{dc_1}{c_1}, \quad \ldots\ldots(4)$$

where $d\phi$ is measured in radians; and hence]

$$\frac{dc_1}{c_1} = 2\frac{dR}{R} + \frac{d\delta}{\delta} - \frac{dP}{P}.$$

Suppose for example v_0 becomes $v_0 + dv_0$, but ϕ and c_1 remain constant; the alteration in the range X is calculated from

$$\frac{dX}{X} = \frac{2\tan\phi}{\tan\omega} \cdot \frac{dv_0}{v_0}.$$

Or if the air density δ alters by $d\delta$, then

$$\frac{dX}{X} = -\frac{\tan\omega - \tan\phi}{\tan\omega} \cdot \frac{d\delta}{\delta}.$$

With the cubic law, $cf(v) = c_2 v^3$, $c_2 = \dfrac{R^2 \pi \delta i g}{1{\cdot}206 P} \lambda_2$, where for example

(4)

$\lambda_2 = 0{\cdot}068$, for values of v between 550 and 600 m/sec, and in equation (1), $Z = c_2 \alpha^2 v_{x_0} X$, where $v_{x_0} = v_0 \cos\phi$. Here α is the mean value of $\sec\theta$; for instance, according to Hélie, in § 23, $\alpha = \sqrt{(\sec\phi)}$, so that $Z = \sim c_2 v_0 X$. In this case Z depends on c_2, X, and v_0. Equation (2) will now be as follows:

$$2\frac{dv_0}{v_0} + \frac{2d\phi}{\tan 2\phi} - \frac{dX}{X} = \frac{1}{B}\frac{\partial B}{\partial X} dX + \frac{1}{B}\frac{\partial B}{\partial c_2} dc_2 + \frac{1}{B}\frac{\partial B}{\partial v_0} dv_0 \quad \ldots(5)$$

$$= \frac{B'}{B} dX \frac{\partial Z}{\partial X} + \frac{B'}{B} dc_2 \frac{\partial Z}{\partial c_2} + \frac{B'}{B} dv_0 \frac{\partial Z}{\partial v_0}$$

$$= \frac{B'}{B}(dX \, . \, c_2 v_0 + dc_2 \, . \, v_0 X + dv_0 \, . \, c_2 X)$$

$$= \frac{B'}{B} Z \left(\frac{dX}{X} + \frac{dc_2}{c_2} + \frac{dv_0}{v_0}\right).$$

Equation (3) is therefore replaced by

$$\frac{B'}{B} \, . \, v_0 c_2 X = \frac{\tan\omega - \tan\phi}{\tan\phi}, \quad \text{or} \quad Z\frac{B'}{B} = \frac{\tan\omega - \tan\phi}{\tan\phi},$$

and so $\quad 2\dfrac{dv_0}{v_0} + \dfrac{2d\phi}{\tan 2\phi} - \dfrac{dX}{X} = \dfrac{\tan\omega - \tan\phi}{\tan\phi}\left(\dfrac{dX}{X} + \dfrac{dc_2}{c_2} + \dfrac{dv_0}{v_0}\right),$

$$\frac{dX}{X} = \frac{3\tan\phi - \tan\omega}{\tan\omega}\frac{dv_0}{v_0} + \frac{\tan\phi - \tan\omega}{\tan\omega}\frac{dc_2}{c_2} + \frac{2\tan\phi\cot 2\phi}{\tan\omega} d\phi. \quad (6)$$

By logarithmic differentiation of the expression for a vacuum, viz.,

$$X = \frac{v_0^2 \sin 2\phi}{g}, \text{ we get } \frac{\Delta X}{X} = 2\frac{\Delta v_0}{v_0} + 2\cot 2\phi \, . \, \Delta\phi.$$

Formulae.

When only one of the quantities v_0, ϕ, c, varies at a time, the relative alteration $\frac{\Delta X}{X}$ of the range X, or its percentage change is given, on the quadratic law, by

$$\frac{\Delta X}{X}=+\frac{2\tan\phi}{\tan\omega}\cdot\frac{\Delta v_0}{v_0}, \quad\text{.........................(1)}$$

$$\frac{\Delta X}{X}=+\frac{2\tan\phi}{\tan 2\phi\tan\omega}\cdot\Delta\phi, \quad\text{......................(2)}$$

$$\frac{\Delta X}{X}=-\frac{\tan\omega-\tan\phi}{\tan\omega}\cdot\frac{\Delta c}{c}. \quad\text{..................(3)}$$

On the cubic law

$$\frac{\Delta X}{X}=+\frac{3\tan\phi-\tan\omega}{\tan\omega}\cdot\frac{\Delta v_0}{v_0}, \quad\text{...............(4)}$$

$$\frac{\Delta X}{X}=+\frac{2\tan\phi}{\tan 2\phi\tan\omega}\cdot\Delta\phi, \quad\text{......................(5)}$$

$$\frac{\Delta X}{X}=-\frac{\tan\omega-\tan\phi}{\tan\omega}\cdot\frac{\Delta c}{c}. \quad\text{..................(6)}$$

Here c is proportional to R^2, δ, i, and varies inversely as P, and so

$$\frac{\Delta c}{c}=2\frac{\Delta R}{R}+\frac{\Delta\delta}{\delta}+\frac{\Delta i}{i}-\frac{\Delta P}{P}. \quad\text{...................(7)}$$

Suppose, for instance, that only the air density δ varies; then $\frac{\Delta c}{c}=\frac{\Delta\delta}{\delta}$; and if this variation is due to an ascent of Δy, then

$$\frac{\Delta\delta}{\delta}=-0{\cdot}00011\,.\,\Delta y,\text{ and }\frac{\Delta X}{X}=+\frac{\tan\omega-\tan\phi}{\tan\omega}\,.\,0{\cdot}00011\,.\,\Delta y,$$

where ω denotes the acute angle of descent.

The formulae (1) to (3) are chiefly employed for howitzers and mortars, and (4) to (6) for guns and rifles.

Examples.

1. Suppose $\phi=38°$, $\omega=53°\frac{1}{2}$, and that $X=8230$ m, the air density being $\delta=1{\cdot}27$ kg/m³.

To determine the increase in range when this normal air density is reduced to 1·22.

Here $\frac{\Delta c}{c} = \frac{\Delta \delta}{\delta}$, with $\Delta\delta = -0{\cdot}05$; and so from (3) or (6)

$$\Delta X = 8230 \frac{\tan 53^\circ\tfrac{1}{2} - \tan 38^\circ}{\tan 53^\circ\tfrac{1}{2}} \cdot \frac{0{\cdot}05}{1{\cdot}27} = 140 \text{ m}.$$

2. With a field gun, at ranges 100, 1000, 2000, 4000, 8000 m, and $\Delta v_0 = 15$ m/sec, the alteration of range would be (*a*) on the quadratic law

$$\Delta X = 5{\cdot}5,\ 54,\ 99,\ 177,\ 307 \text{ m respectively,}$$

(*b*) on the cubic law

$$\Delta X = 4,\quad 49,\quad 83,\ 137,\ 202 \text{ m};$$

or more exactly

$$\Delta X = 4{\cdot}5,\ 49{\cdot}5,\ 91,\ 123,\ 166 \text{ m}.$$

An alteration in ϕ of $\Delta\phi = 5'$ gave, according to the quadratic law,

$$\Delta X = 44,\ 44,\quad 35,\ 23,\ 5 \text{ m},$$

or more exactly

$$\Delta X = 43,\ 43{\cdot}5,\ 36,\ 22,\ 5 \text{ m}.$$

3. With a field howitzer, at the ranges 500, 1000, 2000, 4000, 5000 m, it was found, when $\Delta v_0 = 12$ m/sec, and $v_0 = 295$ m/sec.

(*a*) $\Delta X = 38$, 86, 176, 324, 383 m respectively, on the quadratic law,

(*b*) $\Delta X = 32$, 78, 163, 283, 319 m respectively, on the cubic law,

or more exactly

$$\Delta X = 39,\ 83,\ 162,\ 300,\ 347 \text{ m}.$$

And when $\Delta\phi = 10'$,

(*a*) $\Delta X = 48$, 46, 40, 27·5, 19 m, on the quadratic law,

or more exactly

(*b*) $\Delta X = 60$, 46, 38·5, 26·5, 18 m.

4. With a rifle, at the ranges 500, 1000, 1500 m, and with $v_0 = 875$ m/sec, it was found that, taking $\Delta v_0 = 25$,

(*a*) $\Delta X = 20$, 28·5, 39·5 m, on the quadratic law,

or more exactly

(*b*) $\Delta X = 15{\cdot}5$, 14·0, 16·5 m, on the cubic law,

or repeating the calculation

$$\Delta X = 25{\cdot}5,\ 22{\cdot}5,\ 28{\cdot}0 \text{ m}.$$

So too, taking $\Delta\phi = 3'$,

(*a*) $\Delta X = 129$, 65, 26, 15 m, on the quadratic law,

or more exactly

(*b*) $\Delta X = 110$, 54·5, 27, 14 m.

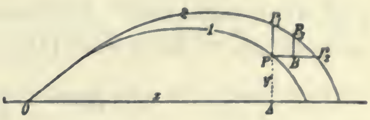
These examples show uncertainty in the employment of the variation formulae.

Consult the Notes on these and other variation formulae, compared with the results in practice.

II. Simple formulae for practical use may now be considered. A trajectory is supposed to be plotted by a series of points for a given value of c, and δ, obtained by the arrangement described in Vol. I,

§ 37, and Vol. III, § 184. The problem is to proceed from point to point, so as to reduce to a normal value c, or a normal density of the air, for the construction of a range table.

For this purpose a trajectory 1 in the figure is to be altered into another trajectory 2, having the same origin O, initial velocity v_0, and the same initial tangent.



We can either start with a point P on the trajectory 1, and move to a point P_1 on the same vertical, having the same $x = OA$, and calculate the corresponding $PP_1 = \Delta y$; or else with constant $y = AP$ we can proceed to a second point P_2, at a further distance $\Delta x = PP_2$ to be calculated; or finally we can make x and y both vary by $PB = \Delta x$, and $BP_3 = \Delta y$.

In the last case an arbitrary assumption must be made as to the direction of PP_3. It can be assumed for example that the step from P to P_3 may be taken over a given slope of the ground, or that it proceeds along a line of equal velocity v, or of equal horizontal velocity $v \cos \theta$.

The calculation of these small alterations of x, or y, or of x and y may be made on various assumptions. Thus for example the trajectory through P may be taken as given by the appropriate rational algebraic function of the third degree, as in the method of Piton-Bressant; and there the corresponding system of equations was

$$y = x \tan \phi - \frac{gx^2}{2v_0^2 \cos^2 \phi}(1 + Kx), \quad \ldots\ldots\ldots\ldots(8)$$

$$\tan \theta = \tan \phi - \frac{gx}{2v_0^2 \cos^2 \phi}(2 + 3Kx), \quad \ldots\ldots\ldots\ldots(9)$$

$$v \cos \theta = \frac{v_0 \cos \phi}{\sqrt{(1 + 3Kx)}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$t = \frac{2}{9v_0 \cos \phi} \frac{(1 + 3Kx)^{\frac{3}{2}} - 1}{K}. \quad \ldots\ldots\ldots\ldots\ldots(11)$$

The empirical factor K is determined by the position of the point P, so that in (8), x, y, v_0, ϕ are known: and from the preceding it follows that K is proportional to the factor c, and this again to the air density; so that

$$\frac{\Delta K}{K} = \frac{\Delta c}{c} = \frac{\Delta \delta}{\delta}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

In the step from P to P_1, when x is left unaltered, there is the alteration of K into $K+dK$

and $$dy = PP_1 = -\frac{gx^3}{2v_0^2\cos^2\phi}dK,$$

and when finite small differences are assumed instead of the exact differential, we have from (12)

$$\Delta y = -\frac{gx^3}{2v_0^2\cos^2\phi}K\frac{dc}{c}. \quad \text{.................(13)}$$

This equation, in which K is to be considered as calculated from (8), can serve for a reduction to the normal air density.

But it involves the difficulty that this calculation of K from (xy) must be repeated: moreover the expressions for Δt and $\Delta(v\cos\theta)$ are not very convenient.

The step from P to P_3 may also be taken so as to keep $v\cos\theta$ constant; thus (10) shows that Kx remains constant; so that

$$\frac{\Delta x}{x} = -\frac{\Delta K}{K} = -\frac{\Delta c}{c}.$$

Moreover it follows from (11) that t is also inversely proportional to K, so that

$$\frac{\Delta t}{t} = -\frac{\Delta K}{K} = -\frac{\Delta c}{c}.$$

From (8) it follows by differentiation, keeping Kx constant, and $\frac{dx}{x} = -\frac{dc}{c}$,

$$dy = \tan\phi \,.\, dx - \frac{g\,.\,2xdx}{2v_0^2\cos^2\phi}(1+Kx)$$

$$= -\tan\phi\,.\,x\frac{dc}{c} + \frac{gx^2(1+Kx)}{2v_0^2\cos^2\phi}2\frac{dc}{c},$$

and so from (8),

$$dy = \frac{dc}{c}[-x\tan\phi + 2(x\tan\phi - y)] = \frac{dc}{c}(x\tan\phi - 2y).$$

Equation (9) can be treated in a similar manner: and so the following system of the difference equations is obtained:

$$\Delta x = -\frac{\Delta c}{c}x, \quad \text{..............................(14)}$$

$$\Delta y = +\frac{\Delta c}{c}(x\tan\phi - 2y), \quad \text{...............(15)}$$

$$\Delta t = -\frac{\Delta c}{c}\, t, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(16)$$

$$\Delta \tan\theta = +\frac{\Delta c}{c}(\tan\phi - \tan\theta). \quad \ldots\ldots\ldots\ldots(17)$$

These equations were first worked out in a different manner by Dr C. Veithen: they serve in a very simple manner to reduce a series of trajectories to normal air density.

They can also be deduced from Siacci's system of solutions given on p. 137; thus when c changes into $c + dc$, while u or $\dfrac{v\cos\theta}{\sigma}$ remains constant, suppose x changes into $x + dx$ and y into $y + dy$.

Since σ and γ were introduced as constants in the solution, the quantities in brackets in the equations of § 23 remain unaltered: and so it follows again that

$$xc, \quad tc, \quad (\tan\phi - \tan\theta)\,c, \quad (x\tan\phi - y)\,c^2$$

remain constant.

The first three of these four conditions lead to equations (14), (16), (17). And from the last it follows by differentiation that

$$(\tan\phi\,.\,dx - dy)\,c^2 + 2c\,dc\,(x\tan\phi - y) = 0,$$

$$dy = \tan\phi\,.\,dx + 2\,(x\tan\phi - y)\,\frac{dc}{c},$$

or, since

$$dx = -x\,\frac{dc}{c},$$

$$dy = \frac{dc}{c}(x\tan\phi - 2y),$$

as in (15); and thereby the equations above, (14) to (17), are again established.

Dr C. Veithen has proved further that this deduction still holds good, when the trajectory is supposed to be calculated in a number of separate arcs, joined together; and so the system of equations holds for high angle trajectories.

Finally, taking the case of the point of descent on the muzzle horizon, C. Veithen has deduced the equations (3) to (6).

Example. Suppose with $v_0 = 580$ m/sec, at $\phi = 45°$, the results were $x = 6200$ m, $y = 3820$ m, $t = 24{\cdot}5$ sec for an air density of 1·20 kg/m³. These results are to be reduced to a normal air density of 1·22, so that $\Delta\delta = +0{\cdot}02$.

$$\Delta x = -\frac{0{\cdot}02}{1{\cdot}20}\,.\,6200 = -103\text{ m}, \qquad x = 6097\text{ m},$$

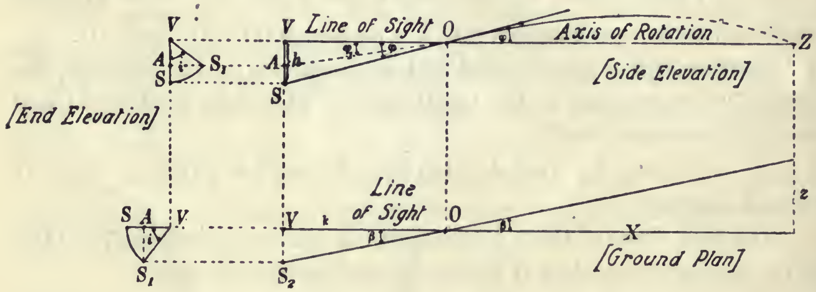
$$\Delta y = +\frac{0{\cdot}02}{1{\cdot}20}(6200\tan 45° - 2\,.\,3820) = -25\text{ m}, \quad y = 3795\text{ m},$$

$$\Delta t = -\frac{0{\cdot}02}{1{\cdot}20}\,.\,24{\cdot}5 = -0{\cdot}4\text{ sec}, \qquad t = 24{\cdot}1\text{ sec}.$$

§ 46. Deviations of the projectile due to slope of the axle of the wheels, or to a canting of the sight of a rifle.

Shooting with a positive elevation and a positive angle of sight, the shell deviates to the side of the lower wheel, or to the dip of the sighting of the rifle.

To calculate the amount of this deviation, consider the weapon first in the exact position, and then in the displaced position.



In the first case, as in the figure, VO is the line of sight directed at the mark Z; SO the axis of the bore, $\phi = VOS$ the angle of elevation, approximately also the angle of departure, neglecting the effect of jump.

In the second case, let S_1O be the axis of the bore in space, while the line of sight VO has preserved its direction, and so acts as an axis of rotation, i denoting the angle turned through.

Suppose a plane drawn perpendicular to the line of sight VO through the sight V. This plane is shown in elevation in the figure; and the point S of the axis of the bore comes to S_1 by the rotation. The new position of the axis of the bore is shown in elevation by AO, in plan by S_2O. The new trajectory lies in the vertical plane through S_2O, with a new angle of elevation ϕ_1; and $\sin\phi_1$ is equal to AV or $h\cos i$, divided by the distance OS, so that

$$\sin\phi_1 = \frac{h}{OS}\cos i = \sin\phi\cos i, \quad \text{..............(1)}$$

and this gives the new angle of elevation ϕ_1 in terms of the original ϕ.

The lateral deviation due to this rotation is shown in plan by the angle β, where

$$\tan\beta = \frac{S_2V}{OV} = \frac{AS_1}{OV} = \frac{h}{k}\sin i,$$

and so

$$\tan\beta = \tan\phi \sin i. \quad \text{.........................(2)}$$

At a range X, the lateral deviation z is

$$z = \sim X \tan\beta,$$

and so

$$z = X \tan\phi \sin i, \quad \text{.........................(3)}$$

and for small values of the angles ϕ and i (in degrees), we have approximately

$$z = \frac{X\phi i}{3280}. \quad \text{.........................(4)}$$

Example: $\phi = 4° \, 35'$, $X = 1800$ m, $i = 5°$,

$$z = \frac{1800 \times 4{\cdot}58 \times 5}{3280} = 12{\cdot}5 \text{ m}.$$

To find the correction for the direction and height, J. Didion supposes the rotation to be made, not about the line of sight, but about the axis of the bore.

Under certain assumptions about the angle of departure the initial tangent of the trajectory remains unaltered and also the position of the point of impact on the vertical target at the range X. The intersection of the prolongation of the line of sight on the contrary is displaced on the target. A shift to the side of the higher wheel is then the requisite correction.

Concerning the different ways of eliminating the effect of the slope of the platform of a gun, the reader should consult the memoir of Ritter von Eberhard (see Note).

§ 47. Deflection due to wind, in a gun at rest or in motion.

So far it has been assumed that the air is at rest, relatively to the gun, and the gun is at rest relatively to the surface of the Earth, and that the rotation of the Earth about its axis is left out of account.

Now the first two assumptions will be altered: and the gun and the air will be assumed to be in motion.

In the compilation of a range table the firing must usually be carried out in a wind; while the results of the ordinary range table refer naturally to a calm. Consequently the measurements made for a range table must first be corrected.

And since alterations in direction and range arise in consequence of the wind, the question is as to the extent of these deviations.

The calculations can be worked out from the general point of view, and then applied to particular cases; but the inductive method is here preferred, and applied in the sequel to various problems.

But these methods are always very uncertain.

The measurement of the wind is taken over a considerable period of time, near the ground: but as a fact the wind blows in gusts, and the velocity is variable within small intervals of time. Moreover the velocity of the wind at the height where the shell is moving is often very different from that near the ground (for numerical examples consult § 111 in Vol. III) and a general law connecting wind velocity with the height is not known and cannot well be laid down.

Further, the direction of the wind at a great height is not the same as at the ground, and it is probable that the wind as seen from below will rotate in a clockwise direction.

Finally it is not impossible that a lifting force arises from a head wind, exerting a supporting force on the surface of an elongated shell in rotation.

On these grounds the formulae must be considered only as a sort of approximation; and for range table purposes, the firing should be carried out in air as calm as possible.

Otherwise the most convenient procedure is to carry out wind measurements by means of the instantaneous anemometer, at a convenient height, or by a succession of pilot balloons.

The resulting wind velocity is that at the average height, which is about $\frac{2}{3}$ of the height of the vertex.

It will be convenient to illustrate the relativity of motion by simple instances.

A railway carriage in fig. 1 is proceeding along the rails BC from B to C with uniform velocity.

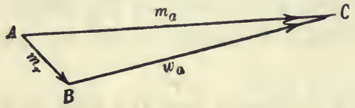


Fig. 1.

A man is moving inside the carriage; and his absolute velocity over the ground is represented by AC or m_a. The relative velocity of the man with respect to the carriage is obtained by impressing on him the reversed velocity w_a of the carriage by vector subtraction.

Thus if BC is the vector of the velocity of the carriage over the ground, and AB the vector of the velocity of the man with respect to the carriage, then the vector AC is the velocity of the man with respect to the ground.

Again let us suppose in fig. 2 a man to be walking on the deck of a ship, moving uniformly through the water; AB is the vector of the man's velocity relatively to the ship, and BC the vector of the ship's velocity through the water, CD the vector of the velocity of the tide. Then the vector sum, or the side AD, is the velocity vector of the man with respect to the earth.

Fig. 2.

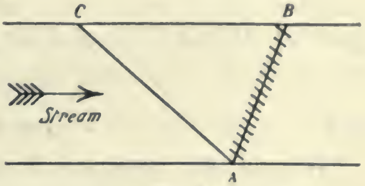


Fig. 3.

When these velocities are uniform, the velocity diagram will give also the displacement diagram. For instance, in fig. 3, a ship sails from A to B. The ship must consequently be steered from A to C as if the water were at rest. This point C is chosen so that while the stream flows from C to B, the ship would move in still water from C to A: then actually the ship moves from A to B; and AC is the course of the ship through the water, CB the drift of the water over the ground in the same time; AB is the actual direction of the ship.

I. Gun at rest on the ground; wind horizontal and parallel to the plane of fire.

The horizontal wind velocity w_p is taken as positive in the direction of the horizontal component of the initial velocity of the shell.

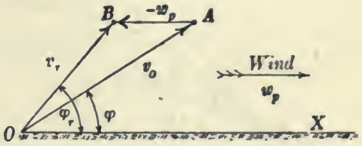


Fig. 4.

After t seconds the horizontal advance x of the shell is observed, as well as the range X and the time of flight T. The initial velocity v_0 is represented by the vector OA, and the angle of departure AOX by ϕ.

The initial velocity reduced to a calm v_r is the vector sum of v_0 and the reversed wind velocity w_p.

In the diagram of fig. 4, OB is the vector of the initial velocity relatively to the wind, BA the vector velocity of the wind over the ground, and the vector OA gives the initial velocity with respect to the ground.

By this construction the angle of departure ϕ_r or BOX is obtained after allowing for the wind.

The diagram of the horizontal projection of the motion of the shell is in this case shown very simply in fig. 5; where $OD = x_r$ is the abscissa of the shell after a time t, and $DE = w_p t$ is the drift of the wind over the ground; $OE = x$ the actual displacement of the shell, where

Fig. 5.

$$x = x_r + w_p t,$$

and
$$X = X_r + w_p T.$$

If after the time t or T, the quantities v_0, ϕ, x, and X are measured, they are supposed to be replaced by v_r, ϕ_r, x_r, and X_r.

Then too, as shown in fig. 4, it is evident that

$$v_r \sin \phi_r = v_0 \sin \phi,$$

$$v_r \cos \phi_r = v_0 \cos \phi - w_p,$$

$$v_r = \sqrt{(v_0^2 - 2v_0 w_p \cos \phi + w_p^2)}, \quad \text{..............(1)}$$

$$\tan \phi_r = \frac{v_0 \sin \phi}{v_0 \cos \phi - w_p}, \quad \text{....................(2)}$$

and also

$$x_r = x - w_p t, \quad X_r = X - w_p T. \quad \text{..............(3)}$$

In a head wind, w_p must be taken as negative.

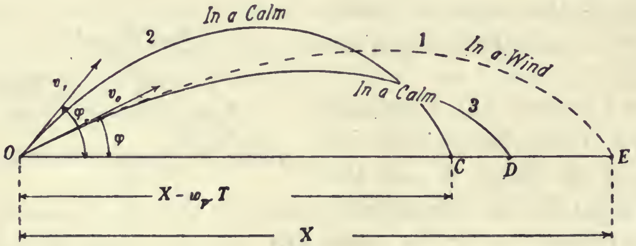


Fig. 6.

The geometrical solution is carried out, as shown in fig. 6; for clearness the two trajectories are drawn out of scale and

$$X_r = X - w_p T.$$

Ordinarily the correction is required in a form where the initial velocity v_0 and the angle of departure ϕ have given values.

Trajectory 2 still requires to be modified to trajectory 3, with the initial values of v_0 and ϕ.

In passing from 2 to 3, with a following wind w_p, the initial velocity v_r is increased to v_0, and at the same time the angle of departure ϕ_r decreased to ϕ.

Here the range X is increased by ΔX_r, so that in dealing with the values of v_0, ϕ, $OC+\Delta X_r$, further calculation is required for a reduction to a calm.

The value ΔX_r is obtained approximately by F. Siacci in the following manner: Suppose $\left(\frac{w_p}{v_0}\right)^2$ neglected in comparison with 1, seeing that w_p is small compared with v_0; then

$$v_r = \sqrt{(v_0^2 - 2v_0 w_p \cos\phi + w_p^2)} = \sim v_0 \sqrt{\left(1 - 2\frac{w_p}{v_0}\cos\phi\right)}$$

$$= \sim v_0\left(1 - \frac{w_p}{v_0}\cos\phi\right);$$

and
$$v_0 - v_r, \text{ or } \Delta v_0 = + w_p \cos\phi.$$

Moreover, $$\tan\phi_r = \frac{v_0 \sin\phi}{v_0\cos\phi - w_p} = \sim \tan\phi\left(1 + \frac{w_p}{v_0\cos\phi}\right),$$

and
$$\tan\phi - \tan\phi_r, \text{ or } \Delta\tan\phi, \text{ or } \frac{\Delta\phi}{\cos^2\phi} = -\frac{w_p\tan\phi}{v_0\cos\phi},$$

$$\Delta\phi = -\frac{w_p}{v_0}\sin\phi,$$

or
$$\Delta v_0 = + w_p\cos\phi, \quad \Delta\phi = -\frac{w_p}{v_0}\sin\phi.$$

According to § 45, on the quadratic law, we have

$$\frac{\Delta X_r}{X_r} = \frac{2\tan\phi}{\tan\omega\tan 2\phi}\Delta\phi + \frac{2\tan\phi}{\tan\omega}\frac{\Delta v_0}{v_0},$$

in which ω, the angle of descent, must be obtained by some method of approximation.

Introducing the values of Δv_0 and $\Delta\phi$, and taking $X = X_r$, we have

$$\frac{\Delta X_r}{X} = \frac{w_p\tan\phi}{v_0\tan\omega\cos\phi}. \quad \text{......................(4)}$$

Thence in trajectory 3 reduced to a calm, with initial values v_0 and ϕ, and the range OD or X_r',

$$X_r' = X_r + \Delta X_r = X - w_p T + \frac{w_p X\tan\phi}{v_0\tan\omega\cos\phi}. \quad \text{............(5)}$$

More generally, the coordinates (x, y) of any point of the trajectory, as well as the time of flight, can be measured on a photogrammetric method in a following wind w_p, and the reduction carried out.

But so far no convenient simple formula for practical use has been given.

Something of the kind can be obtained by replacing the trajectory by a corresponding parabola of the 3rd order, and taking for trajectory 2 in fig. 6 the form assumed by Piton-Bressant,

$$y = x \tan \phi - \frac{gx^2}{2(v_0 \cos \phi)^2} - \frac{gx^3 K}{2(v_0 \cos \phi)^2}, \quad \text{.............(6)}$$

in which K can be considered constant for the point (xy) for a small alteration of the trajectory.

Then x may be left unaltered, and a calculation is made of the change Δy with a following wind w_p, due to a simultaneous increase of the initial velocity v_0 by Δv_0, and a decrease $\Delta\phi$ in the angle of departure. According to the above

$$\Delta (v_0 \cos \phi) = + w_p,$$

and

$$\Delta (\tan \phi) = -\frac{w_p \tan \phi}{v_0 \cos \phi};$$

and we have

$$\Delta y = x\Delta (\tan \phi) - \tfrac{1}{2} gx^2 (1 + Kx) \Delta (v_0 \cos \phi)^{-2}$$

$$= -\frac{xw_p \tan \phi}{v_0 \cos \phi} + \frac{gx^2 w_p}{(v_0 \cos \phi)^3} + \frac{gx^3 K w_p}{(v_0 \cos \phi)^3},$$

or

$$\Delta y = -\frac{w_p}{v_0 \cos \phi}(2y - x \tan \phi). \quad \text{.......................(7)}$$

Here x and y should refer to trajectory 2. But since it is not usual to make up a range table from practice in a strong wind, the corrections to be introduced are relatively small; and so it will be sufficient to take x and y from the Range Table practice in the calculation of Δy in equation (7).

The accuracy of formula (7) may be tested by taking the special case of the point of descent on the muzzle horizon.

Here $x = X$, $y = 0$, so that in (7)

$$\Delta y = \frac{w_p}{v_0 \cos \phi} X \tan \phi;$$

and this is the vertical ordinate at the point C between trajectories 2 and 3.

Thus

$$CD = \frac{\Delta y}{\tan \omega},$$

and $$CD = \frac{w_p X \tan\phi}{v_0 \cos\phi \tan\omega},$$

and $$X_r' = X - w_p T + \frac{w_p X \tan\phi}{v_0 \cos\phi \tan\omega},$$

as in equation (5).

Rule for correction for the wind.

(*a*) For the point of descent on the muzzle horizon:

Suppose the range X and time of flight T observed with a following wind w_p; v_0 and ϕ as before.

To make the reduction to a calm, X is replaced approximately by

$$X_r' = X - w_p T + \frac{w_p X \tan\phi}{v_0 \cos\phi \tan\omega}, \quad \text{.................(8)}$$

on the assumption of the quadratic law, for howitzer and mortar fire; and on the cubic law, for guns and rifles,

$$X_r' = X - w_p T + \frac{w_p X \tan\phi}{v_0 \cos\phi \tan\omega}\left(1 + \cos^2\phi - \frac{\tan\omega \cos^2\phi}{\tan\phi}\right) \text{....(9)}$$

The increase of range with a following wind w_p is then in the first case

$$\Delta X = w_p T - \frac{w_p X \tan\phi}{v_0 \cos\phi \tan\omega}, \quad \text{..................(10)}$$

in which ω, the acute angle of descent, is to be obtained by some approximation. In a head wind w_p is to be taken as negative.

(*b*) For any point (xy) of the trajectory, in a following wind w_p, after a time of flight t, replace x by $x - w_p t$, and at the same time, y by

$$y - \frac{w_p}{v_0 \cos\phi}(2y - x\tan\phi). \quad \text{..................(11)}$$

A further rule will be, that in taking wind influence into account on a trajectory, an assumption can be made that the air density is not the actual observed density δ, but a smaller value

$$\delta\left(1 - \frac{w_p T}{X}\right).$$

Numerical example. Take $v_0 = 580$ m/sec, $\phi = 30°$, $X = 10400$ m, $T = 40$ sec, $\omega = 45°$, $x = 6600$ m, $y = 2000$ m, $t = 20$ sec.

Taking a wind velocity $w_p = 6$ m/sec, the alteration of range according to (10) is 170 m.

Moreover, according to (11), instead of $x = 6600$ m and $y = 2000$ m, after allowing for the wind, $x = 6480$ m, $y = 1997·6$ m.

II. Weapon at rest on the ground: wind blowing horizontally and perpendicularly to the plane of fire.

It is assumed in the velocity diagram of fig. 7 that the wind is perpendicular to the plane of fire, and blows with velocity w_s, from left to right looking down the range.

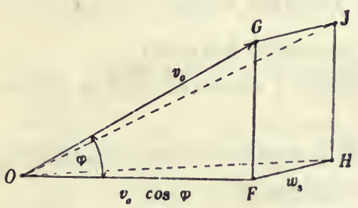


Fig. 7.

The vector of the initial velocity being OG, and the plane OGF the plane of fire, the angle GOF is the angle of departure.

The vector of the wind velocity w_s is HF, perpendicular to OF.

Compound v_0 with the reversed wind velocity $-w_s$ or the vector GJ; the vector sum OJ is the reduced initial velocity v_r, JOH the reduced angle of departure ϕ_r.

The vertical plane OJH, making an angle HOF or ψ with the plane of fire OGF, is the plane in which the motion of the shell must be supposed to take place.

Obviously

$$FG = HJ, \quad OG^2 + GJ^2 = OJ^2,$$

$$\tan\psi = \frac{HF}{OF}, \quad \tan\phi_r = \frac{HJ}{OH};$$

or

$$v_0 \sin\phi = v_r \sin\phi_r, \quad v_r^2 = v_0^2 + w_s^2, \quad \tan\psi = \frac{w_s}{v_0 \cos\phi},$$

$$\tan\phi_r = \frac{v_0 \sin\phi}{\sqrt{(v_0^2 \cos^2\phi + w_s^2)}}; \quad \text{.................(12)}$$

these equations serve to determine the initial velocity v_r and angle of departure ϕ_r, and the relation

$$\tan\psi = \frac{w_s}{v_0 \cos\phi} \quad \text{........................(13)}$$

serves to determine the vertical plane OHJ, in which the movement of the shell takes place.

In the displacement diagram of fig. 8 for the horizontal projection of the motion of the shell, the plane of fire appears as the straight line OK. The straight line OM is drawn making with OK the angle ψ.

Taking the values of v_r and ϕ_r calculated from (12), as well as the ballistic coefficient c, suppose the abscissa x_r of the shell after the time t, and the range $X_r = OM$ in the time of flight T, to be calculated.

At x_r suppose the vector $w_s t$, and at X_r or OM the vector $w_s T$ or ML to be drawn parallel to the wind direction, and perpendicular to OK.

Then L is the actual position of the shell after the time T. In fact OM is the horizontal motion with respect to the wind, ML the drift of the wind over the ground, and OL is the path of the shell relatively to the ground.

The range in the wind in the plane of fire is

$$OK = X = X_r \cos \psi, \quad \text{.....................(14)}$$

whence ψ is obtained from (13).

The lateral deviation of the shell, measured perpendicularly to the plane of fire, is

$$KL = ML - MK \quad \text{or} \quad z = w_s T - X_r \sin \psi. \quad \text{.........(15)}$$

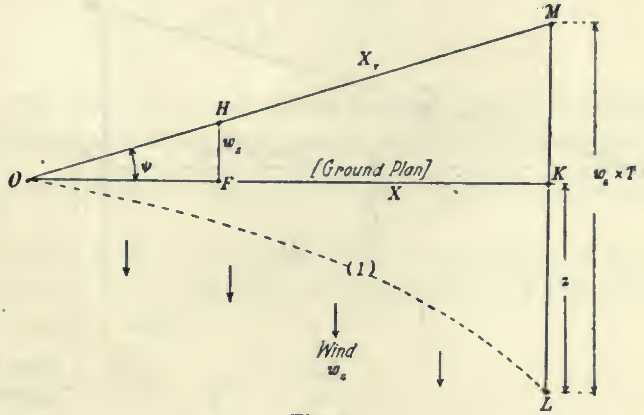


Fig. 8.

This treatment holds evidently for any point of the trajectory The coordinates of such a point after a time t in a side wind w_s being denoted by x and y, the lateral displacement by ζ, and the abscissa after allowance for the wind by x_r; then

$$x = x_r \cos \psi, \quad \zeta = w_s t - x_r \sin \psi. \quad \text{...............(16)}$$

Since ψ is obtained from $\tan \psi = \dfrac{w_s}{v_0 \cos \phi}$, and is in reality a very small angle, it is allowable to replace x_r by x, and $\sin \psi$ by $\tan \psi$.

Thus in general the formula to be employed for the lateral drift of the shell due to wind, perpendicular to the plane of fire, is

$$\zeta = w_s t - x \frac{w_s}{v_0 \cos \phi}, \quad z = w_s T - X \frac{w_s}{v_0 \cos \phi}. \quad \text{......(17)}$$

Example. As above, let $v_0=580$ m/sec, $\phi=30°$, $x=6600$ m, $t=20$ sec, $X=10400$ m, $T=40$ sec; let the wind blow across the range with velocity $w_s=+6$ m/sec.

Here $\cos\psi=0{\cdot}9999$; equation (17) shows that

$$\text{after time } t=20 \text{ sec}, \quad \zeta=6\times 20-\frac{6600\times 6}{580\cos 30}=41 \text{ m},$$

$$\text{after time } T=40 \text{ sec}, \quad z=6\times 40-\frac{10400\times 6}{580\cos 30}=116 \text{ m}.$$

Remark 1. As stated already, the shell moves in the vertical plane OM, with v_r and ϕ_r, as if there were no wind.

If with given v_r, ϕ_r and ballistic coefficient c, the coordinates x_r and y_r are to be calculated at any point of the trajectory after a time of flight t, then, as shown in fig. 9, the axis of an elongated shell, neglecting at first the nutational and precessional oscillations, remains parallel to the plane of fire OK.

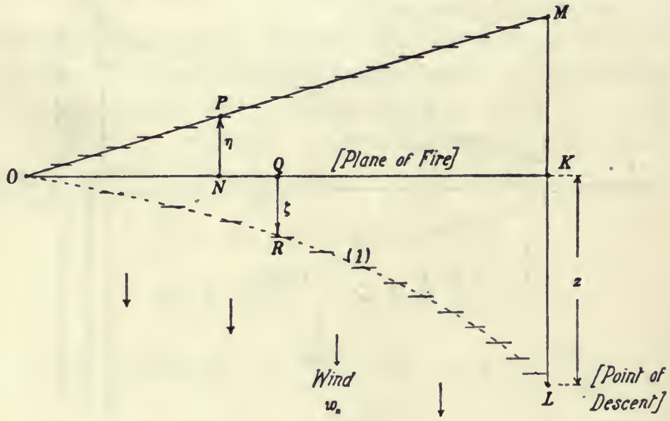


Fig. 9.

The ballistic coefficient c, employed for the motion of the shell OM, thus requres to be somewhat increased.

Remark 2. The lateral deviation can also be obtained, by taking into account the wind pressure which tends to drive the shell out of the vertical plane of fire OK, so that it describes a curve of double curvature over the ground, of which the horizontal projection is the curve OL in fig. 9.

In this motion, perpendicular to the plane OK, the shell moves broadside. Suppose $2R$ to be the calibre, and the longitudinal section of the shell to have an area $2RL$ or $(2R)^2 l$ m².

Let the coordinate of the shell perpendicular to the plane OK after time t be denoted by ζ, and let $\frac{d\zeta}{dt}=u$. The velocity of the wind relatively to the shell is w_s-u. Then the wind pressure is equal to

$$0{\cdot}122\,(2R)^2\, l\,(w_s-u)^2$$

if the surface is flat. On the cylindrical surface of the shell it is about two-thirds of this.

But the wind pressure is $\frac{P}{g}\frac{du}{dt}$, where P denotes the weight of the shell; so we have the differential equation

$$\frac{P}{g}\frac{du}{dt}=\frac{2}{3}\times 0{\cdot}122\,(2R)^2\,l\,(w_s-u)^2;$$

and then in the twofold integration, the constants are determined so that $t=0$, $u=0$, $\zeta=0$.

The lateral deviation, due to wind, is then given by

$$\zeta=w_s t-\frac{1}{C}\log_e(1+Cw_s t), \qquad \text{(18)}$$

and

$$z=w_s T-\frac{1}{C}\log(1+Cw_s T), \qquad \text{(19)}$$

where

$$C=\frac{2}{3}\frac{0{\cdot}122\,(2R)^2\,lg}{P}.$$

III. The weapon at rest on the ground: the wind blowing horizontally at an acute angle with the plane of fire.

Suppose the wind to make an angle α with Ox; the wind velocity being w.

The initial velocity v_0 is represented in perspective in fig. 10 by the vector OC; and so the angle of departure ϕ is COB; the plane COB is the plane of fire.

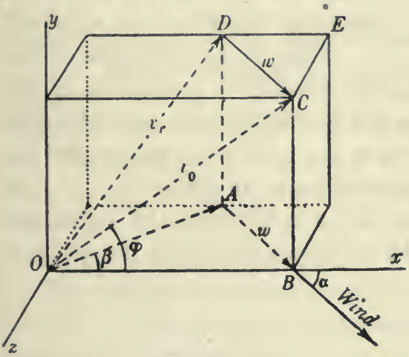


Fig. 10.

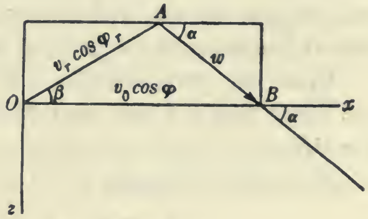


Fig. 10*a*.

The vector CD of reversed wind velocity is combined with OC, and so the resultant vector is OD and the angle DOA or ϕ_r is the reduced angle of departure.

The plane DOA, making an angle β with the plane of fire COB, is then the vertical plane in which the shell moves with the initial elements v_r and ϕ_r: it may be called conveniently the reduction plane.

From fig. 10, and its plan shown in fig. 10a, it is seen at once that

$$OB = v_0 \cos\phi,\quad OA = v_r \cos\phi_r,\quad AD = BC = v_0 \sin\phi = v_r \sin\phi_r,$$

$$AB = DC = w,\quad \tan\phi_r = \frac{AD}{OA},\quad \tan\beta = \frac{w\sin\alpha}{OB - w\cos\alpha},$$

$$OA^2 = v_0^2\cos^2\phi - 2v_0 w\cos\phi\cos\alpha + w^2,\quad OD^2 \text{ or } v_r^2 = OA^2 + AD^2.$$

The equations are then

$$v_r = \sqrt{(v_0^2 - 2v_0 w\cos\phi\cos\alpha + w^2)}, \qquad (20)$$

$$\tan\phi_r = \frac{v_0\sin\phi}{\sqrt{(v_0^2\cos^2\phi - 2v_0 w\cos\phi\cos\alpha + w^2)}}, \qquad (21)$$

$$\tan\beta = \frac{w\sin\alpha}{v_0\cos\phi - w\cos\alpha}, \qquad (22)$$

and these equations give v_r, ϕ_r, β.

From the initial elements v_r and ϕ_r, with the ballistic coefficient c, the coordinates x_r and y_r at any time t can be calculated, as well as those for the time of flight T. Lay off X_r in the line OA_1 from O at an angle β with the x-axis, and from the end A_1 draw A_1J equal to the set of the wind wT.

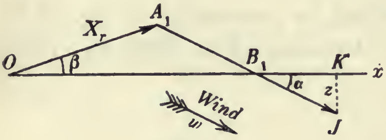


Fig. 11.

Then in the diagram of displacement in fig. 11, representing the horizontal projection of the motion of the shell, OA_1 represents the horizontal motion of the shell relatively to the air, A_1J the motion of the wind over the ground; so that OJ represents the path of the shell over the ground.

Draw the perpendicular JK to the x-axis; then OK represents the range and KJ the wind deflection perpendicular to the plane of fire OK.

From fig. 11 we get

$$X = X_r\cos\beta + wT\cos\alpha, \qquad (23)$$

$$z = wT\sin\alpha - X_r\sin\beta. \qquad (24)$$

Obviously too for any other point, after a time t,

$$x = x_r\cos\beta + wt\cos\alpha,\quad \zeta = wt\sin\alpha - x_r\sin\beta.$$

IV. Dropping a bomb out of an aeroplane.

The aeroplane is supposed to be moving horizontally with uniform velocity v_0 over the ground, and to be at O at the moment when the bomb is dropped, at a height Y.

The movement of the bomb is such that it has a horizontal velocity v_0.

The former methods can be employed to calculate the range X, time of flight T, striking velocity v_e and angle of descent ω; the question of relative velocity need not come into consideration.

In fig. 12 suppose B to be the point on the ground: then

$$\tan\beta = \frac{Y}{X}.$$

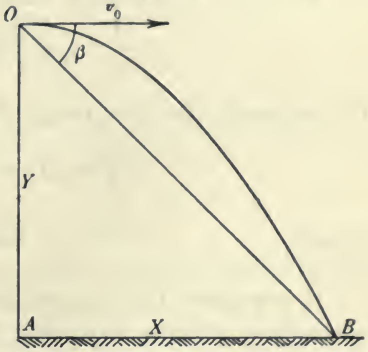


Fig. 12.

P. Charbonnier has calculated tables for a vacuum; and also for air, taking spherical bombs, 15 cm in diameter, weighing 7·5 kg: the calculation is made on the Euler-Otto method for various heights, from 250 m up to 2000 m.

V. Dropping a bomb from an aeroplane; wind blowing in the same direction as the motion of the aeroplane.

As before, the velocity of the aeroplane is v_0 and of the wind w; and the velocity of the aeroplane in a calm is denoted by v_r, as given by the revolutions of the propeller; then with a following wind

$$v_0 = v_r + w. \qquad (25)$$

The calculation of the range X_r and time of flight T_r with initial velocity v_r, and an angle of departure 0°, is carried out by methods described in §§ 20 to 43.

The range in the wind is then

$$X = X_r + wT_r. \qquad (26)$$

When the assumption is made that the movements in the horizontal and vertical directions can be calculated independently, and when the quadratic law of air resistance is employed, then the equation of motion in the horizontal direction is

$$\frac{dv}{dt} = -cv^2,$$

with $v = v_r$ when $t = 0$; then the horizontal velocity v or $\frac{dx_r}{dt}$ at the time t is given by

$$v = \frac{v_r}{1 + cv_r t}.$$

And since $x_r = 0$ when $t = 0$; then at time t

$$x_r = \frac{1}{c} \log_e (1 + cv_r t).$$

Consequently the total range X is given by

$$X = wT_r + \frac{1}{c} \log (1 + cv_r T_r), \quad \text{...............(27)}$$

where $$\tan \beta = \frac{Y}{X}.$$

It is possible that $v_r = w$, and so $v_0 = 0$; that is to say, the aeroplane does not advance against the wind, but seems to be stationary to an observer on the ground. In such a case, the range in the wind is

$$X = X_r - wT_r = \frac{1}{c} \log (1 + cwT_r) - wT_r. \quad \text{.........(28)}$$

This is the same as formula (19).

VI. Dropping a bomb out of an aeroplane with side-wind from behind.

Suppose the aeroplane to be at O at the moment of release, and moving with velocity v_0 over the ground; the wind velocity on the ground being w and its direction making an angle α with the direction of the track.

We compound v_0 and $-w$ to obtain v_r; then on the velocity diagram of fig. 13, OA is the vector of the velocity v_r relatively to the wind, AB the vector of the wind over the ground; OB the vector of the velocity of the aeroplane over the ground. From fig. 13 we have

$$v_r^2 = v_0^2 - 2wv_0 \cos \alpha + w^2, \quad \text{.................(29)}$$

$$\tan \beta = \frac{w \sin \alpha}{v_0 - w \cos \alpha}. \quad \text{.....................(30)}$$

The reduction plane makes an angle β with the vertical plane through OB, and β is obtained by construction, or from equation (30).

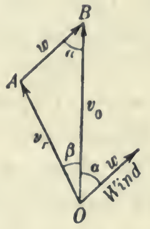


Fig. 13.

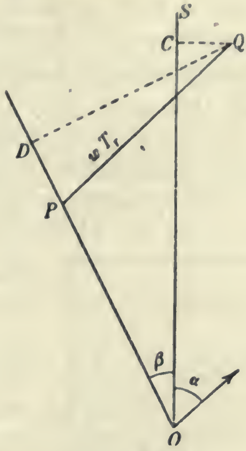


Fig. 14.

The range X_r and time of flight T_r are then calculated, and the displacement diagram of fig. 14 is drawn, in plan.

The line OP is drawn equal to X_r, and PQ is parallel to the wind and equal to the wind displacement wT_r. Then Q is the actual point of fall of the bomb.

If the perpendicular QC is dropped on OS, then OC is the range along OS and CQ is the deflection.

The direction OP is that in which the axis of the flying machine is directed, and the perpendicular DQ from Q on OP denotes the deviation of the point of fall from the vertical plane through the axis of the flying machine.

Thus the wind may be blowing at right angles to the direction of the track; then AB is perpendicular to OS, $\alpha = 90°$, and

$$v_r^2 = v_0^2 + w^2, \quad \tan\beta = \frac{w}{v_0}.$$

The lateral deviation with respect to the direction of flight OS is given by

$$z = wT_r - X_r \sin\beta = wT_r - X_r \frac{w}{v_r}, \quad \text{............(31)}$$

or approximately by

$$z = wT_r - \frac{1}{c}\log(1 + cwT_r), \quad \text{..................(32)}$$

in agreement with (15), (17) and (19).

VII. Firing from an aeroplane, with no wind.

The aeroplane is at O at the moment of firing. The motion is horizontal with velocity s.

The flying machine is at a height Y, above the ground, and moves along the track OS, as in figs. 15, 16, 17. Suppose the initial velocity v_0 is known from the range table; and the angle ϕ is determined from the sighting and the range. These then are the values of v_0 and ϕ which refer to a gun at rest.

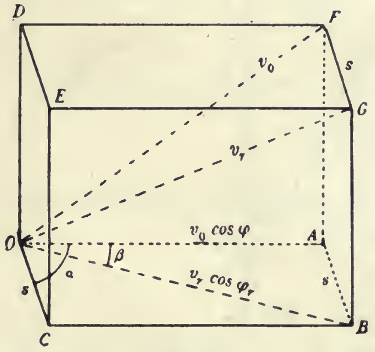


Fig. 15.

The plane of fire, $OAFD$, in fig. 15 makes an angle α with the vertical plane through the direction of motion, OC.

In fig. 15, OF represents the initial velocity v_0 of the projectile relatively to the gun; FG is the velocity s of the gun over the ground, so that the resultant OG is the initial velocity v_r of the projectile over the ground.

The slope of OF is the angle of departure ϕ; but relatively to the ground or the air, the angle of departure is GOB or ϕ_r. The corresponding velocity diagram is drawn in fig. 16 for the horizontal components.

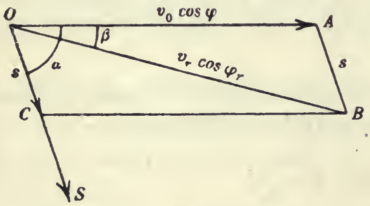


Fig. 16.

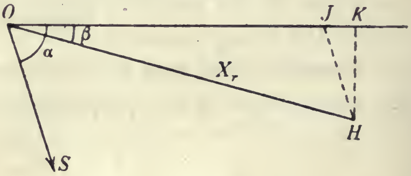


Fig. 17.

But as it is assumed that the air is at rest relatively to the ground, the projectile moves in its actual plane of flight without being affected by the wind, and is influenced only by air resistance, acting in the plane OBG.

But the motion of the shell, relatively to the ground, takes place actually in the plane OBG; and if no account is taken of deviations due to rotation or the slanting position of the axis, the trajectory is a plane curve, and not one of double curvature.

The ballistic calculation is then to be applied to the reduction plane OBG, employing the initial values v_r and ϕ_r, as well as the ballistic coefficient c; the range X_r and time of flight T_r are to be calculated.

As to the ballistic coefficient, it must be slightly increased, since the axis of the projectile is parallel to the plane OAF.

These values of v_r and ϕ_r, as well as the angle β between plane of fire and reduction plane, are given by the following equations: in fig. 15

$$AF = BG, \text{ and } OB^2 = OA^2 + 2OA \,.\, AB\cos\alpha + AB^2,$$

$$v_r^2 \sin^2\phi_r = v_0^2 \sin^2\phi, \quad v_r^2\cos^2\phi_r = v_0^2\cos^2\phi + 2v_0 s\cos\phi\cos\alpha + s^2,$$

and
$$\tan\beta = \frac{AB\sin\alpha}{OA + AB\cos\alpha}.$$

Thus

$$v_r = \sqrt{(v_0^2 + 2v_0 s\cos\phi\cos\alpha + s^2)}, \quad \text{.................(33)}$$

$$\tan\phi_r = \frac{v_0\sin\phi}{\sqrt{(v_0^2\cos^2\phi + 2v_0 s\cos\phi\cos\alpha + s^2)}}, \quad \text{......(34)}$$

$$\tan\beta = \frac{s\sin\alpha}{v_0\cos\phi + s\cos\alpha}. \quad \text{..........................(35)}$$

In the displacement diagram of fig. 17 OJ, OH, and OS are parallel to OA, OB, and OC respectively. The angle JOH is the angle β calculated from (35); OH is made equal to range X_r calculated from v_r, ϕ_r, c.

The perpendicular HK is drawn to OJ; and HJ is parallel to OS.

OJ is the range obtained in the plane of fire without any movement of the gun, and so it is the range of the projectile relatively to the gun; $JH = sT_r$ is the travel of the gun over the ground in the same time; OH or X_r is consequently the range of the projectile relatively to the ground.

The lateral deviation of the projectile from the plane of fire $OAFD$, due to the velocity of the gun, is

$$KH = X_r\sin\beta = sT_r\sin\alpha;$$

and the range

$$OK = X_r\cos\beta.$$

The increase of range, due to the movement of the gun, is then JK or $sT_r \cos \alpha$.

Take the case $\alpha = 90°$, $\cos \alpha = 0$, $\sin \alpha = 1$; then

$$v_r = \sqrt{(v_0^2 + s^2)}, \qquad (36)$$

$$\tan \phi_r = \frac{v_0 \sin \phi}{\sqrt{(v_0^2 \cos^2 \phi + s^2)}}, \qquad (37)$$

$$\tan \beta = \frac{s}{v_0 \cos \phi}. \qquad (38)$$

The influence of the wind, set up by the aeroplane, may also be considered by referring the motion of the projectile, not relatively to the gun, but to the ground, over which the air is at rest.

The range X_r and time of flight T_r are then calculated from v_r and ϕ_r, and OH in fig. 18 is drawn at an angle β with the vertical plane through OA, making $OH = X_r$.

Then H is the point of impact on the ground, $OJ = X_r \cos \beta$ is the range, and $JH = X_r \sin \beta$ is the lateral deviation from the plane of fire due to the motion of the aeroplane.

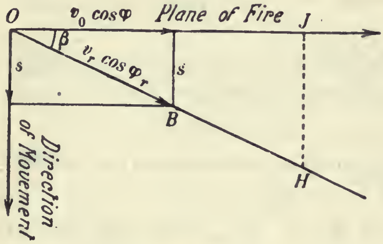


Fig. 18.

If the range is to be calculated from v_0 and ϕ as initial data, a correction for the wind is not made. Since the usual calculations in External Ballistics are made on the assumption of a calm, it is better to employ that vertical plane in the calculation of the motion of the projectile, which refers to the motion in a calm, and here this is the vertical plane through OH.

VIII. Firing from an aeroplane: direction of fire perpendicular to the direction of motion: the wind blowing across the direction of motion at an acute angle.

Denote the initial velocity of the projectile, referred to the gun at rest, and the angle of departure by v_0 and ϕ respectively.

As shown in fig. 19, let the plane of fire be the vertical plane through OA. Along OS and perpendicular to OA is the horizontal motion of the gun: denote the velocity by $s = AC$, relatively to the ground; and denote the horizontal velocity of the wind over the ground by $w = BC$; and the acute angle between the positive wind direction and the positive direction of motion by α.

The velocity s_r of the gun through the air is first determined, by calculation or graphically: in the triangle ABC, $w = BC$ and $s = AC$.

Consequently s_r is given by the vector AB, and thus also β, the angle between s_r and s, is determined.

Thence the relative velocity with regard to the wind, v_r, and the corresponding departure angle, ϕ_r, are found.

Draw the series of lines $OABCO$ in fig. 19; where $OA \# v_0 \cos\phi$, AB is equal and parallel to the velocity s_r, $BC \# w$.

Then OA is the horizontal velocity of the shell relatively to the

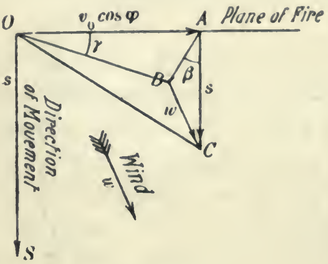


Fig. 19.

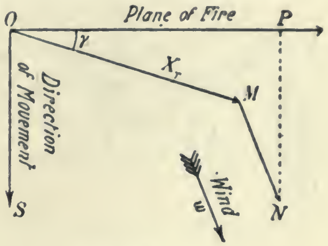


Fig. 20.

aeroplane, AB of the aeroplane relatively to the air; BC the horizontal velocity of the air over the ground: thence the line OC represents the initial horizontal velocity of the projectile over the ground.

The movement of the projectile through the air takes place in the vertical plane through OB; this is then the reduction plane. In the diagram of fig. 19, OB represents the horizontal velocity component $v_r \cos\phi_r$, and this component is thus known. Thence since $v_0 \sin\phi = v_r \sin\phi_r$, we have

$$\tan\phi_r = \frac{v_0 \sin\phi}{OB},$$

and so v_r and ϕ_r are known.

Taking v_r, ϕ_r and the ballistic coefficient c, the calculation is made, for a calm, of the range X_r on the level of the target, and the time of flight T_r.

Then in fig. 20, which shows the horizontal projection, a straight line OM is drawn through the origin O parallel to the vector OB of the velocity diagram, and made equal to the calculated range X_r

Through M draw the vector MN parallel to BC and equal to the set of the wind wT_r. Then N represents the point of descent. From O to N a trajectory of double curvature is shown.

Draw through O a line parallel to OA and through N a line parallel to AC; these lines intersect in a point P; and OP is the range in the plane of fire: PN is the deflection from the plane of fire due to wind and travel.

It is evident that the foregoing may be made to apply to the case where the firing is at any angle to the direction of travel.

These remarks must apply to Case II, where s_r is equal and opposite to w, AB coincides with CB, and AC is zero.

In the triangle OAB of fig. 21, OA is the horizontal component, $v_0 \cos \phi$, of the initial velocity of the projectile relatively to the machine; AB is the velocity

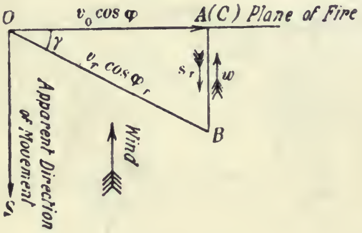


Fig. 21.

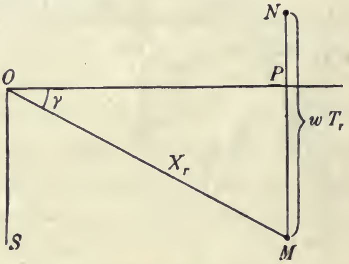


Fig. 22.

of the machine relatively to the air, that is, the velocity given by the revolutions of the propeller in a calm; so that OB is the horizontal component, $v_r \cos \phi_r$, of the projectile through the air.

Similarly in fig. 22 OM is parallel to OB, and the calculation of the range X_r is made from v_r, ϕ_r and the ballistic coefficient, to which the time of flight T_r applies. Moreover through M a parallel MN is drawn to BA and made equal to wT_r. Then OP is the range, and PN is the lateral deviation due to wind.

IX. A ship is moving on a stream. A flying machine is moving in the air, having a definite velocity relatively to the ship. The wind is blowing aslant of the travel of the machine. The firing takes place in a slanting direction.

The case is stated merely to show how with these apparently complicated relative velocities a simple treatment is possible with the help of the velocity and displacement diagrams. A practical application of the theory is not contemplated here.

In the velocity diagram $OABCDE$ of fig. 23, let the vertical plane through OA represent the plane of fire; and let OA represent the horizontal component of the initial velocity relatively to the machine;

AB the horizontal component of the velocity of the machine through the air; BC the horizontal component of the air relatively to the ship; DE the velocity of the stream relatively to the land, and CD the velocity of the ship, relatively to the river.

Consequently CE is the velocity of the ship relatively to the earth; BD the velocity of the air over the stream; BE that of the air

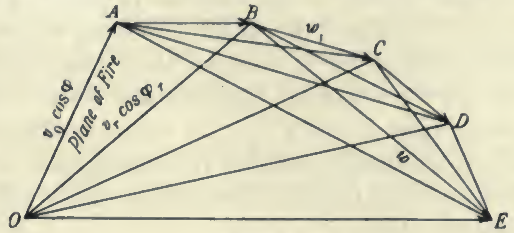


Fig. 23.

over the land; AC that of the machine relatively to the ship; AD is that relative to the stream; AE is that relative to the land; OB is the horizontal component $v_r \cos \phi_r$ of the initial velocity of the projectile through the air; OC the same relative to the ship, OD relative to the stream, OE relative to the land.

The plane in which the motion of the projectile appears to take place is the vertical plane through OB. Given $v_r \cos \phi_r$, v_r and ϕ_r can be obtained. Thence with the aid of the ballistic coefficient the range X_r and time of flight T_r are calculated.

Then in the displacement diagram, fig. 24, draw OA_1A_2 parallel to OA in fig. 23, and make $OB_1 = X_r$ and the angle $A_1OB_1 = AOB$.

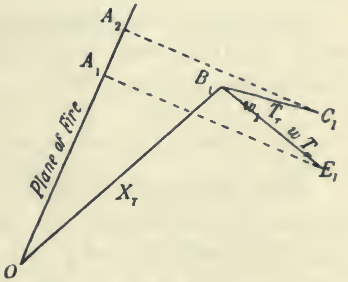


Fig. 24.

Draw through B_1 a parallel to BE equal to the set of the wind, wT_r. Then the point E_1 is the point of impact of the projectile on the ground. Draw the perpendicular E_1A_1 on OA_1, then OA_1 is the range on the plane of fire relatively to the ground; A_1E_1 the lateral deviation due to travel and wind.

Or draw through B_1 a parallel to BC, equal to w_1T_r, where w_1 is the velocity BC of the wind relative to the ship; then the point C_1 is the point of impact relative to the steamer, the perpendicular C_1A_2 is the lateral deviation, OA_2 is the range on the plane of fire, relative to the steamer; and so on.

§ 48. Deviation of the projectile due to rotation of the Earth.

Suppose a projectile is fired northwards from a point O on the equator: the gun has a velocity to the East of about the same magnitude as the infantry bullet M. 71 at the beginning of its trajectory.

After a certain number of seconds the projectile strikes a point O_1 further to the North, and this point is moving to the East with a smaller velocity. The projectile does not then remain in the meridian through O, but travels out of the meridian and more to the East.

On the other hand, if firing from O_1 in the northern hemisphere to the South, the projectile remains, in consequence of the smaller eastward velocity, always behind the meridian drawn through O_1, and so it deviates to the West.

These deviations give rise also to alterations in the range, height of vertex, and time of flight.

As to the quantitative determination of the deviation of the projectile due to rotation of the Earth, calculation for extreme cases in a vacuum gives a lateral deviation of several hundred metres and somewhat less in the range: in air, it appears from calculation that these deviations are of such a magnitude that they may be neglected in comparison with other deviations arising in practice.

An experimental determination of such deviations, due to rotation of the Earth, is beset by great difficulties; and would not be worth the trouble and expense.

(A) *Theory for a vacuum.*

Take the point of departure of the projectile as the origin of a system of coordinates: the axis of x is the horizontal tangent of the parallel of latitude and positive to the East; the axis of y is horizontal and a tangent to the meridian, positive to the North; the axis of z is the line from the centre of the Earth, and positive upwards.

Denote the latitude of O by γ, and the angular velocity of the Earth, $2\pi \div 24 \times 60 \times 60$, by n.

The equations of motion of the projectile are then

$$x'' = 2n \sin\gamma \,.\, y' - 2n \cos\gamma \,.\, z', \qquad (1)$$

$$y'' = -2n \sin\gamma \,.\, x', \qquad (2)$$

$$z'' = 2n \cos\gamma \,.\, x' - g, \qquad (3)$$

the dashes denoting differentiation with respect to t.

Therefore $$x' = 2n \sin\gamma \,.\, y - 2n\cos\gamma \,.\, z + a, \quad \text{......(4)}$$
$$y' = -2n\sin\gamma \,.\, x + b, \quad \text{......(5)}$$
$$z' = 2n\cos\gamma \,.\, x - gt + c. \quad \text{......(6)}$$

Substituting these values of y' and z' in (1), we have

$$x'' + 4n^2 x = 2n\cos\gamma \,.\, gt + 2n\sin\gamma \,.\, b - 2n\cos\gamma \,.\, c. \quad \text{......(7)}$$

Here is a linear differential equation between x and t; it can be integrated by the method of the Complementary Function, taking the constants of integration such that the initial conditions are $t = 0$, $x' = a$, $x = 0$.

Thence x is known as a function of t, and x' also, and thus (2) and (3) can be integrated; and so the position of the projectile at any time is known.

The results of the calculations are as follows:

Statement of results.

$$x = \frac{p}{2n}(1 - \cos 2nt) + \frac{q}{2n}\left(t - \frac{\sin 2nt}{2n}\right) + \frac{a}{2n}\sin 2nt, \quad \text{......(8)}$$

$$y = bt - p\sin\gamma\left(t - \frac{\sin 2nt}{2n}\right) - q\sin\gamma\left(\frac{t^2}{2} + \frac{\cos 2nt}{4n^2} - \frac{1}{4n^2}\right)$$
$$+ \frac{a\sin\gamma}{2n}(\cos 2nt - 1), \quad \text{......(9)}$$

$$z = ct - \frac{gt^2}{2} + p\cos\gamma\left(t - \frac{\sin 2nt}{2n}\right) + q\cos\gamma\left(\frac{t^2}{2} + \frac{\cos 2nt}{4n^2} - \frac{1}{4n^2}\right)$$
$$- \frac{a\cos\gamma}{2n}(\cos 2nt - 1), \quad \text{......(10)}$$

and here, $p = b\sin\gamma - c\cos\gamma$, $q = g\cos\gamma$.

In most cases we can put

$$\frac{1-\cos 2nt}{2n} = nt^2, \quad \frac{1-\cos 2nt}{4n^2} = \tfrac{1}{2}t^2, \quad \frac{2nt - \sin 2nt}{2n} = \tfrac{2}{3}n^2t^3.$$

Example. In the northern hemisphere:

Falling freely; $a=0$, $b=0$, $c=0$, the deviation is to the East, x positive; and to the South, y negative.

Shooting vertically upward; $a=0$, $b=0$, $c=v_0$, the deviation is to the West and to the South.

In a flat trajectory to the East, $a=v_0\cos\phi$, $b=0$, $c=v_0\sin\phi$, there is a deviation in the point of descent to right and an increase of range.

In a flat trajectory to the North, $a=0$, $b=v_0\cos\phi$, $c=v_0\sin\phi$, there is a deviation to the right and a slight alteration of range.

For a numerical example, take initial velocity $v_0 = 820$ m/sec, angle of departure $\phi = 44°$, firing to the North in latitude $\gamma = 54°$. Then $a = 0$, $b = 820 \cos 44°$, $c = 820 \sin 44°$, $\gamma = 54°$; and in a vacuum and without taking into account the rotation of the Earth, the time of flight to a point of descent on the muzzle horizon is 116·13 sec.

Substitute this number $t = 116$ sec, and the other data in (8), and we have $x = +350$ m, to the right.

Thus in this case there is a deviation to the East, due to rotation. Later it will be shown that this deviation is reduced to about one-half, when the influence of air resistance is taken into account.

The southerly deviation in a body, falling freely, can be explained geometrically in the following manner: Consider the cone with vertex at the centre of the Earth, M, and having for base the parallel of latitude through the origin of motion of the falling stone; consider also the tangent plane to the cone along the generating line MO.

Since the stone is being carried round by the Earth to the East, it has at the beginning of its fall an easterly velocity in the direction OT of the tangent to the parallel of latitude. The only force acting on the stone is that of gravity in the direction of the radius of the Earth.

Both lines lie in the tangent plane of the cone, and so the whole motion of the stone must take place on this tangent plane, and the stone describes in fact in this plane a very flat ellipse with a focus at the centre of the Earth.

The stone can then be said to swerve in falling neither to the south nor the north, so long as it remains on the surface of the cone; on the other hand, it can be said to swerve to the north when it comes inside the cone, and to the south when it goes into the space outside.

But this tangent plane, in which the stone falls, coincides with the cone only along the line OM, and so the stone proceeds after the start into the space outside the cone, and so deviates to the South.

(B) *General case, in a space full of air.*

The properties of the trajectory are to be treated first without taking into account the rotation of the Earth, in the manner of Chap. VIII, or in the graphical method of § 31, for a space filled with air.

The coordinates of the trajectory are then known at any time t; suppose them denoted by ξ, η, ζ; when firing to the East ξ denotes x in Chaps. III—VIII, ζ the vertical height, η the horizontal drift of the projectile due to its rotation; η will be negative with a right-handed rifling. Then let (x, y, z) denote the coordinates when rotation of the Earth and air resistance are taken into account.

Then, evidently,

$$x'' = 2n \sin\gamma \,.\, y' - 2n \cos\gamma \,.\, z' + \xi''(t), \quad \text{.........(11)}$$

$$y'' = -2n \sin\gamma \,.\, x' + \eta''(t), \quad \text{.........(12)}$$

$$z'' = 2n \cos\gamma \,.\, x' + \zeta''(t), \quad \text{.........(13)}$$

and so, by integration,

$$x' = 2n \sin\gamma \,.\, y - 2n \cos\gamma \,.\, z + \xi'(t), \quad \text{.........(14)}$$

$$y' = -2n \sin\gamma \,.\, x + \eta'(t), \quad \text{.........(15)}$$

$$z' = 2n \cos\gamma \,.\, x + \zeta'(t). \quad \text{.........(16)}$$

Then if we return to the case where the Earth has no rotation, and $n = 0$, as in the assumption of Chaps. III—VIII, we have

$$x'' = \xi'', \quad y'' = \eta'', \quad z'' = \zeta'',$$

and also $$x' = \xi', \quad y' = \eta', \quad z' = \zeta',$$

and the equations of motion for (x, y, z) and (ξ, η, ζ) are identical.

Here n and γ are known constants, and ξ, η, ζ with their derivatives are supposed to be known from the provisional calculation of the problem when the Earth's rotation is left out of account.

Substitute from (15) and (16) in (11), and we have

$$x'' + 4n^2 x = f(t), \quad \text{.........(17)}$$

where $$f(t) = 2n \sin\gamma \,.\, \eta'(t) - 2n \cos\gamma \,.\, \zeta'(t) + \xi''(t).$$

This differential equation can be solved by the method of the variation of constants; or by the employment of the Abdank-Abakanowitz integrator, it can be solved mechanically.

Consequently, x is found as a function of t, and x' as well. A further mechanical integration of (12) and (13) will give y and z as functions of t, and so the position of the projectile is known at any time under the influence of gravity, air resistance, and the rotation of the Earth.

Integration of (17) gives

$$x = \frac{1}{2n} \sin 2nt \int_0^t f(t) \cos 2nt\,dt - \frac{1}{2n} \cos 2nt \int_0^t f(t) \sin 2nt\,dt,$$

and thus we have $$x = \xi + \Delta x, \quad \text{.........(I)}$$

where

$$\Delta x = +\,2n \sin\gamma \sin 2nt \int_0^t \eta \sin 2nt\,dt + 2n \sin\gamma \cos 2nt \int \eta \cos 2nt\,dt$$
$$-\,2n \cos\gamma \sin 2nt \int \zeta \sin 2nt\,dt - 2n \cos\gamma \cos 2nt \int \zeta \cos 2nt\,dt$$
$$-\,2n \sin 2nt \int \xi \cos 2nt\,dt - 2n \cos 2nt \int \xi \sin 2nt\,dt\,;$$

$$y = \eta + \Delta y, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

$$\Delta y = -\,2n \sin\gamma \int_0^t x\,dt\,;$$

$$z = \zeta + \Delta z, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(III)}$$

$$\Delta z = 2n \cos\gamma \int_0^t x\,dt,$$

where the value of x in (I) is to be substituted.

The integrals,

$$\int (\xi,\ \eta,\ \zeta) \begin{matrix}\sin\\ \cos\end{matrix}\ 2nt\,dt,$$

are to be evaluated by the Integraph.

No definite function of the law of air resistance need be assumed here in this method, which has been employed by the author since 1909.

It is evident that the expressions in (I), (II), (III) hold good also in a vacuum.

Example. A shell, 30·5 cm calibre, weight 445 kg, length 3·5 calibres, radius of point 2 calibres, is fired with initial velocity $v_0 = 820$ m/sec at an angle of departure 44° to the North, in latitude $\gamma = 54°$; barometric height on the ground 760 mm, temperature of the air 15°·5 C, relative humidity 50 °/₀; final rifling of the gun, 25 calibres.

Leaving out of account the rotation of the Earth, the calculations gave: in a vacuum, range 68500 m, height of vertex 16500 m; in air, range 33900 m, abscissa of vertex 19400 m, ordinate of vertex 10980 m.

If the influence of the Earth's rotation is to be calculated, then in the preceding notation, η and ζ are calculated as functions of t according to Chap. VIII, § 41, and ξ in § 56: and

$$\eta = \lambda v_0 \tan\Delta \,.\, \frac{1}{l}\left(\phi \,.\, t - \int_0^T \theta\,dt\right),$$

where Δ is the final angle of twist, l the length of the shell in calibres, and λ an empirical factor, taken as $\lambda = 1{\cdot}158$; and thus it was found by the Integraph that

$$\int_0^T \eta \sin 2nt\,dt = 15872, \qquad \int_0^T \eta \cos 2nt\,dt = 1824000,$$
$$\int \zeta \sin 2nt\,dt = 4544, \qquad \int \zeta \cos 2nt\,dt = 688000,$$
$$\int \xi \cos 2nt\,dt = 80720, \qquad \int \xi \sin 2nt\,dt = 2264.$$

Thence at the end of the trajectory

$$\Delta x = 0{\cdot}025 + 215{\cdot}4 - 0{\cdot}0053 - 59{\cdot}0 - 0{\cdot}160 - 0{\cdot}33 = 156 \text{ m}.$$

Further

$$\int_0^T x\,dt = 84282,$$

and so

$$\Delta y = -9{\cdot}96.$$

Thus by the rotation of the Earth, the range is shortened by 10 m, and a deviation is given to the right of 156 m.

Remark 1. S. D. Poisson employs the quadratic law in the equations of motion with respect to a combination of air resistance and the rotation of the Earth: that is, he assumes in (11) to (13)

$$\left.\begin{aligned} \xi'' &= -c\left(\frac{ds}{dt}\right)^2 \frac{dx}{ds} = -c\frac{ds}{dt}\frac{dx}{dt} = -cs'x', \\ \eta'' &= -cs'y', \quad \zeta'' = -cs'z' - g \end{aligned}\right\} \quad \text{(18)}$$

He then takes both influences into account simultaneously; thereby a double integral arises, for which the limits alone can be assigned. For example, he finds for a bomb of weight 51 kg, and 27 cm calibre, fired at $\phi = 45°$ with $v_0 = 120$ m/sec, in the latitude of Paris, a deviation to the right of 0·9 to 1·2 m, in a range of 1200 m.

And with a shell of weight 90 kg, and 33 cm calibre, at $\phi = 45°$, and in a range of 4000 m he obtains, shooting to the East, a deviation to the right between 5 and 10 m.

Remark 2. St Robert employs the system of coordinates in which the x and y axes lie in the plane of fire; the axis of x horizontal and positive in the direction of fire; the axis of y vertical upward; the z-axis positive to the right of the plane of fire.

The plane of fire is taken at an angle β with the meridian, and directed to the South.

The equations corresponding to (1), (2), (3) are obtained by a rotation of our former xy plane about the vertical; and then on this notation the coordinates are in the form

$$x'' = -2n(\sin\gamma\,.\,z' + \cos\gamma\sin\beta\,.\,y'), \quad \text{(19)}$$

$$y'' = 2n\cos\gamma(\sin\beta\,.\,x' - \cos\beta\,.\,z') - g, \quad \text{(20)}$$

$$z'' = 2n(\sin\gamma\,.\,x' + \cos\gamma\cos\beta\,.\,y'). \quad \text{(21)}$$

This is for a vacuum; and in air the equations become

$$x'' = -2n(z'\sin\gamma + y'\cos\gamma\sin\beta) + \xi'', \quad \text{(22)}$$

$$y'' = +2n\cos\gamma(x'\sin\beta - z'\cos\beta) + \eta'', \quad \text{(23)}$$

$$z'' = +2n(x'\sin\gamma + y'\cos\gamma\cos\beta) + \zeta'', \quad \text{(24)}$$

and here (ξ, η, ζ) have their former signification, where rotation is not taken into account; ξ the horizontal distance, η the height of flight, and ζ the lateral deviation due to the rotation of the shell.

Integration gives

$$x' = -2n\sin\gamma\,.\,z - 2n\cos\gamma\sin\beta\,.\,y + \xi', \quad \text{(25)}$$

$$y' = 2n\cos\gamma\sin\beta\,.\,x - 2n\cos\gamma\cos\beta\,.\,z + \eta', \quad \text{(26)}$$

$$z' = 2n\sin\gamma\,.\,x + 2n\cos\gamma\cos\beta\,.\,y + \zeta'. \quad \text{(27)}$$

Insert these values of x', y', z' from (25) to (27) in (22) to (24), and neglect the term involving n^2, by comparison with n, since $n^2 = \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2$; and we obtain

$$x'' = -2n \sin\gamma \,.\, \zeta' - 2n \cos\gamma \sin\beta \,.\, \eta' + \xi'',$$

and corresponding results in the other equations; then by integration

$$x = \xi - 2n \sin\gamma \cdot \int_0^t \zeta\, dt - 2n \cos\gamma \sin\beta \int \eta\, dt,$$

and so too for the others.

This method of solution has been used lately by N. Sabudski, and the problem reduced to finite formulae, for which he has calculated convenient tables.

It might possibly be objected to the St Robert-Sabudski method, that proof is required as to whether the omission of the term with n^2 is allowable in comparison with the term in n.

Because the method implies that $-2nz \sin\gamma$ and $-2ny \cos\gamma \sin\beta$ in comparison with ξ', as well as $2nx \cos\gamma \sin\beta$ and $2nz \cos\gamma \cos\beta$ in comparison with η', and finally $2nx \sin\gamma$ and $2ny \cos\gamma \cos\beta$ in comparison with ζ', may be neglected.

The method of the author is free from these approximations; as a matter of fact the example shows that the two methods give results in reasonable agreement, so that in this case the tables of Sabudski may be employed.

§ 49. The lateral deflection of the infantry bullet with bayonet fixed.

It was observed for a long time that if the bayonet was fixed on the right-hand side of the barrel, the bullet was deflected to the left.

In the official text-books of the seventies and eighties of the last century (Stacharowski, Neumann, Weygand, Hentsch) this result was treated as a definite fact, without any very obvious cause.

It was first supposed that the gases exercised a reactive pressure, which was due to the blade of the bayonet: but this was soon found to be insufficient. The deflection disappears, if the rifle is fired close to a wall and in a direction parallel to it, the bayonet having been removed. Further the deviation is sometimes greater, if the bayonet is fixed at right angles to the bore.

Formerly, about 15 or 18 years ago, the recoil of the rifle and bayonet was treated as the motion of a rigid system due to the reaction of the gas pressure, the rotation being to the right in consequence of the position of the centre of gravity of the whole body.

A mathematical explanation based on these mechanical principles was given by the author in 1885, and in 1888 it was worked out by F. Kötter, employing the principle of angular momentum.

Later the author became acquainted with facts which could not be reconciled with these explanations.

The deviation to the right with bayonet on the left, and *vice versa*, is not universal, but depends on the design of rifle.

It was pointed out by A. Ch. Minarelli-Fitzgerald that in the Austrian rifle, the deviation was sometimes to the right, and sometimes to the left. The same was found to be the case with the German rifle M. 88.

Moreover if the recoil of the rifle M. 71 was prevented, and if the bayonet was fixed not to the right, but below, a deviation to the left was still observed, although it was to be expected from the second of the explanations that deviation would be absent.

The real cause is found in the elastic deformation due to the initial transversal vibrations of the barrel; consult Vol. III, §§ 154 and 183. The vibrations of the barrel of the rifle M. 71 were investigated by photographic methods by K. R. Koch and the author, and the following results were obtained:

Just after the explosion the barrel begins a transverse vibration, in the fundamental tone and its overtones. The muzzle then starts to move up and down, right and left, and so performs elliptic vibrations.

As far as concerns the horizontal swing, the muzzle moves first to the left, then back through the position of equilibrium to the right, and so on. The bullet leaves the barrel while the muzzle is somewhat to the right, and the moment at which the bullet issues is registered on the spark photograph. This bending of the muzzle throws the bullet somewhat to the right. This happens when no bayonet is fixed.

But now when the bayonet is fixed, the vibrations are naturally somewhat slower, because the swinging mass is greater; and it might be supposed that the forepart of the barrel is still bent to the left while the bullet is leaving the muzzle.

As a matter of fact the photograph shows both the slowing of the vibrations and the deflection of the muzzle to the left at that instant. As for the direction of the bullet, whether to right or left, the second overtone in these horizontal vibrations appeared to be the chief factor, combined too with the first or third overtone. Without the bayonet the time of vibration was about 0·0016 sec; it was 0·0036 sec when the bayonet was fixed. The bullet threw to the left, corresponding to the deformation of the barrel and the phase of the vibration.

The explanation of the effect in question can be found in the fact that the elastic deformation of the barrel in the initial transversal vibrations is altered by the mass of the bayonet attached.

By decreasing the amount of the powder, and using always the same rifle, it was found that the bullet could be made to leave the bore at a later phase of the second harmonic: consequently, with the bayonet always attached in the same way, it was found possible to have deflections both to the right and to the left.

CHAPTER X

Lateral deviations due to rotation of the shell

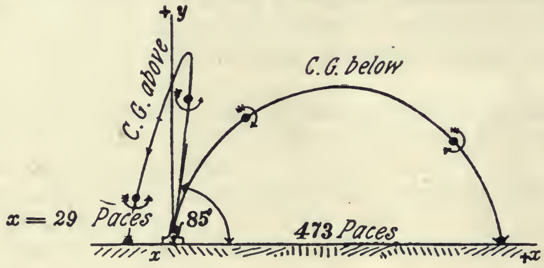
§ 50. Deviations of spherical shells.

The deviations of this nature with the old type of spherical shell were irregular, and were produced by the rotation of the cannon balls.

These rotations were caused by the fact that some play remained between the cannon ball and the wall of the bore, and the ball rebounded several times at the wall of the bore; beside this the escaping gases forced a way past the ball from time to time, and this friction caused the ball to rotate.

With the object of controlling the rotation of the ball so as to give a definite constant deviation, eccentric cannon balls were employed about the year 1830.

If such a ball were placed in the bore with the centre of gravity downwards, for instance, and fired off, the ball rotated, at first at least, about a horizontal axis, with the upper side going forward and the under side backward. This happened because the resultant pressure of the gases was directed through the geometrical centre of the ball, and this was situated above the centre of gravity.



Thence a deviation of the centre of gravity of the ball downwards, a shortening of the trajectory, and generally an increase in the angle of descent followed as natural consequences.

With centre of gravity upwards, the ball experienced a deviation upward, and with angle of departure under 45°, it had a lengthening of the trajectory: on the other hand there was a shortening of the range with very great angles of departure.

In some extreme cases observed by Heim in 1840, the ball fell behind the mortar, as shown in the figure.

1. Experiments at Metz, 1839, carried out by Didion, Morin and Piobert: calibre 22 cm, angle of departure 4° 6′, eccentricity 0·0015 m to 0·0020 m: thence the lifting force, with centre of gravity above, was about 7·1 kg on a 27·9 kg shell.

Weight of powder	Weight of ball	Normal range without eccentricity	Range with eccentricity	
			Centre of gravity at first under	Centre of gravity at first above
1·5 kg	26·6 kg	708 m		
1·5 „	29·9 „	708 „	518	950, with 8·6 turns/sec
1·5 „	27·9 „	708 „	548	941, „ 8 „
1·5 „	26·6 „	869 „		
1·5 „	29·9 „	869 „	712	1163, with 8·6 turns/sec
1·5 „	27·9 „	869 „	731	1009, „ 8 „
3 „	26·6 „	1170 „		
3 „	29·9 „	1170 „	1072	1557, with 8·6 turns/sec
3 „	27·9 „	1170 „	1117	1320, „ 8 „

2. Experiments of Heim, Ulm 1840, with eccentric shell fired from a ten-pounder mortar (p. 310).

3. Experiments of Heim with eccentric shell fired from a ten-pounder howitzer (p. 311).

[The ranges are given in paces of 2·75 feet. The mean lateral deviation is the difference between the sum of the deviations to one side and the other, divided by the number of the shots.

The breadth of the space of the lateral deviations is taken as the sum of the greatest deviations to one side and the other when the shots fall to both sides, and as the difference of the greatest and least deviation when the shots all fall to one side.]

The non-coincidence of the centre of gravity and of the centre of the sphere was effected by hollowing out the cannon ball. The position of the centre of gravity inside the ball was then determined by two observations.

The ball was first allowed to swim in quicksilver, and the point was marked that was uppermost; the line joining it to the centre of the sphere gave the diameter in which lay the centre of gravity; the direction OSM (p. 311) was thus known. Further to determine the distance OS between O the centre of the sphere and the centre of gravity S, and thus to find the so-called eccentricity, a weight q was hung at a fixed point of the surface, and the new position of equilibrium was observed, as in the figure on p. 311.

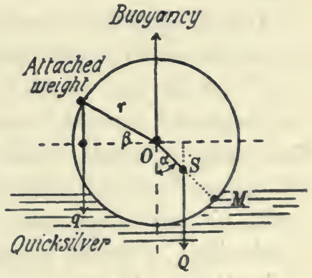
Charge	Angle of departure	Position of centre of gravity in the bore	Number of tests	Mean range	Difference between maximum and minimum range	Mean drift	Breadth of drift	Number of drifts to either side
12	35	above centre	10	622	126	1·2 to right	22	6 to right 4 to left
12	35	below centre	10	517·5	91	5·9 to left	25	2 to right 8 to left
12	35	to right	10	561·5	126	41·2 to right	18	10 to right
12	55	above centre	10	540	73	18·9 to left	38	10 to left
12	55	below centre	10	523	59	13·3 to left	38	1 to right 9 to left
12	55	to right	3	538	57	56·3 to right	24	3 to right
12	55	to left	7	529	106	40·9 to left	30	7 to left
12	20	above centre	10	393	81	1·5 to left	5	6 to left 4 without drift
12	20	below centre	10	311	72	3·4 to left	11	2 to right 8 to left
12	70	above centre	10	322	61	4·1 to left	46	4 to right 5 to left 1 without drift
12	70	below centre	10	369	60	1·3 to left	36	3 to right 6 to left 1 without drift
24	80	above centre	3	132	64	14·3 to left	83	1 to right 2 to left
24	80	below centre	5	611	113	23·8 to right	102	4 to right 1 to left
24	80	to right	1	375	—	258 to right		
24	85	above centre	3	−29·3	68	21 to right	85	2 to right 1 to left
24	85	below centre	2	473	14	7·5 to right	3	2 to right
24	85	to right	1	255	—	306 to right		

Charge in lbs.	Angle of departure	Position of centre of gravity in bore	Number of tests	Mean range	Difference between maximum and minimum range	Mean drift	Breadth of drift	Number of drifts to either side
½	13·5	in front	10	717	266	0·7 to left	21	4 to right 6 to left
½	13·5	behind	10	720	154	2·1 to left	26	3 to right 1 without drift 6 to left
½	13·5	above centre	10	907	223	7·6 to left	24	1 to right 9 to left
½	13·5	below centre	10	568	71	1 to left	19	3 to right 7 to left
½	13·5	to right	5	646	114	38 to right	18	5 to right
½	13·5	to left	5	613	78	30 to left	11	5 to left
1¼	10	behind	10	1328	543	9 to left	92	5 to right 5 to left
1¼	10	above	11	2316	244	15·9 to right	173	6 to right 5 to left
1¼	10	below	10	1055	84	2 to left	21	4 to right 6 to left
1¼	10	to left	10	1424	92	133·8 to left	39	10 to left

The three forces, weight q of the suspended mass, the upward thrust, and the weight Q of the ball alone, considered as concentrated in the centre of gravity, maintain the equilibrium: the angles α and β are measured, and then from the equation of moments we have

$$qr \cos \beta = Qx \sin \alpha,$$

from which the magnitude of x, the eccentricity, can be inferred.

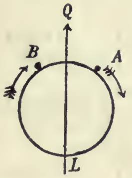


The rotational velocity is determined, either by calculation from the gas and atmospheric pressures, or with more certainty by measurement.

A screen was set up at a short distance from the muzzle; marks placed on the shell made a record in its passage through the screen, and so the rotational position of the ball was known. For example, it was found that with balls of 15 cm calibre, and with a charge of 0·5 kg of powder, the ball at the start was making 18·8 turns a second; this corresponds to an initial circumferential velocity of rotation of 5 m/sec, equal to $\frac{1}{52}$ of the velocity of translation.

The explanation of the deviation of shells, due to rotation, has been debated by ballisticians since the middle of the eighteenth century, partly because the Berlin Academy proposed this subject for a Prize Essay.

The names of the following may be mentioned: Robins 1742, on the alteration of the direction of the air resistance due to the rotation: Euler 1745, on the effect of the irregular rounding of the ball, and of the gyroscopic effect: Lombard in 1783, and Poisson later: Rohde in 1795, who sought for the effect in the driving force of the fuse: Hutton 1812, Gassendi 1819, Paixhans 1822, Terquem 1826, Neumann, on the centrifugal force of the shell as the cause of the deviation: Timmerhans 1841, who argued correctly that the cause of the deviation must lie outside the bore, because the drift increased more rapidly than the range. Didion 1841 and Otto 1843 came very nearly to the right explanation; Otto remarked that "If two elements of surface A and B are taken, equidistant from the diameter lying in the direction of the motion of flight, the element B, which has its velocity increased by the rotation, condenses the air more than the element A, which has its velocity diminished by the rotation." Otto has not assumed that the layer of air, in contact with the surface of the ball, must adhere to it, which would have given the solution for the case of the spherical shell.



The methods of Poisson (1839) and Magnus (1852) must now be considered.

S. D. Poisson proceeded in a purely theoretical manner; he calculated the influence of the rotation of the Earth, and proved that it did not suffice to explain so great a deviation.

Next he took into account the differences of air density round the shell, which may be supposed to be rotating about a horizontal axis, with the upper side going forward relatively to the lower side.

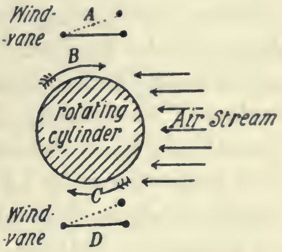
The air density in consequence is greater in the front than in the rear of the shell; and so he concluded that the friction between the shell and the air would be greater in front than behind. The ball will be urged upward in consequence, just like a ball rotating in the same way in contact with a rough cushion in front, and seeking to roll upward on the cushion. (Similar effects are well known with bowls, billiard balls, etc.) This effect is called the Poisson effect, or cushioning. Poisson recognised that this effect of the tangential air friction could not account for the amount of the actual deviation, but he knew little of the effects of the influence of air density on air friction.

With reference to this it may be pointed out that the Newton-Maxwell law on the independence of air friction and density has only been proved experimentally within the limits of atmospheric pressure; the friction seems to diminish with an increase in the density and temperature.

Moreover, in the case of a sphere, rotating about a horizontal axis with the upper side moving forward, it is deflected not upwards but downwards.

Poisson attempted also to take into account the effect of an irregular rounding of the shell.

A satisfactory explanation was first given in 1852 by the well-known physicist G. Magnus, based on experiments with a rotating cylinder, against which a stream of air was blown, as in the figure.



The ball is supposed to be moving horizontally forward, from left to right; or conversely, the centre of gravity of the ball is held at rest, and the stream of air is directed at it from right to left.

The sphere can be set in rotation, about a horizontal axis through the fixed centre of gravity, in the direction shown by the arrows.

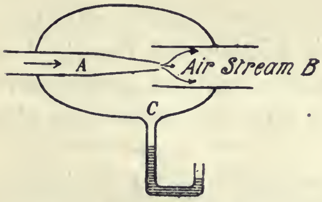
The air, adhering to the sphere, is now carried round with it; on the under side, C, in the same direction as the current of air, on the upper side, B, in the opposite direction.

Then on hydrodynamical principles, there ensues an increase of pressure above, and a diminution below.

To prove the fact that the pressure is increased at B and diminished at C, let us put two little flags at A and D; the flag at A is driven away from the sphere, and the flag at D is attracted to it.

At B the two streams are opposed, the air particles are driven to the side, and an increase of pressure is caused in the neighbourhood. At C a swift stream overtakes a slower, and a drop in pressure is caused, just as in the air-pump on the next page, where a diminution of pressure occurs in the stream of air AB.

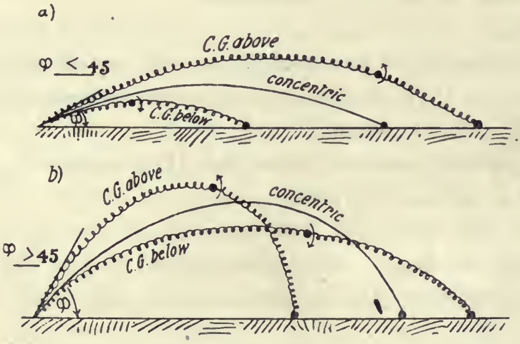
The consequence is an excess of pressure above the ball, and a deflection of the centre of gravity of the sphere downwards, as if the weight of the ball were increased.



The result will then be as follows: Deflection downward (or upward) if the sphere rotates about a horizontal axis and turns forward above (or backward); and deflection to the right (or left) if the sphere rotates about a vertical axis from the left forward (or backward).

Lately the matter has been described in a different way by F. W. Lanchester, England. He takes into account the surface of instability which stretches to the rear of the flying body, but it is not possible to enter into further details.

A quantitative theory of the deviation of a sphere under the influence of the adhering air (Magnus effect) was attempted later by Hélie and also by Tait. Tait started on the analogous deviation of a golf or tennis ball, and assumed a deviating force proportional to the product of the velocity of translation, v, and of the velocity of rotation, the latter to be treated as constant. The deviating force is then proportional to v alone; and for a rotation about a horizontal axis, from above forward, g must be replaced by $g - \mu v$, as pointed out already by Didion.



Calculation shows that with this sort of rotation a path is obtained, which at the ends is concave upwards. And it is not impossible that the curve may have a vertex in the shape of a sharp point directed upward.

In consequence, in a discussion of the case of a spherical shell when rotation is taken into account, the figures will require to be altered to some extent. For further details and the literature of the subject, consult the Report made by G. W. Walker in the *Encyclopaedia of Mathematical Sciences*, Vol. IV, 9, 2*c*, "Games and Sport."

Some experiments were made too by Tait on rotating wooden balls, suspended by wires.

Colonel Ludwig constructed in 1853 a machine, by means of which the influence of the rotation of a sphere on the shape of the trajectory could be demonstrated as in the figure.

§ 51. Deviation of rotating elongated shell. Results of experiment.

The effort to obtain results by an increase in the mass of the shell without a simultaneous increase in the calibre, led up to the systematic employment of elongated shells; and this led again to the introduction of helical grooves in the bore, whereby the shell received a rotation, more or less rapid, about the long axis, as a means of ensuring stability of flight through the air.

Even before the introduction of firearms, with spears and arrows, a rotation about the long axis was provided for the same reason.

In the case of rotating elongated shells, deviations arise of definite direction, due to the rotation of the shell, and these are both in the plane of fire and perpendicular to it.

The latter, as the most important, will receive closer attention in the sequel.

With right-handed rifling, the deviation is generally to the right, and to the left in left-handed rifling, as for instance in the Italian field gun.

The deviation increases more rapidly than the range; the trajectory is consequently a curve of double curvature; and with right-handed rifling, at least in the vast majority of cases, the curve runs to the right of the plane of fire and the horizontal projection, seen from the plane of fire, is in general convex.

These lateral deviations, as they are of considerable magnitude, are well known in artillery work and are corrected by adjusting the sights.

As for the absolute magnitude of the lateral deviation for direct fire, with an angle of departure between 0° and 45°, some numerical values are here given, which were obtained with the French guns.

According to Hélie the French 16 cm gun was used, calibre 162·3 mm, and final angle of twist 6° 30′; weight 30·4 kg, length 371 mm, and semi-angle of the point of the ogive $\gamma = 41° 51'$. The initial velocity v_0 was 334 m/sec, error in the angle of departure (jump) 12′.

In the following table the range is given, the angle of departure ϕ, the lateral deviation due to the rotation of the shell, and the number of rounds.

Range X metres	Angle of departure ϕ	Lateral deviation z metres	Number of rounds n
1806	5° 24′ 18″	7·2	70
3108	10° 17′ 43″	29·0	90
5688	25° 12′ 0″	182·0	80
6579	35° 12′ 0″	324·5	60

The lateral deviation increases more rapidly than the range, and thus the trajectory is a curve of double curvature; this is seen further from the following table for the lateral deviation given by the French field gun M. 1897; calibre 75 mm, constant angle of rifling 7°, maximum gas pressure 2400 kg/cm², weight of the shrapnel 7·24 kg, length of shell 290 mm, initial velocity $v_0 = 529$ m/sec, jump + 7′; right-handed rifling.

Range X	Angle of departure ϕ	Drift, right, z	Range X	Angle of departure ϕ	Drift, right, z
1000	1° 6′	0·4	6000	14° 3′	54·0
2000	2° 43′	2·2	7000	18° 50′	95·0
3000	4° 46′	6·5	8000	25° 53′	172·9
4000	7° 16′	14·9	8500	32° 41′	264·3
5000	10° 19′	29·3			

Experiment has shown that, under given conditions, the lateral drift on the muzzle-horizon increases with the calibre, the final angle of twist, the initial velocity of the shell, the angle of departure, and the bluntness of the point of the shell. As regards the angle of departure this holds only within certain limits, which will be investigated in the sequel. It decreases, however, when the weight of the shell is increased.

So also with infantry bullets, fired from rifled weapons, there must be a lateral deviation, in consequence of the rotation of the bullet.

But with the small angles of departure, ordinarily employed, this deviation is scarcely to be recognised, and may easily be concealed by the natural scattering of the bullets.

The small mass of the rifle bullet is more sensitive than the larger mass of the artillery shell.

The measured range of rifles does not extend beyond 2000 m. Theoretically, the infantry rifle M. 71 should give at 1000 m a lateral deviation of 1·7 to 3·4 m. On the other hand the breadth of the 50 per cent. probability circle (according to Hebler) would be about 3·1 m: so it is clear that for the rifle M. 71 the drift could not easily be settled with certainty.

Observations with rifle bullets are few in number (Thiel, Krause, Quinaux). G. Thiel found by a method, not altogether free from objection, a 36 cm drift at 300 m with the rifle M. 71. According to Krause, with the rifle M. 88, the drift to the right at 1000 m range was 1 m.

In high angle trajectories, that is with an angle of departure between 45° and 90°, other effects are noticed.

If the charge is kept unaltered, but if the angle of departure is increased more and more, then at first the drift increases.

But after a certain angle of departure is exceeded, the lateral drift alters its direction in an irregular manner, so that with right-handed rifling a drift to the left may be observed, and *vice versa*.

The critical angle, where this effect begins, lies somewhere between 55° and 85°, according to the nature of the weapon and its projectile.

When the angle of departure is increased still further beyond this limit, the deviation is to the left with right-handed rifling, and to the right with left-handed twist, until finally the drift disappears with vertical fire.

Some measurements, quoted by Hélie, are as follows:

(*a*) French 16 cm gun; calibre 162·3 mm, final angle of twist 6° (left-handed); weight of shell 31·49 kg, initial velocity of the shell 323 m/sec; length 371 mm; semi-angle of opening $\gamma = 41° \ 51'$; experiments in 1869.

(*b*) French 10 cm gun; calibre 100 mm; final angle of twist 6°; ogival head of shell with a semi-angle of opening $\gamma = 44° \ 26'$. Experiments of 1881, with two different weights, lengths, and initial velocities.

Angle of departure ϕ	Lateral deviation with left-handed rifling z
45°	386 m, left
60°	562 ,, ,,
70°	715 ,, ,,
75°	328 m, right
80°	290 ,, ,,

Weight 12 kg, Length 343 mm, $v_0 = 535{\cdot}5$ m/sec		Weight 14 kg, Length 392 mm, $v_0 = 504{\cdot}5$ m/sec	
Angle of departure ϕ	Lateral deviation, left-hand twist z	Angle of departure ϕ	Lateral deviation, left-hand twist z
42° 4′	525·9 left	42° 4′	430·2 left
57° 4′	533·9 ,,	57° 4′	421·5 ,,
69° 4′	829·0 ,,	69° 4′	640·8 ,,
80° 4′	365·0 right	80° 4′	420·3 right

Consequently, with these two guns, the deviation changes sign between 70° and 80°.

At the same time it was observed, that with left-hand twist and drift to the right, or right-hand twist and drift to the left, the shell struck the ground base first.

With bullets the same change must be present; this takes place at 82° with the bullet of the French infantry. At this elevation the maximum drift to the right must be unstable and change into the maximum drift to the left.

In consequence of this irregularity, the change will show itself in equal drifts to the right and left at a certain definite elevation, and at a greater elevation the left-hand drift will begin to preponderate.

Shooting vertically upward, the regular lateral deviation must vanish again. In fact, it is only the influence of the wind and the error in the angle of departure that can produce a deviation from the vertical direction of motion.

Besides, the shell in such a case remains for the most part parallel to itself, and so arrives again with the base downward.

§ 52. What is the nature of the flight of an elongated rotating shell?

Some of the facts concerning the lateral deviations due to the rotation of the shell and the position of the point of descent on the muzzle horizon have already been mentioned.

Before investigating these lateral deviations, the question arises as to what is the reason why an ordinary artillery projectile, fired from a rifled gun at an angle of elevation between 0° and 50°, always, or nearly always, strikes the ground with the point foremost?

The shell is like a spinning top. It might thus be expected that the axis of the shell will remain parallel to itself; and that the shell will strike the ground in a slanting position, so that the base will be the first part to touch the ground.

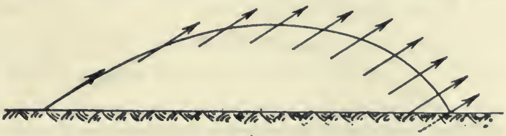
Three cases arise, according to the degree of stability of the shell.

(*a*) *Stability of the shell too great.*

In this case the axis of the shell appears to remain parallel to itself, as represented in fig. *a*.

If the initial velocity is small enough, the position of the axis of the shell in the air can be followed clearly by the eye, and the shell strikes the ground with a part of the base.

But this parallelism of the axis is only apparent; actually the point of the shell, as seen from the gun, must point somewhat to the right with a right-hand twist, and the axis of the shell must make an angle with the horizon slightly less than the departure angle.



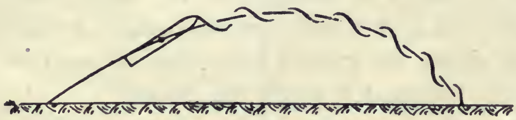
a)

In this case the drift is great; the range is in general somewhat less than in a normal case. On the other hand, it seems that in consequence of the frictional drag of the air against the shell in this sort of flight, the range in extreme cases can prove to be greater than in a vacuum; this has been attested by Minarelli and N. Sabudski. Such would be the case when the rotation of the shell or its moment of inertia about the axis of figure is too great, or if the length of the shell has been chosen too small in relation to the angle of the rifling.

(*b*) *Stability of the shell too small.*

This state is present when, for instance, the angle of twist is too small, or the length of the shell has been made too large, or unsymmetrical lateral shock arises from the escaping gases as the shell leaves the muzzle of the gun.

The axis of the shell then makes violent nutational gyrations about the centre of gravity, as shown in the figure.



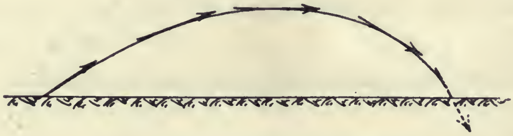
b)

There are visible conical vibrations about the instantaneous tangent of the trajectory as axis of the cone: a closer description of the results will follow later on.

In extreme cases the shell appears as a large disc, when observed from the gun; the range is much too small: the errors are great. A strong whizzing is to be heard and there are powerful periodic impulses in the flight through the air.

(*c*) *Stability of the shell correct.*

The shell flies so that the axis coincides with the instantaneous tangent to the path of the centre of gravity. When the initial



c)

velocity of the shell is small enough, and the trajectory is observed from the side, the point of the shell is seen directed upward as far as the vertex of the trajectory; at the vertex the axis of the shell is horizontal: and then the point of the shell sinks down, till finally it arrives on the ground with the slope of its axis equal to the angle of descent.

It is often considered that this sort of motion is similar to that of an arrow.

It is characteristic of the motion of an arrow that if its axis is deflected from the direction of the tangent to its path from any cause, a reaction arises at once from the air resistance, urging the axis of the arrow back again into the direction of the tangent, and the moment of the forces increases with the angle made by the axis of the arrow with the tangent. The arrow would fly automatically in the direction of the tangent if the centre of gravity were near the front, at a distance from the rear end of about $\frac{2}{3}$ to $\frac{3}{4}$ of the whole

d)

arrow length, and if the rear end was of appropriate shape. In this case the resultant of the air resistance meets the axis behind the centre of gravity, and so the moment of the reacting couple increases with the angle between the axis of the arrow and the tangent, as shown in the figure.

But such an arrow-like action cannot exist here, as in an elongated shell of usual form and distribution of mass, the intersection of the resultant air resistance with the axis always lies ahead of the centre of gravity, even when the axis of the shell makes an angle with the tangent to the path, reaching from 0° up to 85° or 90°. This is shown from the calculations and experiments of E. Kummer (see § 12), at low velocities, and can easily be extended to high velocities by a test of the following nature: Take an infantry bullet and place it so as to be free to rotate about an axis perpendicular to the axis of figure, and set the axis of the bullet at angles of 0°, 5°, 10°, 15°, 20°, ... with an outflow of compressed air from a receiver, with the point of the bullet towards the flow of air; then open the air valve. The direction of the first impulse of the air received on the axis of the bullet shows clearly that the point of action of the air resistance lies between the head of the bullet and the centre of gravity.

Moreover, the arrow-like action of a rotating elongated projectile is merely apparent, and the nature of the motion of the elongated shell is in reality not identical with the flight of an arrow.

As a matter of fact the axis of the shell does not remain exactly in the tangent to the path; but the point of the shell is found alternately above and below the tangent, and, neglecting nutation, it is permanently to the right of the vertical plane through the tangent. These details of the movement of the shell are to be taken into consideration later on, in § 54, etc.

§ 53. Various explanations of the lateral deviation or drift of a rotating elongated shell.

A purely qualitative explanation of the regular lateral deviation of an elongated projectile, fired from a gun rifled with a right-handed twist, may now be given: a quantitative treatment will follow in § 56.

In a vacuum the axis of the shell would retain its original direction in space, provided it were a principal axis and also the axis of original rotation, if nutations due to shock were ignored. Gravity acts solely in producing a curvature of the path of the centre of gravity in the plane of fire, and cannot produce a double curvature; this would have to be due to some other force, acting at right angles to the plane of fire.

As to the "centrifugal force of the shell," acting as a deviating force, many erroneous opinions have been expressed.

A centrifugal force can only be taken into account when a body rotating about an axis reacts on another body, and conversely.

This is the case, for instance, when the mutual action between two particles of the same rotating shell is considered, as in the treatment of the rigidity of the body; or else when the movement of shrapnel-shell, or explosive shell is considered, bursting after the explosion. In an instruction book published in 1894 it is stated: "Concerning the cause of the drift to the right, different views are given. It can be explained, for example, as follows: Since the bullet rotates from left to right, seen from behind, the rotation in the left-hand half acts against the attraction of the Earth, whereas it works with gravity in the right-hand half.

"The swing to the right is thus stronger, and the bullet deviates out of the vertical plane to the right, and this increases the further the bullet flies."

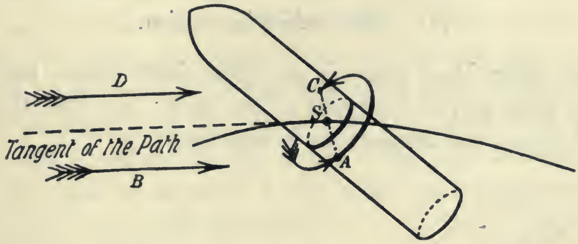
If this explanation had any truth in it, a millstone, or the fly-wheel of a machine, or a circular saw, would jump out of their bearings, if the covers were removed.

There are, as a matter of fact, three main causes at work, and these will be considered separately.

(*a*) *The action of the air adhering to the shell.*

Suppose an elongated shell to be fired at an angle of 50°, and that the centre of gravity has reached the neighbourhood of the vertex *S*. There seems no cause at work to alter the direction in space of the axis of the rotating elongated shell: and so it makes an angle of 50° with the tangent to the trajectory, as in the figure.

Instead of this it can be assumed that the centre of gravity of the shell is at rest in space, and that the air streams past it with a corresponding velocity, parallel to the tangent to the path.



a)

This stream of air is shown in the figure by the curves *B* and *D*, with the stream *B* on the near side, and the stream *D* on the far side of the rotating shell.

But then, in addition, the air adhering to the shell will be whirled round by it. This adhering air moves round the near side of the shell in the direction of the arrow *A* and so has a component of velocity in the same direction as the current *B* of the first-named air stream. On the far side of the shell on the contrary, the adhering air, moving in the direction of the arrow *C*, has a component which opposes the air streaming against the shell.

The resultant effects are then similar to those described already in § 50 for cannon balls; this effect may be described under the name of the "Magnus effect."

The effect consists in a rarefaction on the left of the shell, and a condensation on the right. In consequence with the right-hand rifling there is an excess of pressure from right to left, and so a deviation ensues to the left.

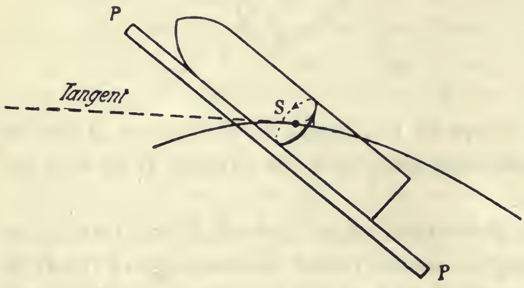
As a matter of fact the deviation which arises is to the right and not to the left; so that this effect alone cannot explain the actual result. Other influences must be at work to alter it.

It may be well to mention a theory, brought forward by A. Dähne, in 1884. He assumed that this adhering air, revolving with the shell, was the main cause, but that the resultant of the air resistance cut the axis of the shell behind the centre of gravity. Consequently an excess of pressure existed from right to left between the centre of gravity and base of the shell. The base is thereby driven to the left, and the point of the shell to the right. The air resistance then acts against the shell as against a sail set aslant, and presses the shell on the whole to the right.

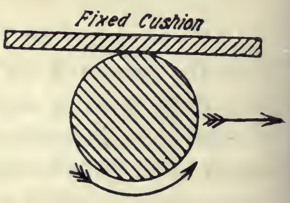
This theory sounds plausible; but it is unconvincing, seeing that the point of application of the resultant air resistance lies actually in front of the centre of gravity, as shown in § 52.

(*b*) *The cushioning action.*

Assume again that there is an appreciable angle between the axis of the shell and the tangent to the path; and that the point of the shell is above the tangent, as in the figure.



b)



c)

Assume further that the shell is rotating in a fixed position, and that the air is streaming past it, as before.

A condensation is produced in the air on the front of the shell, that is on the side towards the target; and on the rear side, towards the gun, there is a rarefaction of the air. The friction of the air in consequence is greater on the front than on the rear of the shell; and the effect is then as if a cushion of air *PP* were pressed against the forward side of the rotating shell, as in fig. *b*.

A reaction is thereby set up, as can be observed often with a billiard ball, as in fig. *c*; the shell rolls on the fixed cushion, and actually in our case to the right.

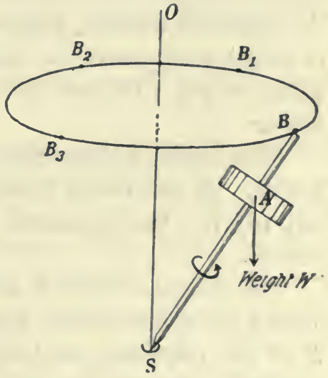
With a right-hand rifling, and the point of the shell above the tangent to the path, a deviation to the right arises in consequence.

(*c*) *The gyroscopic action.*

Concerning the possibility of explaining the drift of an elongated shell in rotation by gyroscopic action, a calculation was undertaken in 1839 by S. D. Poisson.

But G. Magnus was the first in 1852 to show the main phenomena, by experiments on models in a laboratory.

When a top, as in the figure, is rotating about a fixed support *S*, and rotates rapidly in the direction of the arrow about its axis *SB*, it does not fall down, when it is set free, but on the contrary it exhibits the well-known gyroscopic effects. If the centre of gravity lies above the point of support, the axis of the top preserves its position in space, even when the cup, *S*, in which the top is placed, is carried slowly round the room. On the other hand in the case shown in the figure, when the centre of gravity is found above and outside the point of support *S*, the upper end of the top (leaving nutation out of account) describes a horizontal circle $BB_1B_2B_3\ldots$ round the vertical *SO* through *S*.



The axis of the top describes a right circular cone about *SO* as the axis of the cone. Instead of the axis of the cone falling down under gravity in the plane *OSB* about *S*, and tilting over to the ground, the axis moves at right angles to this plane.

This motion is called precession, or conical revolution.

This is similar to the motion of the Earth under the attraction of the Sun.

The axis of the Earth describes a complete cone in 26,000 years, with a semi-vertical angle of $23\frac{1}{2}$ degrees; and the pole, or the point where the axis of the Earth produced meets the sky, is in perpetual motion, and describes a circle round the pole of the ecliptic as its centre. This is the cause of the Precession of the Equinoxes.

When these ideas are applied to the rotating shell, considered as a spinning top, and it is assumed, as before, that the centre of gravity of the shell is at rest, and that the air streams past it with reversed velocity, and so reacts on the shell, the following facts represent the conditions.

The point of support of the top is to be considered as being the centre of gravity of the shell, supposed fixed; SO is the direction of the motion of the centre of gravity or of the tangent to the path; thus the air streams past the shell parallel to OS, with its centre of gravity supposed fixed; SB is here the axis of the shell, making at the moment an angle OSB with the tangent SO of the path. The shell is supposed to rotate about its axis in the direction of the arrow, that is with a right-hand twist as seen from S.

The weight W, concentrated at the centre of gravity A, is to be replaced by the resultant air resistance W, arising from the air streaming past; the point of application is at A, which is between S the centre of gravity of the shell and B the point.

The point B of the shell must consequently move at right angles to the plane OSB, and to the right, as seen from S.

If however the point of the shell moves to the right, the air, streaming past an elongated shell of the usual shape, presses more against the left side of the shell than against the right. The air acts, as explained already, against the shell, like the wind against a sail or board, and presses the shell on the whole towards the right of the plane of fire. Drift is thereby given to the right with right-hand twist.

This is the explanation of the drift of a rotating elongated projectile. It can easily be shown that of the three causes in action, (*a*), (*b*), (*c*), the gyroscopic effort exceeds the other two in magnitude.

An elongated shell is fired at an angle of about 40° from a gun with a rapid final twist; let us consider it when it is near the vertex S of the trajectory, and situated so that the point of the shell is somewhat above the tangent to the path.

The action (*a*) of the adhering air (Magnus effect) would account only for a drift to the left. But as the drift is actually to the right, the gyroscopic effect (*c*) is greater than this Magnus effect (*a*).

The action of (*c*) is greater too than the cushioning effect (*b*) (Poisson effect), the only other one which accounts for a drift to the right. The two causes (*a*) and (*b*) (Magnus effect and Poisson effect) are certainly present with spherical rotating shells. But in reality a rotating sphere, as shown in § 50, is subject to the effect of the adhering air, and not to the cushion effect, as this last effect alone would give the wrong direction to the drift.

The following conclusions are therefore drawn:

gyroscopic effect (*c*) > effect (*a*) of the adhering air,

effect (*a*) of the adhering air > cushion effect (*b*),

gyroscopic effect (*c*) > > cushion effect (*b*).

At the utmost the gyroscopic effect can only be decreased or increased slightly by the other two effects; but the chief influence is the gyroscopic action.

Other questions remain to be considered.

In the first place the following point may be raised: if the comparison is to be complete, between the action of gravity in producing precession in the top and the shell, the point should describe a complete circle round the direction of the air resistance, and so approximately round the tangent to the path.

Relatively to the vertical plane the point of the shell must first move to the right and downward, then to the left, and again upward. It might be expected that a drift would ensue alternately to the right and left. What is the reason why the drift to the right does not change later into a drift to the left?

These difficulties were considered by G. Magnus, 1850, A. Paalzow, 1867, and E. Kummer, 1875.

As in the cases of the angle of departure between 0° and 50°, right-hand drift alone arises with right-hand twist, they assumed arbitrarily that the conical motion of the axis of the shell was so slow that the axis had only time to move to the right and somewhat downward, before the shell struck the ground.

But this assumption does not apply universally. When the appropriate formulae of gyroscopic motion are applied to this case, so that the moment of air resistance replaces the moment of gravity, and the impulse of projection of the shell replaces the gyroscopic impulse, the result is that the time of a complete revolution of precession is greater than the total time of flight only in the case of an excessive twist; in fact in a well-designed gun numerous complete turns of precession are made during the total time of flight.

Thus for example:

	Time of a complete precessional revolution	Number of revolutions per second
In the old heavy field gun:		
(*a*) at the beginning of the trajectory	0·7 sec	1·4
(*b*) at the end of the trajectory	0·3 „	3·3
In a mortar	3·7 „	$\frac{1}{4}$
In the infantry rifle	0·11 „	9·1

The correct solution of the difficulty is as follows.

The comparison between the motion of a rotating elongated shell and the motion of a symmetrical top moving about a fixed support under the influence of gravity alone, is not such that the impulsive momentum of the top is replaced merely by the impulse received by the shell, and the moment of gravity by that of the air resistance.

The idea of stabilising an elongated shell in flight through the air by means of a rotation is derived from the oldest practical application of the gyroscopic principle; among all the various technical cases of gyroscopic motion as applied to the bicycle, monorail, gyrocompass, etc., the movement of a shell is certainly the most complicated. For while in the ordinary spinning top, the force of gravity producing the motion of precession is constant in magnitude and direction, in the case of the elongated shell the resistance of the air is variable in magnitude and in direction, and in point of application.

The consequence is that the axis of the shell does not describe a circular cone, and the point of the shell does not move round in a complete circle about the initial tangent or any other position of the tangent. Neglecting nutation, the point of the shell describes in space a cycloidal curve, and the axis of the shell a cycloidal cone, lying on the right-hand side of the vertical plane through the tangent.

Otherwise, considered relatively to the tangent, the point of the shell is found always to the right of the tangent, and alternately above and below.

The fact that the point of the shell remains on the right-hand side of the vertical plane through the tangent to the path, is the reason why the right-hand drift does not change later into a drift to the left.

And the fact that the point of the shell, after completing a cycloidal arc, always comes into coincidence with the tangent to the path, is the reason why the shell strikes the ground with its point.

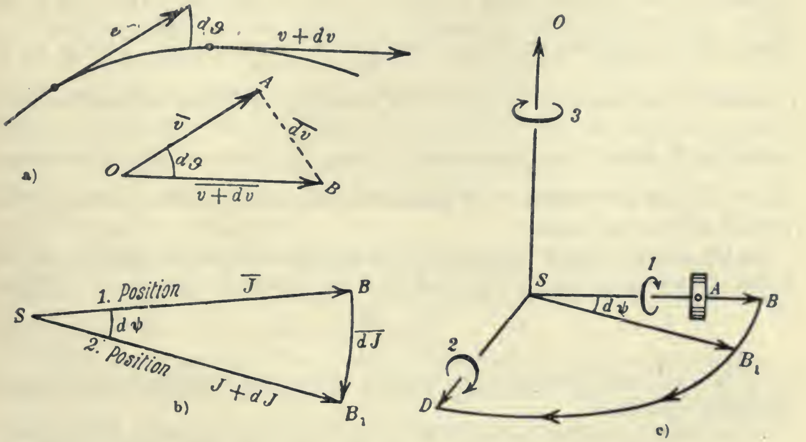
The movement of the shell about its centre of gravity will be examined more closely in §§ 54, 55.

Finally the simple facts of gyroscopic theory determine in every case the direction which the axis of the shell is made to assume.

The idea, introduced by F. Klein and A. Sommerfeld in their *Kreiseltheorie*, of the Impulse or Impulse Moment is of value.

Impulse is to be understood to mean the angular momentum of all the separate particles of a body rotating about a fixed point.

When a top is considered with strong spin, or a rapidly rotating flywheel, or a shell in rapid rotation, the remaining rotations may usually be neglected in comparison with the rotation about the axis. In such cases it is permissible



with sufficient accuracy to replace the impulse of the top by the so-called impulse J, that is, the product of the moment of inertia of the top about its axis of figure and the angular velocity about this axis. This is the impulse set up in the top by hand with the string, or by mechanical action, and in the shell by the gas pressure in combination with the motion imparted in the bore by the grooves.

Consider then a top SB, rotating about S as in fig. c, where the axis is horizontal: C is its moment of inertia about SB, and r is the angular velocity of the top about SB, so that $Cr=J$ represents the impulse which can be represented in magnitude and direction by SB; the vector is drawn in the positive direction if the top appears to rotate clockwise. A man may be supposed to stand in SB with his feet at S; and then he sees the rotation to the right in the direction of the arrow.

If a definite system of coordinates is not necessary, a vector may be employed to represent any quantity, such as a velocity, acceleration, a force, angular velocity, a couple, and so forth, and its direction is shown by an arrow.

The corresponding quantity can be denoted by a stroke drawn over it. Thus for example in fig. a, a point is moving with a variable velocity. Let the velocity at the time t be represented in magnitude and direction by OA or $\bar{v}$, at time $t+dt$ by OB or $\overline{v+dv}$. Then AB or $\overline{dv}$ is the vectorial change of velocity (geometrical, not algebraical). Divided by dt this gives the acceleration in magnitude and direction; and $m\frac{\overline{dv}}{dt}$ is the force.

Further suppose a top has a definite position of its axis SB; and let $SB=\bar{J}$ denote the impulse in magnitude and direction. After the time dt suppose the axis to be in the position SB_1. To give this alteration of position a definite impulse change $\overline{dJ}$ (fig. b) is required. The chief law in gyroscopic theory is enunciated by Klein and Sommerfeld in the following convenient form, in analogy with Newton's Law of Acceleration: "The top moves so that the change of velocity of the impulse vector in magnitude and direction is equal to the moment of the external forces about the point of support." Here $\overline{dJ}$ is the change of the impulse vector, and its rate of change is $\frac{\overline{dJ}}{dt}$. This is to be equated to the moment, $\frac{\overline{dJ}}{dt}=M$. The extremity B of the impulse vector is called by F. Kötter the "twist-point." Then $\frac{\overline{dJ}}{dt}$ is the velocity of the twist-point, and the law states: "The moment of the external forces is equal to the velocity of the twist-point."

If the impulse vector has turned through the angle $d\psi$ in the time dt, the travel of the twist-point is $\bar{J}d\psi$; and since, in absolute magnitude, $J=Cr$, then

$$\frac{d\psi}{dt}=\frac{M}{Cr}.$$

This is the well-known formula which is of the most frequent use in practical gyroscopic problems.

Now if the axis SB of the top in fig. c is moving in the direction of the arrow 1 about SB, and is brought into the position SB_1, the rotation is about the axis SO perpendicular to SBB_1, and a rotation in the direction of arrow 3 is required; the moment Mdt is employed in this infinitesimal rotation $d\psi$; the axis of this moment is SD, parallel to BB_1.

This moment arises when the top is set free, and the centre of gravity of the top is not above the point of support S, but in A, where $SA=a$. Let W be the weight of the top; then the moment of the weight is $M=Wa$. Gravity draws the top about the horizontal line SD, perpendicular to SB, and parallel to BB_1; and to an observer looking from D to S, in the direction of the hands of a clock, as in arrow 2.

The moment of gravity must then be represented by a vector drawn from S to D; and it generates a precession of the top from B to B_1 and on to D; and the axis of the top, ignoring nutation, describes a horizontal plane.

The precessional angular velocity is then

$$\frac{d\psi}{dt} = \frac{Wa}{Cr},$$

where C is the moment of inertia of the top about its axis of figure; r the angular velocity of rotation about its axis; W the weight, a the distance between the point S and the centre of gravity A.

The following then is the rule for finding the direction of motion of the axis of the top, when the axis is set free: the initial impulse J of the top is to be drawn in the correct direction from the point of support S along the axis SB, as a vector; and is positive to the side where the direction of rotation is with the hands of the clock. The same is done for the moment M of the external forces, producing the precession.

Then imagine a small additional vector $dJ = Mdt$, drawn through the end, B, of the impulse vector $\bar{J}$, and parallel to the vector of the turning moment M. The line joining the centre of rotation S with the end B_1 of the additional vector gives the position of the axis of the top at the time $t + dt$.

This rule is to be applied in the following treatment of the problem.

Let the plane of the paper be taken as the plane of fire. The shell is flying from right to left, or the centre of gravity S of the shell may be supposed at rest, while the air is streaming from left to right against the shell, which is rotating about its long axis with angular velocity r. The resultant air resistance is denoted by W. The point of intersection of the resultant with the axis is at A, near the point of the shell.

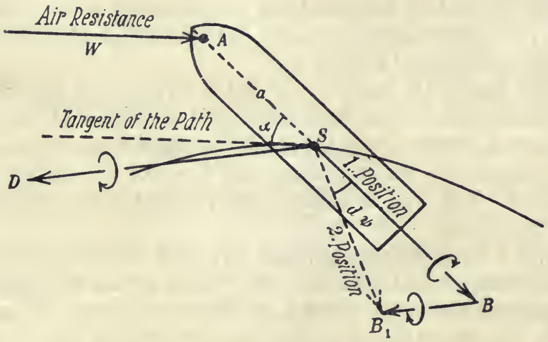
In consequence of the stability of the shell owing to the rotation, the axis makes an angle α with the tangent to the path, but the axis is still situated in the plane of fire.

The actual impulse $J = Cr$ is represented by the vector SB drawn from S, the centre of gravity, along the axis. With right-handed rifling this vector is to be drawn backward, because the shell seen from behind is turning like the hands of a clock.

The moment of the air resistance is tending to turn the shell about a horizontal axis SD drawn through S, the centre of gravity, and in the figure this axis is drawn forwards perpendicular to the plane of the paper; because, as seen from D, the air resistance tends to turn the shell in the clockwise direction.

A vector BB_1 must then be supposed to be drawn through B, the end point of the impulse vector, parallel to and in the same direction as the moment-vector $\bar{M}$ or SD, and of magnitude $\bar{M}dt$;

and then SB_1 is the new position of the impulse vector and consequently of the axis of the shell after the time dt. The base of the shell moves forward in consequence, and the point of the shell is to the further side of the plane of the figure; or seen from the gun the point of the shell goes towards the right of the plane of fire.



The shell starts in consequence a motion of precession with the angular velocity $\frac{M}{Cr}$, or $\frac{Wa}{Cr}$, in the direction of the twist.

This causes, as stated already, a motion of the shell towards the right.

The further course of the motion of the shell about the centre of gravity S will be described in § 54.

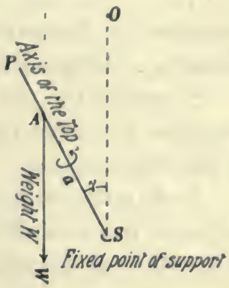
§ 54. The movements of precession and nutation of an elongated shell in rotation. Rectilinear motion of the centre of gravity.

The explanation of the gyroscopic movements of a rotating elongated projectile will begin with a simple case, where the analogy with the theory of the common spinning top under gravity is most evident.

A. *Shell fired vertically upward or downward without lateral impulse.*

The shell is of the shape of an elongated figure of revolution, with an initial angular velocity r about the axis, and fired vertically upward, so that the centre of gravity as a first approximation moves in a vertical line. At the start suppose the long axis to make an angle γ_0 with the vertical.

The shell is supposed to have received no impulse perpendicular to its long axis. Further, let the following assumptions be made: the shell is so shaped that the resultant air resistance W on the shell acts approximately in the direction of motion of the centre of gravity, and is therefore vertical; and its point of intersection with the axis is at a constant distance from the centre of gravity S (Assumption 1).



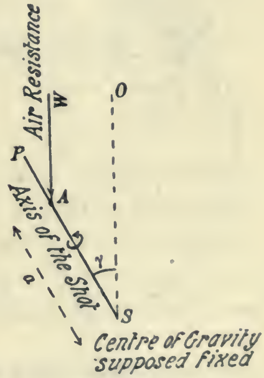
Further, suppose the motion of the centre of gravity of the shell to be divided into separate parts; and in the part considered for the moment, suppose the air resistance, which actually varies with the angle γ between the long axis and the vertical and with the velocity of the centre of gravity, to be taken as constant, and as known in magnitude (Assumption 2).

The angular velocity r of the shell is then also constant.

Finally, the motion of the shell about its centre of gravity, moving upward with velocity v, may be taken to be the same, as if the centre of gravity were fixed in space, and as if the air were flowing past it downward with velocity v (Assumption 3).

After these preliminaries the analogy is complete with the spinning top which has its point placed in a fixed cup S, and its centre of gravity situated at a distance a from S, in the case where the top is under the same initial conditions and set free without shock, and friction is supposed to be absent.

The results arrived at by calculation are given here without proof:



Suppose a sphere is described about S the centre of gravity of the shell with radius SP, where P is the point of the shell; and let the vertical through S cut the sphere in O. The movement of SP, the axis of the shell, is represented in a simple manner by following the motion of the point P over the surface of this sphere.

The point P of the shell describes a double motion.

In the first place the inclination γ of the axis of the shell to the vertical varies periodically between its smallest value γ_0 (arc OP_0) and its greatest value γ_1 (arc OP_1); or the point of the shell fluctuates continually between two circles, $P_0P_2P_4 \ldots$ on one side, and $P_1P_3P_5 \ldots$ on the other, which are described on the surface of the sphere with centre O and radial arcs γ_0 and γ_1. This motion is called nutation.

Secondly, the arc P revolves about O, varying periodically, and assumes in series the positions OP_0, OP_1, OP_2, ... This motion is called precession.

As to nutation, the instantaneous angle of inclination γ or OSP between the axis of the shell and the vertical, or the arc OP on the surface of the sphere, is given at any time by

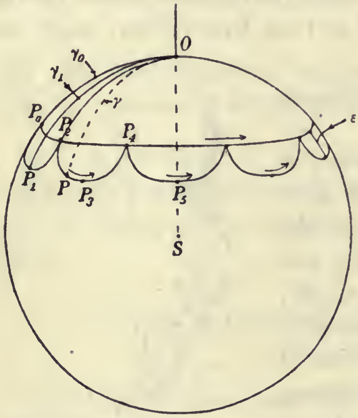
$$\frac{d\gamma}{dt} = \sqrt{\left[(\cos\gamma_0 - \cos\gamma)\left\{\frac{2Wa}{A} - \frac{C^2r^2}{A^2\sin^2\gamma}(\cos\gamma_0 - \cos\gamma)\right\}\right]}. \quad \ldots(1)$$

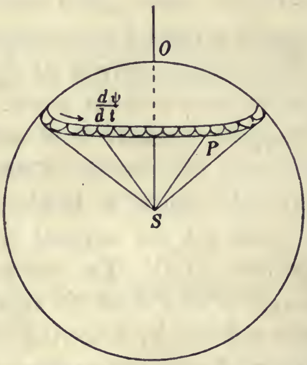
Here C denotes the moment of inertia of the shell about its long axis SP, A the moment about any axis through S perpendicular to the long axis; A is consequently supposed to be greater than C, as is actually the case with an elongated shell.

Then $\gamma_1 = OP_1 = OP_3 = \ldots$ and

$$\cos\gamma_1 = \sigma \pm \sqrt{(\sigma^2 + 1 - 2\sigma\cos\gamma_0)}, \quad \sigma = \frac{C^2r^2}{2WaA}. \quad \ldots\ldots\ldots(2)$$

As σ increases, so the outer limiting circle lies nearer to the inner circle, determined by the initial conditions of the shell; and





the difference between γ_1 and γ_0 is smaller and also the arcs of nutation $P_0P_1P_2$, $P_2P_3P_4$, ...; on this account σ is called the Stability Factor.

The axis of the shell will sometimes sink down to the horizontal plane through S, and will then be square to the vertical path, if $\gamma_1 = \frac{1}{2}\pi$. This occurs when $\sigma = \frac{1}{2}\sec\gamma_0$.

When the nutational oscillations are very small so that the higher powers of $\gamma_1 - \gamma_0$ may be neglected, the amplitude ϵ of the nutation is approximately

$$\epsilon = \frac{1}{2\sigma}\sin\gamma_0 = \frac{2WaA\sin\gamma_0}{C^2r^2}. \qquad (3)$$

The time of a complete arc $P_0P_1P_2$, or $P_2P_3P_4$, ... is then

$$T_1 = \frac{2\pi A}{Cr}. \qquad (4)$$

The plane through SP, the axis of the shell, and the vertical SO then revolves about SO with the angular velocity

$$\frac{d\psi}{dt} = \frac{Cr}{A}\frac{(\cos\gamma_0 - \cos\gamma)}{\sin^2\gamma}.$$

The angular velocity is thus sometimes zero, that is when the point of the shell arrives at $P_0, P_2, P_4, \ldots$ on the inner limiting circle, of radius γ_0; on the other hand, it is a maximum at the points

$$P_1, P_3, P_5, \ldots.$$

The average angular velocity of precession is

$$\omega = \frac{Wa}{Cr}. \qquad (5)$$

This increases as the moment Wa of the air resistance increases, and the impulse Cr about the long axis decreases. The time T in which the point of the shell describes a complete circle about O is then

$$T = \frac{2\pi Cr}{Wa}. \qquad (6)$$

If the shell is spinning with right-hand twist (r positive), that is, so that an observer, looking from S, the centre of gravity of the shell, towards the point, sees it rotating clockwise, and if the air resistance cuts the axis in front of the centre of gravity, the precession takes place in the same direction as the rotation of the shell about its long axis.

In this case, when the point of the shell with right-hand twist is inclined to the left of the plane of the figure at the beginning of the motion, and when the air resistance acts in front of the centre of gravity, the point of the shell moves from the left forwards, and then to the right backwards, and the inclination of the axis to the vertical alters periodically at the same time.

The air resistance acts on the shell as against a sail set aslant, and the shell moves alternately to the right, backwards, left, and downwards.

The assumption that the centre of gravity of the shell is moving in the vertical is then to be considered only as a first approximation. In a second approximation the centre of gravity describes a certain helix about the vertical, through the starting-point.

Example. Let the initial velocity of the centre of gravity of the shell in a vertical direction be $v_0 = 442$ m/sec; the final angle of twist $\Delta = 3° 36'$, the half calibre $R = 0{\cdot}044$ m, and so the angular velocity of rotation of the shell about the long axis (§ 100)

$$r = \frac{v_0 \tan \Delta}{R} = \frac{442 \tan 3° 36'}{0{\cdot}044} = 632 \text{ rad/sec.}$$

Let the moment of inertia about the long axis be $C = 0{\cdot}00065$ kg-m-sec^2, and then $Cr = 0{\cdot}41$; the moment of inertia about a cross axis through the centre of gravity $A = 4{\cdot}2C$, $a = 0{\cdot}081$ m, $Wa = 3{\cdot}7$ m-kg; then the stability factor $\sigma = 4{\cdot}1$, the time of a circuit of precession

$$T = \frac{2\pi Cr}{Wa} = 0{\cdot}7 \text{ sec};$$

that is 1·4 revolution/second.

The time of a nutational oscillation is on the above assumptions

$$T_1 = \frac{2\pi A}{Cr} = \frac{4{\cdot}2}{100} \text{ sec};$$

corresponding to 24 oscillations per second.

B. *Shooting vertically upward or downward, with initial lateral impulse.*

As in case A; but at the beginning of the motion an impulse is given to the axis of the shell.

At $t = 0$ let $\frac{d\gamma}{dt} = w_s$, and $\frac{d\psi}{dt} = w_t$; that is, at the start, we give the axis an angular velocity w_s about a horizontal axis through S, as though the point P of the shell received a blow outwards in the direction of the radial arc OP perpendicular to the boundary circle $P_0P_2P_4 \ldots$; and in addition let the axis of the shell receive an angular velocity w_t about the vertical SO, whereby P is driven in the direction of the tangent to that circle.

In this case the angle of inclination γ between the axis of the shell and the vertical is given as a function of t by the equation

$$A^2 \sin^2\gamma \left(\frac{d\gamma}{dt}\right)^2 = A \sin^2\gamma \,[2\,Wa\,(\cos\gamma_0 - \cos\gamma) + Aw_s^2 + A \sin^2\gamma_0\, w_t^2]$$
$$- [A \sin^2\gamma_0 \,.\, w_t + Cr\,(\cos\gamma_0 - \cos\gamma)]^2. \;\ldots\ldots\ldots\ldots(1)$$

This equation can be integrated in particular cases by means of the Integraph, when $\cos\gamma$ is introduced as the new variable. Then ψ is given at time t by

$$A \sin^2\gamma \frac{d\psi}{dt} = A \sin^2\gamma_0 \,.\, w_t + Cr\,(\cos\gamma_0 - \cos\gamma). \;\ldots\ldots\ldots(2)$$

The greatest and least values of γ are obtained from (1) by putting $\frac{d\gamma}{dt} = 0$.

Writing $\quad \cos\gamma = u, \quad \cos\gamma_0 = u_0, \quad \sin\gamma_0 = u_1,$

$$A\,(1 - u^2)\,[2M\,(u_0 - u) + Aw_s^2 + Au_1^2 w_t^2]$$
$$- [Au_1^2 w_t + Cr\,(u_0 - u)]^2 = 0, \;\ldots\ldots\ldots\ldots(3)$$

where Wa is put $= M$. This is a cubic equation for u, and the solution gives the least and greatest angle $OSP = \gamma$; since two roots of u or $\cos\gamma$ lie between -1 and $+1$, this gives real angles.

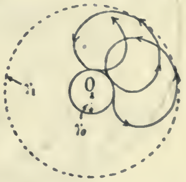
The equation may be written

$$u^3 - u^2\,(u_0 + i_1 u_1^2 + i_2 + i_3) + u\,(-1 + 2i_3 u_0 + i_4 u_1^2)$$
$$= i_3 u_0^2 + i_1 u_1^4 + i_4 u_1^2 u_0 - i_2 - i_1 u_1^2 - u_0, \;\ldots\ldots(4)$$

where $\quad i_1 = \frac{Aw_t^2}{2M}, \quad i_2 = \frac{Aw_s^2}{2M}, \quad i_3 = 2\sigma = \frac{C^2 r^2}{2MA}, \quad i_4 = \frac{Crw_t}{M}.$

In the special case, where a tangential impulse alone takes place, and so $w_s = 0$, then $u_0 - u$ is a factor of (3) or (4), and $\gamma = \gamma_0$ is one solution.

The nutational arc in this case touches the precession circle of radial arc γ_0, as in the figure. Then (4) reduces to



$$u^2 - u\,(i_3 + i_1 u_1^2)$$
$$= 1 - i_3 u_0 - i_4 u_1^2 + i_1 u_0 u_1^2; \;\ldots\ldots(5)$$

and the root $u = \cos\gamma$, lying between -1 and $+1$, gives the other boundary circle of the nutation arcs.

It is assumed that the impulse w_t is so arranged and the quantities A, W, a, C, r are so chosen that $i_1 u_1^2$ can be neglected in comparison with i_3, and $i_1 u_0 u_1^2$ in comparison with 1; and then the small nutations may be replaced by the approximate values given by Klein and Sommerfeld: u^2 is replaced by its initial value u_0^2, and then, since $u_1^2 = 1 - u_0^2$,

$$-ui_3 = u_1^2 - i_4 u_1^2 - i_3 u_0,$$

or

$$\frac{u_0 - u}{u_1} = \sin\gamma_0 \left(\frac{2MA}{C^2r^2} - \frac{2Aw_t}{Cr}\right).$$

Denoting the greatest amplitude of the nutation arc by ϵ, and putting $\gamma = \gamma_0 \pm \epsilon$, and expanding for small values of ϵ, then since $u = u_0 \mp u_1\epsilon$, the amplitude

$$\epsilon = \sim \sin\gamma_0 \left(\frac{2MA}{C^2r^2} - \frac{2Aw_t}{Cr}\right) \text{.....................(6)}$$

A nutation will then be completed in the time

$$T_1 = \sim 2\pi A \div \sqrt{(C^2r^2 - 4AM + A^2w_t^2u_1^2)}, \text{ ....(7)}$$

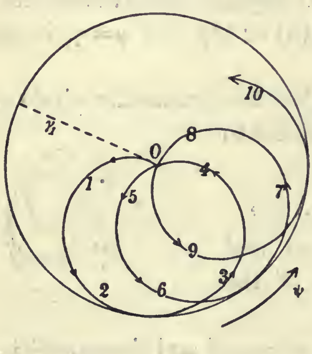
and when γ_0 and w_t are not too large, this can be replaced by

$$T_1 = 2\pi A \div \sqrt{(C^2r^2 - 4AM)}. \text{.....................(8)}$$

C. *Continuation.*

A more important case is where the axis of the shell starts vertically, with the initial direction SO coincident with that of the motion of the centre of gravity; but where in addition a lateral impulse has acted on the axis of the shell.

In this case $\gamma_0 = 0$; the axis of the shell receives the impulse $A\gamma_0'$, and has an angular velocity γ_0' about an axis through the centre of gravity, perpendicular to the axis of the shell, in consequence of the vibration of the muzzle or some effect of the escaping gases. The point of the shell then describes loops.



The figure is drawn so that the motion of the point of the shell is seen from above.

The loops pass always in succession through the prolongation of the initial tangent, or if no error of departure or jump is present, through the prolongation of the axis of the bore, so that the axis of the shell is always brought back to the vertical path of the centre of gravity by retracing a loop.

The maximum deviation between the axis of the shell and the vertical is given by

$$A^2(\gamma_0')^2 = \tan^2\frac{\gamma_1}{2}[C^2r^2 - 2AM(1+\cos\gamma_1)]. \quad \ldots\ldots(1)$$

At any given time t, when the angle of deviation is small and the stability is great, that is when $\frac{1}{\sigma^2}$ can be neglected in comparison with 1,

$$\gamma = \gamma_1 \sin\frac{\sqrt{(C^2r^2 - 4AM)}}{2A}t = \sim \gamma_1 \sin\frac{1}{2}\left(\frac{Cr}{A} - \frac{2M}{Cr}\right)t \quad \ldots(2)$$

and a single loop will be described in the time

$$T_1 = \frac{2\pi}{\frac{Cr}{A} - \frac{2M}{Cr}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and this is independent of the relative magnitude of the initial impulse.

The axis of the shell revolves thus about the vertical through the centre of gravity in the direction of the rotational velocity r, received from the rifling, and with the average angular velocity

$$\frac{d\psi}{dt} \doteq \frac{Cr}{2A}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Remark. If the axis of the shell at the end of the trajectory lies very nearly in the tangent of the path, and the shell strikes a target, it will receive a blow at its front end.

If this blow is given from below and upwards, with right-hand twist, the point of the shell swerves upwards and then to the right; if the blow is from left to right, the point of the shell swerves to the right and down; if the shell is struck at the front end from above downwards, the point swerves downwards and to the left: finally if the blow is from right to left, the point swerves to the left and upwards.

At the impact of the shell on the ground, the blow in most cases will be given from below and upwards, and so the shell swerves to the right.

If the shell penetrates into a bank of earth, the powerful nutation will in some cases cause a complete reversal of the path of the shell.

D. *Rectilinear motion of the centre of gravity, with inaccurate distribution of the mass of the shell.*

This case will arise in the immediate neighbourhood of the muzzle of the gun.

While the shell is moving up the bore of the gun, the pressure of the gases preponderates, and thereby the shell undergoes an acceleration right up to the muzzle, and even for a short distance beyond.

A point must then exist where the shell undergoes neither an acceleration in the direction of the prolongation of the axis of the bore, nor a retardation; and where it can be assumed approximately that no external force acts on the shell, leaving gravity out of account, as that does not affect the motion of the shell about the centre of gravity.

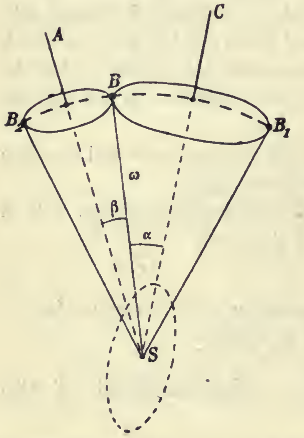
The corresponding motion of the axis of the shell can be considered in different ways. Concerning this, two modes of representation, due to Poinsot, may be mentioned which may be applied to the shell.

The motion of the shell is defined through its momental ellipsoid. Let this be a prolate ellipsoid of revolution, with semi-axes $\frac{1}{\sqrt{C}}, \frac{1}{\sqrt{A}}$, where C denotes the moment of inertia of the shell about its long axis, and A is that about a cross axis through the centre of gravity, and it is assumed that $A > C$.

If the shell is symmetrical, the C axis of the momental ellipsoid coincides with the axis of the shell.

But in the case where the distribution of mass is not symmetrical in relation to the axis of figure, these two axes make an angle with each other.

At the instant when the shell is free after leaving the muzzle, suppose SB to be the direction of the instantaneous axis of rotation, and ω to be the magnitude of the resultant angular velocity.



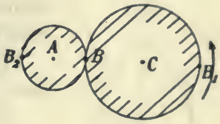
In the case where the mass of the shell is distributed symmetrically about the axis of figure, and the principal axis C lies also in the direction SC, the resultant angular velocity ω is the resultant of r, the angular velocity acquired round the principal axis SC from the rifling, and of s, the angular velocity of the shell, due to some impulse of the gases or the vibration of the muzzle, about an axis through S perpendicular to SC, and to the plane of the paper; and thus $\omega = \sqrt{(r^2 + s^2)}$. The angle BSC or α is thus given by

$$\cos\alpha = \frac{r}{\omega}, \text{ or } \tan\alpha = \frac{s}{r}.$$

But on the other hand if the axis of the figure does not coincide with the principal axis SC, but is denoted by the line SB in the beginning of the motion, and if the shell is free from shock, then the resultant angular velocity ω arises from the fact that the shell in the passage up the bore is forced to move with the axis of figure along the axis of the bore, and in this constrained motion through the grooves an angular velocity ω is given about the axis of figure SB.

The shell is next supposed fixed at the centre of gravity S, with a circular cone BSB_1, fixed in it as in the figure, movable about its apex S, with semi-vertical angle α (the polhode cone), and axis SC.

At the same time another cone BSB_2 is taken, fixed in space, the herpolhode cone, and with semi-vertical angle β, given by



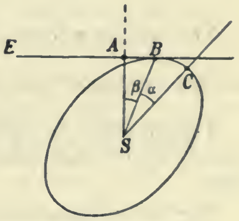
$$\tan\alpha = \frac{C}{A}\tan(\alpha + \beta).$$

The cones touch along the generating line SB.

The subsequent motion of the shell about the centre of gravity S is produced by rolling the cone BSB_1, fixed in the shell, on the cone BSB_2 fixed in space, with constant angular velocity

$$\frac{d\psi}{dt} = \frac{Cr}{A\cos(\alpha + \beta)}.$$

This is the angular velocity with which the plane $ABCS$ revolves about the axis SA of the fixed cone; and in the same direction as r, when $A > C$.



The line of contact SB of the two cones is then the instantaneous axis of rotation.

The angles α and β remain constant, and also the angle ASC; the resultant angular velocity ω is also constant.

Suppose for example A, C, α known; then β can be found from the above: and if r is known as well, either directly, or indirectly from the equation $r = s\cot\alpha$, then $\frac{d\psi}{dt}$ is also known.

The procedure can start also from the momental ellipsoid, shown in the figure; the initial position of SB the instantaneous axis, meeting the ellipsoid in B, is also supposed to be known.

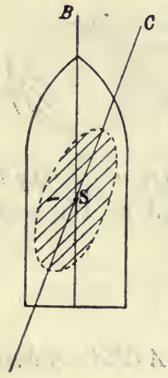
The tangent plane of the ellipsoid is drawn at B, and cuts the vertical through S in A.

Suppose the ellipsoid rolls on the fixed plane E, the line SB to the point of contact gives at any moment the instantaneous axis SB; and so forth.

First example. Suppose in consequence of the irregular distribution of the mass that the axis of figure of the shell does not coincide exactly with the principal axis of the momental ellipsoid, which is an ellipsoid of rotation, with three principal moments C, A, A; $A > C$; and suppose the axis of figure SB to make an angle of 1° with SC, the principal axis of the momental ellipsoid.

Assume a muzzle velocity as in the former example of $v_0 = 442$ m/sec, a final angle of twist 3° 36′, half-calibre 0·044 m, moment of inertia $C = 0{\cdot}00065$ (kg-m-sec²), and $A = 4{\cdot}2C$.

In the passage of the shell up the bore, the axis of figure SB is the axis of rotation, in consequence of the constraint of the grooves; and suppose the shell is leaving the muzzle; and that it has not received a transverse angular velocity through an impulse.



Then SB is the direction of prolongation of the axis of the bore, which is supposed to be at rest.

At the moment of release the principal axis SC of the momental ellipsoid may be supposed to have the forward end directed towards the right; and then we determine the subsequent motion of the shell, on the assumption that no external force is acting.

The angle $BSC = \alpha = 1°$; and the angular velocity about SB is on the previous assumption equal to

$$\omega = \frac{v_0 \tan \Delta}{R} = \frac{442 \times \tan 3° \, 36'}{0{\cdot}044} = 632 \text{ radians/sec};$$

$$\tan(\alpha + \beta) = \frac{A}{C} \tan \alpha = 4{\cdot}2 \tan 1°, \quad \alpha + \beta = ASC = 4° \, 10'.$$

The angular velocity r about SC is $\omega \cos \alpha = \sim 632$; $\dfrac{d\psi}{dt} = \dfrac{Cr}{A \cos(\alpha + \beta)} = 151$;

and the number of revolutions per second $= \dfrac{151}{2\pi} = 24$.

The moving cone BSB_1 with the semi-vertical angle 1° then rolls on the fixed cone BSB_2, of which the axis is directed to the left-hand side of the axis of the bore.

The axis of figure SB is not the instantaneous axis, but the point B of the shell describes an epicycloid, lying on the circle BB_2.

The point B at first describes an arc towards the right, and then moves to the left. The angle between the axis SA, fixed in space, and the axis of figure will fluctuate between 3° 10′ and 5° 10′.

The instantaneous axis describes a cone about SA with the semi-vertical angle 3° 10′, and the principal axis SC describes a circular cone about the same axis, of semi-vertical angle 4° 10′. The rolling cone fixed in the shell, rolls 24 times a second round the cone fixed in space.

Rapid vibrations of this sort often take place in rotating shafts and fly-wheels, when they are not centred exactly; according to the above they are present in shells, when the mass is not properly distributed.

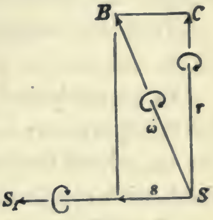
Second example. Suppose the shell has the proper distribution of mass, so that the axis of figure coincides with the principal axis SC.

At the start of the free motion, let SC be the direction of the prolongation of the stationary axis of the bore, as well as the axis of figure and the principal axis; but suppose the shell to receive at this instant an angular velocity about an axis through the centre of gravity perpendicular to the axis of the shell. We desire to determine the magnitude and direction of the blow, so that the motion of the principal axis SC may be the same as in the former example.

The axis of figure or the principal axis describes in this case a circular cone round SA of semi-vertical angle $ASC = \gamma = 4° 10'$ in $\frac{1}{24}$ sec; and if this is the case, the angle between the instantaneous axis SB and SC must be an angle $BSC = 1°$ at the beginning of the motion.

The velocity of the impulse

$$s = r \tan 1° = \sim 632 \tan 1° = 11 \text{ radians/sec};$$



and this must be the angular velocity of the impulse about a line SS_1 perpendicular to the axis of the shell through S the centre of gravity, and under the assumption of right-hand twist the impulse must act towards the further side of the drawing. Such an angular velocity might originate in the vibration of the muzzle upwards.

§ 55. Motion of Precession and Nutation of an elongated rotating shell. Curvilinear motion of the centre of gravity.

It has been assumed in the preceding that the motion of the centre of gravity is rectilinear, as in vertical fire, and approximately for a very short path in curved fire; the usual trajectory may now be considered with a curved path of the centre of gravity.

It is well known that the motion of the shell can be resolved into the motion of translation of the centre of gravity, which proceeds as if all the external forces acted through it, and into a rotation of the shell about the centre of gravity, as if it were a point fixed in space.

The two motions are dependent, and it is impossible to solve this complicated problem with strict accuracy.

This relative dependence is at once evident: the greater the angle between the axis of the shell and the tangent to the path, the greater the air resistance against the shell; so that the greater the resistance offered by the longer side, the more the trajectory of the centre of gravity is altered. On the other hand, the greater the curvature of the trajectory, the more the angle alters between the tangent to the path at any point and the direction of the initial tangent; therefore the greater must be the amplitude of the gyratory motions of the axis of the shell.

This mutual dependence of the two movements necessitates a procedure of approximation.

The method to be employed consists in an approximate solution of the equations of translation, without consideration of the motion of rotation, and then the corresponding results are employed in the equations of the motion of rotation, which are then integrated.

The integrals, so obtained, are again employed in the calculation of the deviations of the shell in consequence of these rotations; and we examine the equations of translation with a view to taking into account the definite terms of correction.

This is a procedure employed in a similar manner in Astronomy, in the calculation of perturbations.

In an analytical solution, two systems of coordinates are employed for this purpose: one fixed in space, with an origin at the centre of the muzzle of the gun, and another movable in space, but fixed in the shell, with origin at the centre of gravity.

The differential equations of the motion of translation of the centre of gravity are then obtained, and also Euler's differential equations of the rotation of the shell about the centre of gravity, under the forces acting in the preceding cases; and then an integration of the differential equations is attempted.

For a purely analytical solution of the problem of the oscillatory motion of the shell, investigations have been made, in particular by St Robert, N. Sabudski, M. de Sparre, and P. Charbonnier.

The author too has given an analytical solution under certain limiting assumptions in the *Zeitschrift für Mathematik und Physik*, 43, 1898, pp. 133 and 169, as also in the first edition of this volume; the assumptions are explained in the first edition. The problem as stated by N. Sabudski and M. de Sparre is considered there more carefully in detail.

The latter employs the principle, first used by F. Klein in gyroscopic work, of the introduction of the complex variable into the two differential equations, and, after integration, of the separation into the real and imaginary parts.

But N. Sabudski made it clear that these analytical solutions do not supply a complete solution of the problem, even when the gyroscopic motion outweighs by far the action of the adhering air, and the cushioning action, as it was shown in § 53 that it is permissible to assume. The following uncertainties arise in the quantities of the differential equations: the components W_p and W_s of the air resistance W, parallel and perpendicular to the axis of the shell; the distance, a, between centre of gravity and point of application of air resistance on the axis of the shell; the angular velocity r of the shell about its long axis. According to § 12, calculations can be made of W_p and W_s for any angle α between the axis of the shell and the tangent to the path, but the critical remarks at the close of § 12 have shown that these calculations are very uncertain.

The angular velocity of rotation of the shell is not really constant, because friction between the air and the shell acts not only along the surface of the cylindrical surface, but also at right angles (compare Vol. III. § 184, pp. 286—288); and accurate knowledge of the decrease of r is very uncertain.

But an analytical treatment of the problem cannot well be undertaken with advantage, until experimental work has satisfactorily determined the values of some of these unknown quantities.

The need of such a systematic experimental treatment has been pointed out rightly by A. Dähne; and in any case such a research can only be carried out with great trouble and expense.

On these grounds the analytical treatment is omitted, and only the graphical method of approximation is employed, which was given by the author in the article in the *Zeitschrift f. Math. u. Phys.*

Graphical approximate solution by the author, 1898.

Leaving the rotation of the shell out of account at first, a calculation is made in the usual manner, of §§ 5 and 8, of the successive angles of inclination θ of the trajectory, from the beginning of the motion up to the impact of the shell on the ground, for a series of successive small intervals of time Δt; as also of the corresponding velocities of the centre of gravity of the shell, and thence of the air resistance $W(v)$.

Next let us follow the centre of gravity S of the shell: or, fix S and let the air stream past the shell.

Describe a sphere about S with radius 1 m, and draw lines through

S parallel to the various directions of the tangent, corresponding to the separate time intervals Δt.

The intersection of these lines with the sphere may be called the *tangent points*, and denoted, as in fig. *a*, by

$$O, T_1, T_2, T_3, T_4, \ldots.$$

The intersections of the axis of the shell with the sphere are called the *shell points*, and denoted by

$$O, O_1, O_2, O_3, \ldots.$$

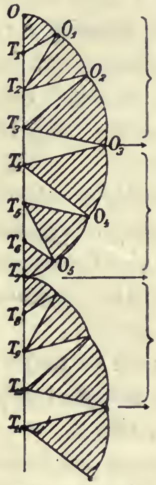


Fig. *a*.

The inside surface of this sphere may serve as the plane of the diagram in the following drawing: the sphere is supposed to be looked at from S; and the construction may be carried out approximately as if it were a flat sheet of paper.

The method depends on the preliminary assumption, that for a small time-interval Δt the precession of the shell about its centre of gravity S is independent of the motion of S in the trajectory, and that the two motions, which actually take place together, may be treated as if one followed the other.

At the beginning of the motion the tangent point is at O; and after the time Δt it has shifted to T_1, after $2\Delta t$ to T_2, and so on. These points are assumed to be known from a preliminary calculation that has been carried out; for the slope of SO is ϕ, of ST_1 is θ after the time Δt, and so on. The points $O, T_1, T_2, \ldots$ thus lie very nearly in the plane of fire, and if a flat sheet of paper is employed for the drawing, they lie approximately in a vertical straight line, though strictly speaking the points $T_1, T_2, T_3, \ldots$ lie a little to the right of the vertical through O; but this is not shown in the drawing.

The shell point is also initially at O, since the shell is projected from the bore direct without constraint; and the initial impulse experienced by the shell from the escaping gases does not produce deviations that need be considered.

After the lapse of the time Δt, and after the tangent has turned from SO into the position ST_1, there is an angle OST_1 between the axis of the shell SO, and the tangent to the path ST_1, because the axis remains parallel to itself in consequence of the stability imparted by the rifling. Precession now takes place; the axis of the shell describes in the time Δt a small part SOO_1 of a circular cone about the tangent ST_1, that is, the point of the shell describes round T_1 a circular arc OO_1 with radius $T_1O = T_1O_1$, because the precessional motion must take place round the direction of the resultant air resistance, and this is approximately parallel to the tangent ST_1 of the trajectory.

Then the angle $OT_1O_1 = \Delta\psi$ is given by $\frac{\Delta\psi}{\Delta t} = \frac{Wa}{Cr}$, in accordance with the fundamental equation of gyroscopic theory. W (kg) is the air resistance on the shell, a (m) the distance between centre of gravity S and point of application of the air resistance on the axis, assumed somewhere near the point of the shell, C the moment of inertia of the shell about its axis of length, $r = \frac{2\pi v_0}{D}$ the angular velocity, D (m) the length of the pitch of rifling; or $r = \frac{v_0 \tan\delta}{R}$, δ the final angle of twist, $2R$ (m) the calibre.

After a further element of time Δt, the tangent point has reached T_2. The angle between the axis of the shell SO_1 and the tangent of the path ST_2 has now become O_1ST_2; and we therefore describe a circular arc O_1O_2 about T_2 with radius T_2O_1, of which the central angle is given as before; and the further construction proceeds as in figure (a).

It is seen then that in a well-designed construction of the gun and shell, the axis of the shell continually moves round the direction of the tangent, or at least moves in its neighbourhood, as for instance at T_7; further, that the point of the shell lies alternately higher and lower than the tangent; and finally, that the point of the shell keeps to the right of the tangent.

This construction thus explains in the simplest manner not only the arrow-like flight of a well-constructed shell, but also the observed fact that with right-hand twist, a drift ensues to the right; that is as long as the angle of departure does not increase beyond a certain amount.

The cycloidal curve $OO_1O_2\ldots$ of the point of the shell is the precession curve: it is analogous to the precession circle described by the head of a top, acted on only by the force of gravity.

As is evident in the course of the construction of the figure, the change of the circle into the cycloid has its origin in the fact that the centre of the precession circle is not at rest on the line $OT_1T_2T_3\ldots$, but is moving: and moreover, the direction of the air resistance, producing the precession, is always altering.

The curve $OO_1O_2\ldots$ can also be considered as generated by the motion of a circle, of variable radius. The centre of the circle moves from O with the velocity $\frac{d\theta}{dt}$ along the straight line $OT_1T_2\ldots$, θ denoting the instantaneous horizontal slope of the tangent.

At the same time the circle is turning with the angular velocity $\frac{d\psi}{dt}$, the describing point on the circumference starting initially from O.

The construction can be carried out under various assumptions, and examined as follows. If the precession $\frac{d\psi}{dt}$ proceeds at a relatively rapid rate, the cycloidal arcs are numerous, and so long as the nutation, which is to be considered later, is small, the heights of the cycloidal arcs, and thence the average movement of the point to the right, are also small.

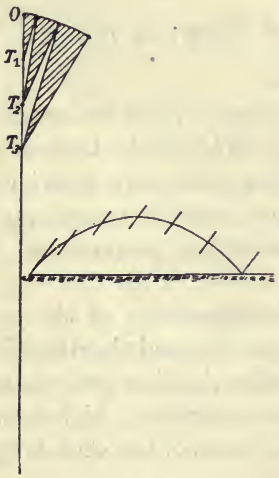


Fig. *b*.

Frequently the axis of the shell approaches very close to the tangent of the path; in such cases the flight of the shell is very similar to that of a well-constructed arrow, and the drift remains small.

On the other hand, when the precession is relatively very slow, as in fig. *b*, the case can arise where the points O, O_1, $O_2\ldots$ come close together in comparison with the points O, T_1, T_2, T_3; consequently the point of the shell no longer keeps close to the tangent.

To an observer, looking at the trajectory from the side, as in figure (*b*), the axis of the shell must appear to remain parallel to itself. The shell comes to the ground with the base forwards; the drift is great, and may in some cases change in sign.

Whether the one case happens or the other, depends on the closeness of the series of points $O, T_1, T_2, T_3, \ldots$ in comparison with the series $O, O_1, O_2, O_3, \ldots$, with equal time intervals Δt, and so on the ratio of $\frac{d\theta}{dt} : \frac{d\psi}{dt}$. But in § 17,

$$\frac{d\theta}{dt} = -\frac{g \cos \theta}{v},$$

and moreover,

$$\frac{d\psi}{dt} = \frac{Wa}{Cr},$$

so that the ratio becomes

$$f = \frac{Crg \cos \theta}{Wav}.$$

Thus in the case of a high angle trajectory when the shell is near the vertex (where $\cos \theta$ is large and v is small), and in addition the length of the shell is moderate and the angle of rifling large (a, small; $\frac{r}{v}$, large), then f has proportionally a large value, and figure (b) represents the case.

Theoretically f can be infinite, if the angle between the axis of the shell and the tangent grows greater, and reaches 90°. Here a is zero, and the precession ceases; the axis of the shell remains parallel to itself, and the shell flies with the base forward on the descending branch, like a shell with a flat head, and with reversed rotation.

In such a case the flight of the shell would be improved, either by a diminution of the angle of rifling, or by an increase of the initial velocity; or by an addition to the length of the shell; or finally, by placing the centre of gravity more towards the base, to make a greater and f smaller.

So far nutation has not been considered; but this is always present, as mentioned in § 54. Nutation is either independent of impulse or due to such shock; in the last case, a lateral initial blow acts on the shell, which does not pass through the centre of gravity.

It appears that in the passage of the shell from the muzzle a lateral impulse nearly always acts, greater or smaller, due to the escaping gases.

The effect of the impulse depends on the gas-pressure at the muzzle, the nature of the powder, the shape of the base of the shell, and finally on the stability factor; also on the angle of rifling, the initial velocity, the two principal moments of inertia, and the position of the centre of gravity.

The nutations are probably damped later by the action of the air immediately behind the shell.

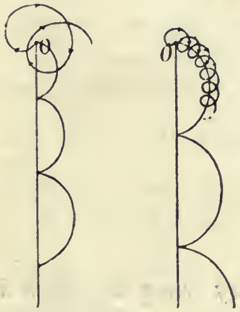


Fig. *c*.

Figure (*c*) shows the progress of these nutations. Just as in the ordinary top under gravity, the precession circle forms the centre of the nutation arcs of the head of the top; so in the same way, the precession cycloid forms the centre of the nutations of the point of the shell.

The direction of rotation of the nutation arcs is the same as that of the curve of rifling; that is, with the right-hand twist, it is from above to the right; then down, and to the left.

It is possible for the precessional arcs to be small while the arcs of nutation are large; and then it is as if the point of the shell were describing a circle about the instantaneous tangent. The nutation is then visible to the eye, provided the velocity of the shell is small; and as the air is set into violent motion by these oscillations of nutation, the sound of these oscillations is frequently heard in periodical vibrations in the air, and the range is thereby diminished.

The expressions in the formulae of § 54 for the nutation, the stability factor, the periodic time of a nutation, and so forth, will hold approximately in the case of a curved trajectory.

§ 56. Calculation of the lateral deviation or drift of an elongated projectile in rotation.

1. *Empirical formula.*

The observations made in France on the final deviation Z of the shell at its point of descent on the muzzle horizon, have been employed by Hélie in the following formula, which is of a purely empirical nature:

$$Z = Av_0^2 \sin^2 \phi, \qquad (1)$$

$$A = 551 \frac{(2R)^3}{P} \tan \Delta \sin \gamma. \qquad (2)$$

Here A denotes a constant for a given gun-system, called the *deviation value*, given by (2); the notation is as before, with Δ the fixed angle of twist, and γ the semi-angle of the cone at the point of the shell.

W. Heydenreich gives, for example, the following numerical values to A; for a turret howitzer firing 21 cm shrapnel, $A = 0{\cdot}0166$, the shell being comparatively short for a considerable twist of the bore; for a heavy field gun, $A = 0{\cdot}0030$, with slight twist for a comparatively great length of shell.

This Hélie formula, although very frequently employed, must not be considered as holding universally, but it may be employed for angles of departure up to 50°. Shooting vertically upward, $\phi = 90°$, and (1) would give a maximum deviation Z, whereas theoretically it should be zero.

E. Bravetta proposes for high initial velocity v_0, to take the factor A as a linear function of the range,

$$Z = (A_1 + A_2 X)\, v_0^2 \sin^2 \phi, \quad \text{.........................(3)}$$

and A_1, A_2 are to be determined from different values of Z, observed at different ranges.

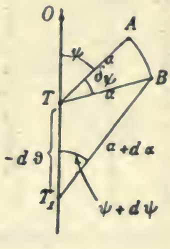
2. *Theoretical calculation of the lateral deviation, or drift, by approximation.*

It has been pointed out already in § 55, that for a measure of the drift, the ratio $\frac{d\theta}{dt} : \frac{d\psi}{dt}$ may be taken; and that

$$f = \frac{g \cos\theta \,.\, Cr}{v \,.\, Wa},$$

with the notation explained before.

This expression for f denotes the average angle between the axis of the shell and the vertical plane through the tangent, along the arc of a cycloid.



This can be proved as follows: the tangent point, as it was called, starts a movement of precession at the point O, as in the figure. The point then retires backward, and reaches T in the time t.

The point of the shell also was situated at O at the start, and will be found after the time t at some point A.

The centre of gravity S is supposed to be above the plane of the diagram, with ST the tangent of the path, and SA the axis of the shell.

The angle between the vertical plane SOT, and the impulse plane SAT (plane through the tangent of the path ST and axis of shell SA) is taken to be ψ after time t; and the angle AST is denoted by a.

If the tangent to the path maintained its deviation in space, and the resistance of the air to the shell were constant in direction and magnitude, the axis of the shell must describe a complete cone about the tangent to the path, or more accurately about a line through S parallel to the direction of the air resistance; and TA must revolve about the fixed point T, with the mean angular velocity

$$\frac{d\psi}{dt}=\frac{Wa}{Cr};$$

and the alteration $\delta\psi$ in time δt is given by

$$\delta\psi=\frac{Wa}{Cr}\,dt.$$

Actually at the same time the end T of the tangent moves downward in the time dt to T_1, and the angle of slope θ diminishes by $d\theta$, so that in the figure $TT_1=-d\theta$.

If it is assumed again that in the small time element dt the two movements are independent of each other, the point of the shell for the same position of T moves first from A to B, and then T moves down to T_1.

The angle TT_1B between the plane of impulse and the vertical plane through the tangent of the path is thereby changed into $\psi+d\psi$ in the time dt: and at the same time the angle between tangent and axis of the shell has become $T_1B=a+da$.

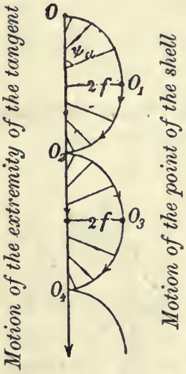
Applying the law of sines to the triangle TBT_1 we have

$$\frac{a+da}{a}=\frac{\sin(\psi+\delta\psi)}{\sin(\psi+d\psi)}, \quad\ldots\ldots\ldots\ldots(a)$$

or

$$d\psi=\delta\psi-\tan\psi\frac{da}{a}=\frac{Wa}{Cr}\,dt-\tan\psi\frac{da}{a}. \quad\ldots\ldots\ldots\ldots(b)$$

Further, if the vertical is drawn from T on T_1B, $da=-\cos\psi\, d\theta$, and since as in § 17,



$$d\theta=-\frac{g\cos\theta}{v}\,dt,$$

we have

$$da=g\cos\psi\frac{\cos\theta}{v}\,dt. \quad\ldots\ldots\ldots\ldots(c)$$

Elimination of dt between (b) and (c) gives, when $\tan a$ is written in place of a,

$$d\psi=\frac{Wa}{Cr}\,\frac{v\,da}{\cos\psi\, g\cos\theta}-\frac{\tan\psi\, da}{\tan a},$$

$$\frac{d\sin\psi}{da}+\sin\psi\cot a=\frac{1}{f}, \quad f=\frac{Crg\cos\theta}{vWa}. \quad\ldots\ldots(d)$$

When the approximation is made of assuming that the ratio f of the two angular velocities may be taken as having a constant mean value along a cycloidal arc, the equation (d) may be integrated as a linear differential equation with a perturbation term; and then

$$\sin\psi=\frac{1}{f}\tan\frac{1}{2}a.$$

When ψ has reached 90°, a has reached its greatest value, $a_{max.}$, given by

$$\tan \tfrac{1}{2} (a_{max.}) = f.$$

For small values of the angle a, the greatest value of a is then $2f$; and the average value of a is f.

With right-hand rifling the shell will drift on the whole to the right, because the axis of the shell is placed aslant at a definite angle to the vertical plane through the tangent, and so the point of the shell is turned to the right.

The deviating force can then in a first approximation be taken as proportional to the sine of this angle, or proportional to the angle itself with small deviation; and so it is equal to Wf.

On the other hand this force is $m \frac{du}{dt}$, where m is the mass of the shell, and $u = \frac{dz}{dt}$ is the instantaneous velocity of the shell at right angles to the plane of fire, and z denotes the lateral deviation at time t; so that

$$m \frac{du}{dt} = \text{const. } Wf,$$

where the constant is to be determined; or, if W is eliminated,

$$m \frac{du}{dt} = \text{const. } \frac{Cr\, g \cos \theta}{av}, \quad \text{.....................(4)}$$

$$du = -\text{const. } \frac{Cr}{ma}\, d\theta. \quad \text{.....................(5)}$$

Strictly speaking, r and a as well as u and θ are functions of the time t, and little is known about them.

In a well constructed system of gun and shell, r and a alter only slowly, and not to a great extent as appears from § 12 and § 184; both diminish as the time increases. Then, if $\theta = \phi$, $u = 0$, and $t = 0$;

and therefore $$u = \frac{dz}{dt} = \text{const.} \frac{Cr}{ma} (\phi - \theta), \quad \text{..................(6)}$$

$$z = \text{const.} \frac{Cr}{ma} \left(\phi t - \int_0^t \theta\, dt \right), \quad \text{..................(7)}$$

and the notation is as before.

A preliminary calculation as in §§ 3 to 8 will give θ as a function of t; and the integral of θ can be worked out by help of the integraph.

The following rough approximation from (6) will give a formula more convenient in practice: In a cylindrical shell $C = m \cdot \frac{1}{2} R^2$; and a mean value of a for small deviation is proportional to the length of the shell L, or $2Rl$, where l denotes the length of the shell in calibres. At first $\theta = \phi$; at the point of descent, $\theta = -\omega$; and so $\phi - \theta$ is 0 at first, and finally $\phi + \omega$, and the arithmetic mean is $\frac{1}{2}(\phi + \omega)$.

Thus the drift Z at the point of descent is given by

$$Z = \lambda \frac{v_0 \tan \Delta}{l} (\phi + \omega) T, \quad \text{.....................(8)}$$

where λ is a factor to be determined by experiment, lying between 0·005 and 0·01 for most guns; and the notation is as before.

Example. Assume for a gun, $v_0 = 440$ m/sec, $l = 2\cdot6$ calibres; and for a range of 1000 m, take $\phi = 3° 20'$, $\omega = 5° 13'$, $T = 3\cdot92$ seconds: angle of rifling $\Delta = 3° 36'$; thus

$$Z = (0\cdot005 \text{ to } 0\cdot01) \frac{440 \tan 3° 36'}{2\cdot6} (3° 20' + 5° 13') \, 3\cdot92$$
$$= 1\cdot7 \text{ to } 3\cdot4 \text{ m}.$$

This is an amount that, as stated already, may easily be exceeded by the natural scattering of the shells.

3. A formula, based on similar hypotheses, for the lateral deviation or drift has been worked out by P. Haupt, 1876, in which Z is made proportional to $(\phi + \omega) T$.

According to Charbonnier, in high angle fire with low initial velocity, we should take

$$Z = \text{const. } \phi T, \quad \text{.............................(9)}$$

but if the initial velocity is great,

$$Z = \text{const. } (\phi + \omega) T. \quad \text{.....................(10)}$$

Proceeding with similar assumptions, E. Hamilton, 1908, has suggested the formula

$$Z = \text{const. } \frac{R^3 \tan \Delta}{m} (\phi + \omega) \sec \phi; \quad \text{............(11)}$$

and E. Muzeau has given the formula

$$Z = \text{const. } X \tan \phi \, . f(\xi), \quad \text{..................(12)}$$

where

$$f(\xi) = \frac{4}{15\xi(\xi - 1)} \left\{ \frac{2}{21} \frac{(3\xi - 2)^{\frac{7}{2}} - 1}{\xi - 1} - 1 \right\},$$

in which ξ is the abbreviation for

$$\xi = \frac{v_0^2 \sin 2\phi}{gX}.$$

Taking into account a very large number of experimental results, P. Bertagna considers that the formula, which suits the case the best, is

$$Z = \text{const. } X \sin\phi. \qquad (13)$$

But this is doubtful.

In addition to the theoretical calculations given above, another may be introduced, proposed lately by Lanchester, for arriving at equation (4) in a simpler way.

The original impulse of the shell, as stated already, is nearly identical with the impulse due to the rotation about the axis of figure, in consequence of its speed; and so, as above in § 53, with right-hand twist the impulse must be represented by a vector SB drawn from the centre of gravity towards the base along the axis of the shell.

The air resistance, acting along the instantaneous direction of the tangent, tends to turn the shell about a node line SD, drawn perpendicular to the vertical plane through the tangent to the left, with angular velocity $\frac{d\theta}{dt}$. By the laws of gyroscopic action, this implies a turning moment, with axis SV perpendicular to SB and SD, and drawn upward.

Actually the shell is turned by this moment, so that its point swerves to the right out of the vertical plane through the tangent, with its base to the left. The magnitude of this turning moment is $Cr\frac{d\theta}{dt}$, and on the other hand it is Ka, where K denotes the component of air resistance which arises in consequence of the swerving of the shell about SV, and is perpendicular to the vertical plane through the tangent; this tends to move the shell horizontally to the right. The acceleration of the shell to the right being $\frac{d^2z}{dt^2}$, and m the mass of the shell, $K = m\frac{d^2z}{dt^2}$, and so

$$ma\frac{d^2z}{dt^2} = -Cr\frac{d\theta}{dt},$$

with the minus sign, because θ decreases while t increases. Here $\frac{d\theta}{dt} = -\frac{g\cos\theta}{v}$; consequently

$$\frac{d^2z}{dt^2} = \frac{Cr}{ma}\frac{g\cos\theta}{v}.$$

This is equation (4); and Lanchester suggests that a should be determined experimentally from observation of the drift z.

The notes at the end of this volume mention the theory proposed by G. von Gleich, which is seemingly of a very general nature.

4. Finally the formula may be quoted, which was suggested by Mayevski-Vallier for the drift z at the range x

$$z = \frac{1}{400}\psi\mu^2\tan\Delta\,\frac{P}{i(v_0)\beta R^2}\cdot v_0 x\left[\frac{B(u)-B(v_0)}{D(u)-D(v_0)} - M(v_0)\right]\sec^3\phi \quad (14)$$

in which μ is the radius of gyration of the shell about its axis, in half calibres, P the weight of the shell in metric tons, ψ is a constant, about 0·41, but it is best determined experimentally for the given system; $i(v_0)$, β, u, $D(u)$, $D(v_0)$ refer to the system of solution in § 41; $M(v_0)$ is given by Landenskiöld's Table, as well as $B(u)$ and $B(v_0)$.

Example. $2R=0·27$: $P=0·18$: $v_0=505$: length of shell$=5R$: $\mu=0·8R$: $\psi=0·41$: $\Delta=4°$. For $\phi=1° 11'$, $x=1000$, and $z=0·4$. For $\phi=14° 10'$, $x=7000$ and $z=49·9$.

VALUES OF THE FUNCTIONS $M(u)$ AND $B(u)$.

(Landenskiöld's Table.)

u	$10^5 M(u)$	$10^4 B(u)$	u	$10^5 M(u)$	$10^4 B(u)$
700	00	00	400	1399	94560
690	14	34	390	1543	107560
680	29	137	380	1706	122670
670	45	317	370	1893	140370
660	62	580	360	2118	162230
650	80	935	350	2391	189670
640	99	1392	340	2724	224600
630	119	1953	330	3133	269600
620	140	2635	320	3640	328400
610	163	3448	310	4270	406000
600	188	4404	300	5060	510300
590	214	5516	290	6050	651000
580	242	6801	280	7234	832500
570	272	8274	270	8651	1065800
560	305	9953	260	10356	1366400
550	340	11863	250	12422	1755300
540	377	14027	240	14950	2262000
530	417	16467	230	17990	2910000
520	460	19216	220	21610	3725000
510	507	22305	210	25950	4746000
500	557	25774	200	31200	6035000
490	612	29659	190	37820	7667000
480	672	34020	180	45550	9744000
470	737	38897	170	55450	12409000
460	807	44357	160	68990	15862000
450	884	50470	150	84090	20390000
440	968	57320	140	105120	26420000
430	1060	64990	130	133140	34550000
420	1161	73590	120	171300	45780000
410	1273	83330	110	224600	61640000
400	1399	94560	100	301300	84690000

§ 57. Experimental demonstration of the oscillation and drift of a shell.

1. Perrodon's apparatus.

A truck is allowed to run along curved rails, of the shape of the trajectory, carrying a Bohnenberger machine with a model of the shell.

The shell hangs freely from a Cardan suspension and is rotated swiftly. The air resistance is represented by a spiral spring, acting in the direction of the axis of the truck, which corresponds to the direction of the tangent to the path.

In this way conical oscillations are set up, and Perrodon seeks in this way to determine the condition of stability of the shell.

2. Lecture apparatus of Pfaundler.

This serves to demonstrate the stability of the axis of a rotating shell, and its conical oscillation.

A pointed shell is placed in a horizontal frame, suspended so as to be easily movable, and the shell is set in rotation. At the rear end wind vanes are attached, and can be moved into any position. The conical oscillations can then be observed, to one side or the other, according to the direction of the rotation, and the setting of the wind vanes.

3. Impulse apparatus of Ludwig.

A small wooden model of the shell is placed at the end of the spindle of the apparatus, which can be set in motion by hand. A blow of a hammer on the other end of the axis will then give the impulse to the model of the shell.

4. Lecture apparatus of A. von Obermayer and V. von Niesiolowski.

A strong blast of air is blown from a powerful fan against a model of the shell, mounted by Cardan suspension so that it can turn about the centre of gravity. The precession due to the air resistance is thereby shown.

Magnetic forces can also be employed to produce a similar effect.

5. The author employs the following method of demonstrating the oscillation and drift of a shell.

A. Throwing a wooden disc or flat stone.

The gyratory motion of a shell can be shown very clearly with a circular disc of wood, about 8 cm in diameter and $\frac{1}{2}$ cm thick, or with a smooth flat stone, by throwing them a considerable distance in an open space.

As usual in throwing such bodies, the disc is to be held between the thumb and middle finger, and partly gripped by the forefinger.

The disc is held at right angles to the plane of flight desired, and is thrown so as to make an angle of 30° with the horizon. In hurling the disc, the forefinger rotates it clockwise, when thrown by the right hand, about an axis of rotation directed upward and backward. This gives the initial position of the impulse-vector.

The air resistance meets the front side of the disc, and tends to turn it clockwise about a horizontal axis, drawn to the left of the plane of projection. Thence we get the position, as in § 53, of the next vector; and then it can be predicted how the disc will turn and to which side it will swerve.

Actually in the case of a disc, a drift to the left will occur, if thrown by the right hand, to the right if thrown by the left; and it is very pronounced even in a range of 40 m.

B. Model mortar with wooden shell.

A wooden base serves as the carriage, on which any one of four Mannesmann tubes may be mounted. These are provided with a cast zinc insertion to represent the rifling, a contrivance for receiving a small charge of black powder, up to 10 g, and the breach is closed by a bayonet joint. Two of the tubes have a right-hand twist, angle of rifling (a) 43° 40′ = 3·3 calibres, (b) 17° 39′ = 10 calibres; the two others have left-hand twist with the same angles of rifling; calibre 7·9 cm.

The shells are made of red beech, and are provided with grooves to correspond with the rifling of the bore. They are of various lengths, 30·27, 35·5, 36 cm; one of the shells is provided with an axial hole, in which a bar of iron can be fixed (either forward, midway, or behind), so that it is possible to observe the influence of the positions of the centre of gravity.

The angle of elevation is measured by a quadrant. The firing of the powder charge is made by a fuse.

For carrying out the experiments, an open space of about 400 metres is required; it must be as nearly calm as possible, as these light wooden shells are influenced powerfully by wind.

The results refer mainly to a shell 30·27 cm in length, calibre 7·9 cm; length of the cylindrical part 23·75 cm, weight 0·930 kg; moment of inertia about the long axis $C = 0{\cdot}00012$ mkg sec²; about the cross axis through the centre of gravity $A = 0{\cdot}000515$; with a charge of 5 g, $v_0 = 23{\cdot}88$ m/sec; with 10 g charge, $v_0 = 41{\cdot}4$ m/sec.

(*a*) Tube with the smaller right-hand twist, angle of rifling 17° 39′; 5 g charge.

The shell flies like an arrow, even up to an angle of departure of $\phi = 71°$.

Looking at the trajectory from the side, the long axis is seen to lie in the tangent to the path; range, at $\phi = 45°$, 107 m, time of flight $T = 5{\cdot}3$ sec. With a gradually increasing angle of departure the drift to the right grows continually.

After $\phi = 71°$, the drift changes to the left, and the shell strikes the ground with the base first.

(*b*) The same tube; charge 10 g.

At $\phi = 45°$, maximum range $X = 321$ m, time of flight $T = 9{\cdot}3$ sec, drift to right $Z = 39$ m.

At $\phi = 70°$, range 183 m, $T = 12{\cdot}4$ sec, $Z = 61$ m.

Change from right to left drift at an angle ϕ lying between 77° and 80°. At $\phi = 80°$, the shell descends with the base first; once this happened too in a flat trajectory, and then the point was directed to the right.

Drift to the left at $\phi = 80°$ was 66 m, and sometimes rather less.

It can be seen with an angle of 80° that the shell has a slight drift to the right up to the vertex of the path, and that the drift to the left begins beyond the vertex and rapidly increases.

(*c*) Tube with the greater right-hand twist, angle of rifling 43° 40′.

The axis of the shell remains apparently parallel to itself for all departure angles; and it seems on close observation as if the point of the shell were a little to the right.

Change from right to left drift at about $\phi = 53°$, with a charge of 5 g, as well as with 10 g.

Range at $\phi = 45°$ with 5 g charge is 55 m, $T = 4{\cdot}26$ sec; at $\phi = 45°$ and 10 g charge, range is 180 m, $T = 8{\cdot}24$ sec.

The angle of rifling is thus too great for this shell.

A longer range was sometimes observed than would be obtained in a vacuum; the supporting influence of the air on the surface of the shell seems to be the cause.

(*d*) The same tube, shooting vertically upward.

The shell flies very nearly parallel to itself, as a sort of diabolo-top, and so arrives, with a strong humming noise, back again in almost vertical position, and strikes the ground with the flat base.

(*e*) Tube with the smaller right-hand twist, angle of rifling 17° 39′; shell of greater length, 35·5 cm.

Very pronounced nutational oscillations ensued; the point of the shell appears to describe complete circles round the tangent to the path; seen from the firing point the shell looks like a great disc. The twist is not enough for this length of shell.

In the employment of the tubes with left-hand twist, there is a corresponding change of sign in drift, direction of the oscillations, precession and nutation, and so forth.

CHAPTER XI

The application of the theory of Probability to Ballistics

§ 58. Introductory.

There are many wholly accidental causes which may tend to deflect a shell from the target. Thus, in spite of all the trouble that may be expended on directing the aim, it may none the less be inaccurate: there may be slight variations in the density of the air or in the direction and velocity of the wind, which cannot be taken into account: with rifles, it is not possible to consider the vibratory motion of the barrel to be an absolutely constant factor, and there may also be small changes in the initial velocity. A deviation due to any one of these causes may be considered to be accidental, and to take place in a wholly irregular manner. But if a large number of shells are fired, these deviations are subject to definite laws, similar to those which hold in all exact measurements of a physical or technical nature. The mathematical theory of probability applies in this case, just as it does in the throwing of dice or in questions of life insurance. If a single shell is fired, it is, generally speaking, impossible to say whether it will go to the left or the right of the target, or whether it will be above or below the mark. Equally it is impossible with the help of Tables of Mortality to say when any individual will die. On the other hand we know the definite fact that out of 100,000 male persons, 17,750 will reach the age of 70, and out of 100,000 females, 21,901 will attain that age: and if 100 bullets are fired from a known rifle at a target, we can also tell how many will fall within a circle with a radius of 20 cm. The following considerations apply to all quantitative measurements, and not only to the deviations of shells: therefore they can be used in connection with accidental variations of any character, such as the measurement of gas pressures or the velocities of shells. But for the sake of simplicity, we shall mostly confine our attention to the deviations of bullets, in so far as ballistical applications are concerned.

The Mathematical Theory of Probability.

The following propositions are here briefly stated, and it will be easily seen how the conclusions are drawn.

1. Let us call the mathematical or absolute probability of an event, a: then if n is the total number of cases, and t is the number of cases in which the event happens, we have $a = \frac{t}{n}$. Thus we see that a is a fraction less than unity, or at any rate not greater than unity. It is equal to 1, if the event is certain, and equal to 0, if the event is impossible.

Example 1. If two dice are thrown, what is the probability that the total will be 7? The number of possible cases is 36, since each of the 6 sides of the one dice may be combined with any one of the 6 sides of the other: therefore $n = 36$. The only cases in which we get 7 are by combining 6 and 1: 5 and 2: 4 and 3: 3 and 4: 2 and 5: 1 and 6. There are thus 6 such cases, and $t = 6$. Therefore $a = \frac{6}{36} = \frac{1}{6}$. The chance that 7 is *not* thrown is $\frac{5}{6}$, since there are 30 possible combinations in which the total is not 7.

If we take the chances of throwing 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, we find that they are respectively $\frac{1}{36}$, $\frac{2}{36}$, $\frac{3}{36}$, $\frac{4}{36}$, $\frac{5}{36}$, $\frac{6}{36}$, $\frac{5}{36}$, $\frac{4}{36}$, $\frac{3}{36}$, $\frac{2}{36}$, and $\frac{1}{36}$. Therefore there is a greater chance of throwing 7 than of throwing any other number.

Example 2. An urn contains 20 balls, of which 7 are white, 5 black, and 8 red. The chance of drawing a black ball is obviously $\frac{5}{20} = \frac{1}{4}$.

Example 3. An urn contains 7 white and 6 black balls. If 5 balls are withdrawn, what is the chance that three will be white and two black? The number of ways in which 5 balls can be withdrawn is $\binom{13}{5}$, where $\binom{13}{5}$ denotes $\frac{13!}{8!\,5!}$. Therefore $n = \binom{13}{5}$. On the other hand, 3 white balls can be withdrawn in $\binom{7}{3}$ ways, and 2 black balls can be extracted in $\binom{6}{2}$ ways, each of which can be combined with any of the methods of withdrawing the white balls. Therefore $t = \binom{7}{3}\binom{6}{2}$. Therefore $a = \dfrac{\binom{7}{3}\binom{6}{2}}{\binom{13}{5}}$.

2. Let us suppose that there are three mutually exclusive events, A, B, and C: the chance of A's occurring is a, of B is b, and of C is c. Then the chance of the occurrence of any one of them is $a + b + c$.

Example 1. What is the chance of throwing either 2 or 3 or 4 with two dice? The absolute probabilities of any one of these events are, as we already know, $\frac{1}{36}$, $\frac{2}{36}$, and $\frac{3}{36}$. Therefore the chance of throwing any one of them is $\frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{1}{6}$.

Example 2. Let us take the case of a target, which is divided into an infinite number of parallel vertical strips: let the chance of hitting the first strip be dy_1, of hitting the second strip be dy_2, and so on. Then the chance of hitting the target is $dy_1 + dy_2 + \ldots = \Sigma\, dy = \int dy$, where the limits of the integral extend from left to right. The application of this proposition will be seen later. The events are considered to be mutually exclusive, on the assumption that the projectile does not break to pieces, in which case the target might be hit in several places by the one shell.

3. Let us take several independent events, $A, B, C, \ldots$, and let their respective chances be $a, b, c, \ldots$. Then the chance that they will occur simultaneously, or in a specified sequence is equal to the product of their separate chances: that is, it is equal to the product $abc\ldots$. But if the events are dependent, then the probability that A occurs, and then B, and then C is abc, on the assumption that b is the chance of B's occurring, after A has happened, and c is the chance of C's occurring, after A and B have happened.

Example 1. Two persons, P_1 and P_2, throw two dice-boxes at the same time. What is the chance that P_1 throws a total of 2, and P_2 at the same time throws a total of 4? The chance that P_1 throws 2 is $a = \frac{1}{36}$. The chance that P_2 throws 4 is $b = \frac{3}{36}$. Therefore the chance of the combined events is $\frac{1}{36} \times \frac{3}{36} = \frac{1}{432}$, while the chance that this does not happen is $\frac{431}{432}$. Therefore the odds against the event are 431 to 1.

Example 2. A boy thinks that the chance of being promoted at Easter is $\frac{2}{3}$, and the probability of getting a bicycle at the same time is $\frac{1}{2}$. Then the chance of both events happening is $\frac{1}{3}$, on the assumption that the two things are independent of one another. This answer will also be correct, if the events are mutually dependent, provided the boy estimates his chance of getting the bicycle at $\frac{1}{2}$, after he has got promotion.

Example 3. A man, A, is 35 years old, and his wife, B, is 28. What are the chances that after 20 years (1) both are alive, (2) one is dead, (3) A is alive and B is dead, (4) A is dead and B is alive, (5) both are dead, and (6) at least one of them is alive? The tables of Mortality state that out of 100,000 males, 51,815 reach the age of 35, and 36,544 attain the age of 55: while out of 100,000 females, 58,647 reach the age of 28, and 46,605 live to 48. Therefore the chance that A survives for 20 years is $a = \frac{36544}{51815}$, and for B is $b = \frac{46605}{58647}$. Then the chance of (1) is $ab = 0{\cdot}56$: and of (2) is $1 - ab = 0{\cdot}44$: and of (3) is $a(1-b) = 0{\cdot}145$: and of (4) is $b(1-a) = 0{\cdot}234$: and of (5) is $(1-a)(1-b) = 0{\cdot}061$: and of (6) is $1-(1-a)(1-b) = 0{\cdot}94$. The sum of the chances of (1), (3), (4) and (5) is unity, since one of these cases must arise.

Example 4. In artillery practice, let the chance of a short range be a, and of a long range be b: and let the chance of a wrong observation be c, which Mangon considers to have the value 0·1. Then the probability that a short range is observed is the same as the chance that either the range is short and actually observed as such or that the range is long and is erroneously reported as being short: this is actually $a(1-c)+c(1-a)$.

Example 5. Let the angle of departure be suitable for a range of 4000 metres: let the chance of a short range $(-)$ be $\frac{3}{4}$, and of a long range $(+)$ be $\frac{1}{4}$. With an elevation for 4100 metres, let the chance of a $+$ be $\frac{4}{5}$, and of a $-$ be $\frac{1}{5}$. Then the chance of getting 4000 $-$ and 4100 $+$ is $\frac{3}{4}\times\frac{4}{5}=\frac{3}{5}$.

§ 59. Continuation.

1. *Probability of repeated experiments.*

Two events, A and B, are to be considered (such as hit and miss), of which one must occur: if their absolute probabilities are a and b, we have $a+b=1$. If the experiment is repeated three times, the following cases are possible. (a) A may occur three times in succession: the probability of this is a^3. (b) A may occur twice and B once. The chance of this is a^2b. On the other hand, if the order is immaterial, we must consider the probability of the sequence A, A, B; or A, B, A; or B, A, A: the chance of this is $3a^2b$. (c) A may occur once, and B twice: if the sequence is immaterial, the chance is $3ab^2$. (d) B may occur three times, and A not at all: the chance of this is b^3. These various expressions are the terms in the expansion of $(a+b)^3$. Moreover $1-b^3$ is the chance that A occurs at least once: and $1-(b^3+3ab^2)$ is the chance that A occurs at least twice. Thus we have the following general law. If A and B are two events, one of which must happen (such as hit and miss), and if their chances are a and b, then the probability that in s trials, the event A happens m times, and B happens $s-m$ times in any order is

$$\binom{s}{m} a^m b^{s-m} = \frac{s!}{m!(s-m)!} a^m b^{s-m}.$$

Example 1. The probability of throwing doubles with two dice is $\frac{6}{36}=\frac{1}{6}$. Thus the probability of throwing doubles exactly three times in 600 throws is

$$\frac{600!}{3!\,597!}\cdot\left(\frac{1}{6}\right)^3\cdot\left(\frac{5}{6}\right)^{597}.$$

Example 2. After how many throws is the chance of throwing doubles at least once equal to $\frac{1}{2}$? $\frac{1}{2}=1-(\frac{5}{6})^s$, and therefore $s=\frac{\log\frac{1}{2}}{\log\frac{5}{6}}=3{\cdot}8$. So that after 4 throws, the chance of throwing doubles is a little more than $\frac{1}{2}$.

Example 3. The probability of a shot being short is denoted by a, and of an over is $1-a=b$, and 5 shots are made. The chance of 3 short and 2 over, in any order, is $10a^3b^2$. The probability of at least 3 overs is the total for 3 or 4 or 5, in any order, and so is $10a^2b^3+5ab^4+b^5$. The chance of 2 overs at most, is the same as for 1 or 2 overs, and so is $a^5+5a^4b+10a^3b^2$.

Example 4. A gun is fired at the target, and the chance of a short (−) and of an over (+) is the same, viz., $\frac{1}{2}$. Six shots are fired at the target. Then the chance, in any order, of

$$3+ \text{ and } 3-, \text{ is } \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^3=\frac{20}{64},$$

$$2- \text{ and } 4+ \quad ,, \quad \binom{6}{2}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2=\frac{15}{64},$$

$$2+ \text{ and } 4- \quad ,, \quad \binom{6}{4}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4=\frac{15}{64}.$$

The chance that one of these cases happens, is then

$$\frac{20+15+15}{64}=0{\cdot}78.$$

In 100 such groups, 78 will then give the range correctly.

2. *Continuation. Law of great numbers.*

The chance, that in $s=m+n$ cases, the event A happens exactly m times, and B happens $n=s-m$ times, is equal to $\frac{s!}{m!\,n!}a^m b^n$ and is generally small; it is greatest when m lies between $sa-b$ and $sa+b$.

The maximum is then when $m=sa$, or if sa is not an integer, when m is the integer between $sa-b$ and $sa+b$. An approximate value of this maximum is $\frac{1}{\sqrt{(2\pi sab)}}$.

The greater the value of s, the closer we come to the combination for which $m:n=a:b$.

If for example there are a million balls in an urn, 100,000 white and 900,000 black, and the probability, a or b, is required of drawing a white or a black ball, then $a=\frac{1}{10}$, $b=\frac{9}{10}$; and when the ball is replaced each time after drawing, the ratio m of the number of the white balls drawn to the black is always nearly $1:9$.

3. *Continuation. Law of Bernoulli-Laplace.*

As stated, the chance of throwing doubles with two dice exactly 100 times in 600 throws is rather small, and may be taken as being equal to

$$\frac{1}{\sqrt{(2\pi\times 600\times\frac{1}{6}\times\frac{5}{6})}}=\frac{1}{23}.$$

Much greater on the other hand is the chance that in 600 throws about 100 doubles are thrown (80 to 120 times for instance).

The following law of Bernoulli-Laplace is useful, and with the help of Stirling's law provides an approximation, which is fairly accurate if s is large.

The probability is

$$P = \frac{2}{\sqrt{\pi}} \int_0^\gamma e^{-t^2}\, dt + \frac{e^{-\gamma^2}}{\sqrt{(2\pi sab)}},$$

that in s cases the event A, having the absolute probability $a = 1 - b$, happens from

$$sa - \gamma \sqrt{(2sab)} \text{ to } sa + \gamma \sqrt{(2sab)} \text{ times,}$$

so that the ratio $\frac{m}{s}$ lies between the limits

$$a \mp \gamma \sqrt{\left(\frac{2ab}{s}\right)}.$$

The following form of the law can be used in simple numerical applications: we assume that m lies between the limits $sa \pm \gamma[\sqrt{(2sab)} - \frac{1}{2}]$, and the probability will then be $P = \frac{2}{\sqrt{\pi}} \int_0^\gamma e^{-t^2}\, dt$.

For the special value $P = \frac{1}{2}$, $\gamma = \rho = 0{\cdot}476936$; in this case $\gamma \sqrt{(2sab)} - \frac{1}{2}$ is called the probable deviation.

A generalisation of this law, given by Poisson, relates to the case where the chance for the occurrence of the event A varies from one case to the other; a_1 in the first case, a_2 in the second, and so on.

Denote the mean chance

$$\frac{a_1 + a_2 + \ldots}{s} \text{ by } a;$$

and further put $b_1 = 1 - a_1$, $b_2 = 1 - a_2$, ..., $b = 1 - a$; finally put

$$\kappa = \sqrt{\left[\frac{2}{s}(a_1 b_1 + a_2 b_2 + \ldots)\right]}.$$

Thus in s events the probability is

$$P = \frac{2}{\sqrt{\pi}} \int_0^\gamma e^{-t^2}\, dt + \frac{e^{-\gamma^2}}{\kappa \sqrt{(\pi s)}},$$

that the event A happens between $as \mp \gamma\kappa \sqrt{s}$ times, or $\frac{m}{s}$ lies between the limits $a \mp \frac{\gamma\kappa}{\sqrt{s}}$.

Thence it follows, as s increases, that $\frac{m}{s}$ approaches a.

If the chances $a_1, a_2, \ldots$, as well as $b_1, b_2, \ldots$, are assumed equal, then $\kappa = \sqrt{(2ab)}$, in agreement with the Bernoulli-Laplace law.

Example. What is the probability that in 600 throws with 2 dice, doubles occur from 80 to 120 times? Here $a = \frac{6}{36} = \frac{1}{6}$, $b = \frac{5}{6}$; $s = 600$, $\gamma\sqrt{(2sab)} - \frac{1}{2} = 20$ $\gamma = 1{\cdot}59$; hence $P = 0{\cdot}97$ is the probability.

4. *Bayes's rule.*

It is known, for example, that a million balls are in an urn, and 400,000 of them are white, and 600,000 of other colours; so that the chance of drawing a white is $a = \frac{2}{5}$, and of the reverse is $b = \frac{3}{5}$; 800 balls have been drawn and replaced. What is the chance that of the 800 balls from 310 to 330 of them are white?

We have already dealt with a problem of this character, but a different question arises if all that is known is that there are a million balls in the urn, of which a certain number are white. After 800 drawings, 320 are white and 480 another colour. With what degree of accuracy can this trial determine approximately the unknown ratio a of the white to the total number of balls? What is for instance the probability that the ratio $a = \frac{320}{800} = \frac{16}{40}$ as determined from the 800 events is true within $\frac{1}{40}$ or $2{\cdot}5\,{}^\circ\!/_\circ$; or in other words, what is the probability that the number of white balls in the urn, is greater than 375,000 and less than 425,000?

In these cases it is required from the observed events to deduce inversely the unknown cause, and to make some conclusion about a and b.

Here the approximate rule of Bayes is employed: when the event A has an absolute probability a, and in $s = m + n$ occurrences, it happens m times and fails n times, the probability is

$$P = \frac{2}{\sqrt{\pi}} \int_0^\gamma e^{-t^2}\, dt$$

that the unknown number a lies between the limits $\frac{m}{s} \mp \gamma \sqrt{\frac{2mn}{s^3}}$, where γ is some arbitrary number.

In the special case of $P = \frac{1}{2}$, then a lies between

$$\frac{m}{s} \mp 0{\cdot}4769 \sqrt{\frac{2mn}{s^3}}.$$

The greater the value of s, the closer is a to the ratio $\frac{m}{s}$; this is the inversion of the law of great numbers.

Example 1. Take the urn problem (Czuber): here $s=800$, $m=320$, $n=480$; and take $\gamma\sqrt{\frac{2\times 320\times 480}{800^3}}=\frac{1}{40}$; thence $\gamma=1{\cdot}0205$, and thus $P=0{\cdot}851$.

With the probability 0·851 it can be assumed that the ratio of the number of white balls in the urn to the total number of balls, can be taken as $\frac{320}{800}$, within 2·5 °/$_\circ$, or that it lies between $\frac{15}{40}$ and $\frac{17}{40}$; or that the number of white balls lies between 375,000 and 425,000.

Example 2. Half the sheets were lost, giving a record of hit and miss. Counting up the numbers preserved, 240 hits were found and 120 misses. It was most likely then that the complete list of 720 shots had 480 hits and 240 misses. How far can this assumption be wrong? Here $s=360$, $m=240$, $n=120$, $P=\frac{1}{2}$; so that

$$a=\frac{240}{360}\mp 0{\cdot}4769\sqrt{\frac{2\times 240\times 120}{360^3}}=\frac{480\mp 38}{720}.$$

Thence the number of hits is most likely to lie between 442 and 518.

The papers of Czuber, as well as those of Sabudski, Eberhard, and von Kozàk should be consulted for the proof of the Law given above.

5. *The probability of predictions.*

Suppose $m+n$ cases for the event A have been observed, which happens m times, and fails n times.

Then the probability that in $p+q$ subsequent cases the event A happens p times, and does not occur q times is

$$\frac{(p+q)!}{p!\,q!}\frac{\int_0^1 x^{m+p}(1-x)^{n+q}dx}{\int_0^1 x^m(1-x)^n\,dx}=\frac{(p+q)!\,(m+p)!\,(n+q)!\,(m+n+1)!}{p!\,q!\,(m+n+p+q+1)!\,m!\,n!}.$$

Remark 1. The formula of Stirling refers to the approximate calculation for high numbers: he assumes that

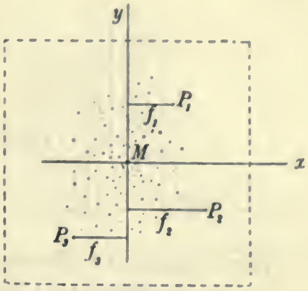
$$n! = \sim n^n e^{-n}\sqrt{(2\pi n)}.$$

This may be employed for high numbers, and even as low as $n=10$.

Remark 2. The definite integrals employed in the theory of Probability will be found in the Tables 14 and 15, Vol. IV.

§ 60. Theory of the deviation of projectiles. Measure of the accuracy.

A number of bullets are fired at a vertical target, the rifle being very carefully adjusted after preliminary practice. The aim is to hit the point M, about which the accidental points of impact P_1, P_2, etc., will arrange themselves in such wise that few will be far from the mark, and most of them will be near M, equally distributed to the right and left, or above and below.



For the present only the lateral deviations are to be considered, right (positive) and left (negative). Denote them by $f_1, f_2, f_3, \ldots$; they are the x coordinates of the various points of impact in the coordinate system, of which the origin is placed at M.

The fact that a small error occurs more frequently than a large one, and that the hits are symmetrical with respect to the y axis, may be inferred from the consideration that these errors $f_1, f_2, f_3, \ldots$ arise from several independent sources of error.

Thus for example there are three sources which may cause error: (a) small alterations of the wind velocity may cause the deviation of $-2, -1, -0$ cm; (b) small alterations of the vibration of the barrel may cause lateral errors, right and left, $-1, 0, +1$ cm; (c) error of aim, $-1, 0, +1, +2, +3$ cm.

Their combination can produce the resultant deviations $-4, -3, -2, -1, 0, +1, +2, +3, +4$ cm in 1, 3, 6, 8, 9, 8, 6, 3, 1 different ways respectively.

For instance, the deviation -4 can arise in one way $-2, -1, -1$.

The error 0 on the other hand can arise in the following ways:

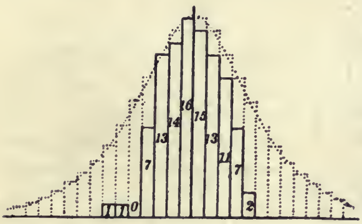
$$-2, -1, +3 \mid -2, 0, +2 \mid -2, +1, +1 \mid -1, -1, +2 \mid$$
$$-1, 0, +1 \mid -1, +1, 0 \mid 0, -1, +1 \mid (0, 0, 0) \mid 0, +1, -1 \mid,$$

i.e., in 9 different ways (Czuber).

Example. Shooting with a rifle the deviations to right and left were measured for 100 shots: between $x=0$ and $x=\pm 4$ cm, 15 positive, to the right, and 16 negative to the left; between $x=4$ and $x=8$ cm, 13 positive and 14 negative; between 8 and 12 cm, 11 positive, 13 negative; between 12 and 16 cm, 7 positive and 7 negative; between 16 and 20 cm, 2 positive, 0 negative; between 20 and 24 cm, 0 positive, 1 negative; between 24 and 28 cm, 0 positive, 1 negative deviation; total 100.

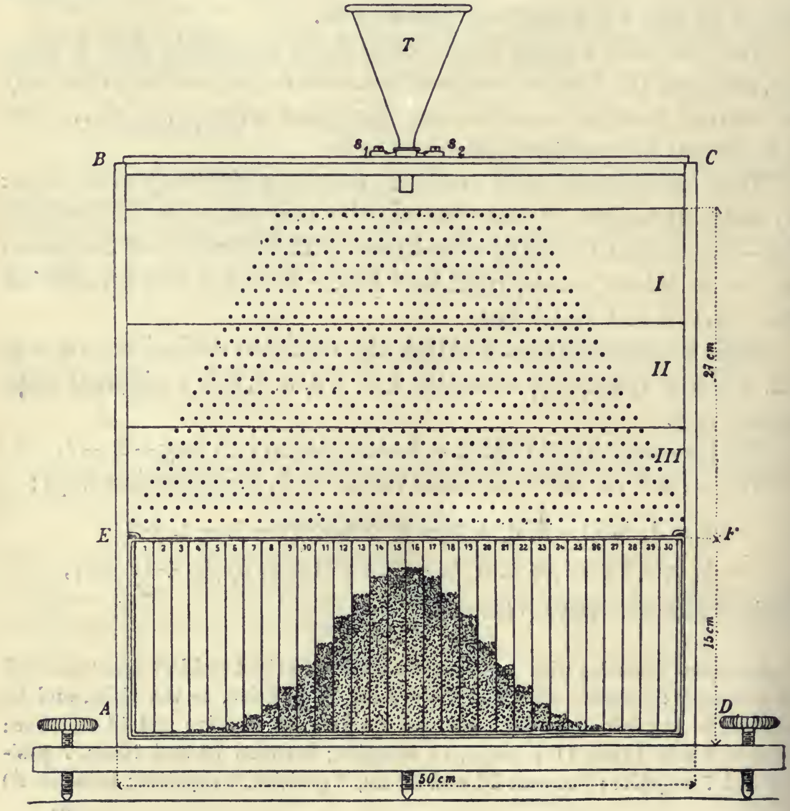
The fact is that the smaller deviations are more frequent than the large.

The same effect is produced in the apparatus shown in the figure at the bottom of the page, where the flying shot are replaced by grains of millet or lead shot, and the scattering is effected by a number of metal pins from which the grains rebound.

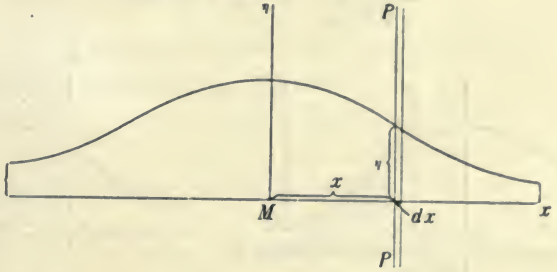


If the deviations in a large number of shots are plotted as a function of x, the curve assumes a certain form.

The equation of the curve was proved by Gauss to be $\eta = ae^{-h^2x^2}$, and can be deduced on theoretical principles. For the present it will suffice to say that the formula has shown itself to be very useful in practice.



At the distance x from the vertical middle line $M\eta$ a narrow strip is drawn, of unit breadth.



The probability of hitting this strip is $\eta = ae^{-h^2x^2}$, or if n is the total number of shots, then $nae^{-h^2x^2}$ shots fall on the strip.

With an indefinitely small breadth of the strip, and with PP at a distance x from M, the number of hits is $nae^{-h^2x^2}\,dx$; in short, $nae^{-h^2x^2}\,dx$ denotes the number of shots which have a deviation x to the right or left of the vertical through M.

It is required next to determine the constants a and h.

An infinite target must always be hit; that is, the total number of shots falling on all the strips of the target of breadth dx, from $x = -\infty$ to $x = +\infty$ is equal to n, or

$$na \int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = n,$$

and since

$$\int_{-\infty}^{\infty} e^{-h^2x^2}\,dx = \frac{\sqrt{\pi}}{h}, \quad a = \frac{h}{\sqrt{\pi}}.$$

Thus $\dfrac{h}{\sqrt{\pi}}\, e^{-h^2x^2}\,dx$ is the probability of a deviation x, and

$$n\,\frac{h}{\sqrt{\pi}} \int_c^d e^{-h^2x^2}\,dx$$

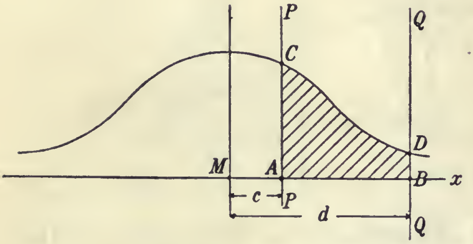
is the number of hits on a strip $PPQQ$ of infinite height of which the sides PP, QQ are at distances c and d from M; this is the area of the curve $ABCD$ (on the next page), multiplied by n the number of shots.

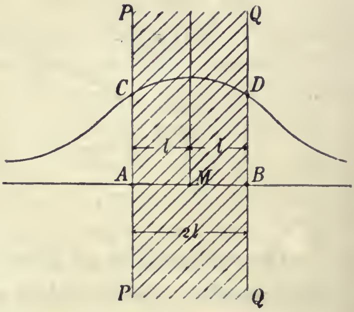
If it is required to hit the space $PPQQ$ on the target, of breadth $2l$ and placed symmetrically with respect to M, the probability of hitting it is given by the area $ABCD$, and is equal to

$$\frac{h}{\sqrt{\pi}} \int_{-l}^{+l} e^{-h^2x^2}\,dx = \frac{2h}{\sqrt{\pi}} \int_0^l e^{-h^2x^2}\,dx;$$

and since $hx = t$, $dx = \dfrac{dt}{h}$, then the probability is

$$= \frac{2}{\sqrt{\pi}} \int_0^{t=hl} e^{-t^2}\, dt = \phi(hl).$$





This integral must be calculated as a function of hl. Since

$$e^{-z} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} \cdots,$$

and in consequence of its uniform convergence

$$\int e^{-t^2}\, dt = \int \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \frac{t^8}{4!} \cdots\right) dt = t - \frac{t^3}{3} + \frac{t^5}{2!\,5} - \frac{t^7}{3!\,7} \cdots,$$

the probability of hitting the strip $PPQQ$ of breadth $2l$ is

$$\frac{h}{\sqrt{\pi}} \int_{x=-l}^{x=+l} e^{-h^2x^2}\, dx = \frac{2}{\sqrt{\pi}} \int_{t=0}^{t=hl} e^{-t^2} dt$$

$$= \frac{2}{\sqrt{\pi}} \left[hl - \frac{1}{3}(hl)^3 + \frac{1}{2!\,5}(hl)^5 \cdots \right] = \phi(hl),$$

as in Table 14.

The constant h is the measure of the precision of shooting with the weapon employed. The precision of hitting, so far as concerns the deviations in the direction of the x axis, is given by the probability of hitting the thin strip that goes through M; and this probability is

$$\frac{h}{\sqrt{\pi}}\, e^{-h^2 \cdot 0}\, dx = \frac{h}{\sqrt{\pi}}\, dx.$$

If this is $\dfrac{h_1}{\sqrt{\pi}}\, dx$ for the first weapon, and $\dfrac{h_2}{\sqrt{\pi}}\, dx$ for the second, the ratio of the measures of precision of the two weapons is $h_1 : h_2$.

In place of h other convenient measures of precision are employed, which can easily be taken off the target diagram.

The following are the various mean values of the deviations that are employed:

The arithmetic mean of the deviations, $\frac{f_1+f_2+\dots}{n}$, is exactly or nearly zero and is therefore useless. We may take on the other hand the so-called "mean quadratic deviation"

$$\mu=\sqrt{\left(\frac{f_1^2+f_2^2+\dots}{n}\right)},$$

or, sometimes, the "mean cubic deviation"

$$\mu_3=\sqrt[3]{\left(\frac{f_1^3+f_2^3+\dots}{n}\right)},$$

or the average deviation,

$$E=\frac{(f_1)+(f_2)+(f_3)+\dots}{n},$$

where the deviations are all taken with the + sign; or finally the probable or 50 °/₀ deviation w, for which the probability is $\frac{1}{2}$; and h can be calculated from any one of these measures of accuracy of fire, in the following manner.

(*a*) The mean quadratic error, $\mu=\sqrt{\left(\frac{\Sigma f^2}{n}\right)}$.

According to the definition of μ, $n\mu^2$ is equal to the sum of the squares of all the deviations $f_1\,f_2\,f_3\dots$, and so is = (deviation x_1)^2 times the number of deviations of magnitude x_1 + (deviation x_2)^2 times the number of deviations of magnitude x_2, and so on. Then according to the above

$$n\mu^2=x_1^2\,\frac{h}{\sqrt{\pi}}\,e^{-h^2x_1^2}\,dx_1\,.\,n+x_2^2\,\frac{h}{\sqrt{\pi}}\,e^{-h^2x_2^2}\,dx_2\,.\,n+\dots$$
$$=\frac{nh}{\sqrt{\pi}}\int_{-\infty}^{\infty}x^2\,e^{-h^2x^2}\,dx.$$

This integral is to extend over an infinite plane, and so from $x=-\infty$ to $x=+\infty$, since the law of Gauss includes errors infinitely great.

But

$$\int_{-\infty}^{\infty}x^2\,e^{-h^2x^2}\,dx=\frac{\sqrt{\pi}}{2h^3},$$

and so

$$n\mu^2=\frac{nh}{\sqrt{\pi}}\,\frac{\sqrt{\pi}}{2h^3},\qquad h=\frac{1}{\mu\sqrt{2}}=\frac{0{\cdot}7071}{\mu},$$

by which h is determined.

(*b*) The average deviation $E = \frac{\Sigma |f|}{n}$.

Here, in the definition of E, nE = sum of the products of the deviation x and the number of deviations corresponding to x, and

$$nE = \int_{-\infty}^{\infty} x \frac{h}{\sqrt{\pi}} e^{-h^2x^2} dx \,.\, n,$$

and since

$$\frac{2h}{\sqrt{\pi}} \int_0^{\infty} xe^{-h^2x^2} dx = \frac{1}{h\sqrt{\pi}},$$

$$E = \frac{1}{h\sqrt{\pi}}, \quad h = \frac{1}{E\sqrt{\pi}},$$

giving another determination of h.

At the same time a relation is given between μ and E, that is

$$E = \mu \frac{\sqrt{2}}{\sqrt{\pi}} = 0{\cdot}79788\mu.$$

(*c*) The probable, or 50 °/₀ deviation w.

This is the deviation for which the probability is $\frac{1}{2}$; or in other words, $2w$ is the breadth of a strip of indefinite height, placed symmetrically about M, which contains half the shots.

Then from the preceding, w is to be determined from the relation

$$\frac{n}{2} = \frac{nh}{\sqrt{\pi}} \int_{x=-w}^{x=+w} e^{-h^2x^2} dx,$$

or

$$\frac{2}{\sqrt{\pi}} \int_{t=0}^{t=wh} e^{-t^2} dt = \frac{1}{2}.$$

This integral is given in Vol. IV, Table 14, and denoted by $\phi(t)$; and $\phi(wh) = \frac{1}{2}$.

The Table gives

$$hw = 0{\cdot}4769363 = \rho;$$

and

$$w = \frac{\rho}{h} = \rho\mu\sqrt{2} = 0{\cdot}6744898\mu.$$

Collecting the results, we have

$$h = \frac{1}{\mu\sqrt{2}} = \frac{1}{E\sqrt{\pi}} = \frac{\rho}{w}, \quad \rho = 0{\cdot}4769363;$$

and the probable deviation error is

$$w = 0{\cdot}6744898\mu = 0{\cdot}8453476E;$$

the mean quadratic error is

$$\mu = 1{\cdot}4826021w = 1{\cdot}2533141E;$$

and the mean average error is

$$E = 0{\cdot}7978846\mu = 1{\cdot}1829372w.$$

In the German Army the "mean error" is taken to mean the average error E; and the "mean deviation" is not the double of E, but the double of the probable error w; care must be taken to notice this distinction.

Remark 1. According to Gauss the probable error w can be determined with less accuracy in the following manner: the deviations $f_1\, f_2\, f_3$ are arranged in order of absolute magnitude, and the middle one is taken if n is odd, or the mean of the two middle ones if n is even and here w is denoted by w_g.

Thus for example, if the deviations are

$$-2{\cdot}8 \,|\, +0{\cdot}9 \,|\, +0{\cdot}4 \,|\, -0{\cdot}2 \,|\, +0{\cdot}3 \,|\, -0{\cdot}4 \,|\, +0{\cdot}1 \,|\, -1{\cdot}6 \,|$$

and they are arranged in the order

$$0{\cdot}1 \,|\, 0{\cdot}2 \,|\, 0{\cdot}3 \,|\, \underbrace{0{\cdot}4 \,|\, 0{\cdot}4} \,|\, 0{\cdot}9 \,|\, 1{\cdot}6 \,|\, 2{\cdot}8,$$

then $$w = w_g = 0{\cdot}4.$$

This method is employed frequently in the cases where a rough value of w will serve.

Remark 2. The question may be asked as to the accuracy of the value of μ, or E, or w_g.

Gauss's approximate rules are given here, and hold only when n is large; the work of Czuber may be consulted for further details.

Probably

$$\mu = \sqrt{\frac{\Sigma f^2}{n}} \text{ is true to within } \frac{0{\cdot}4769}{\sqrt{n}} \times 100 \text{ per cent}$$

$$E = \frac{\Sigma\,|f|}{n} \quad \text{,,} \quad \text{,,} \quad \frac{0{\cdot}5096}{\sqrt{n}} \times 100 \quad \text{,,}$$

$$w_g \quad \text{,,} \quad \text{,,} \quad \frac{0{\cdot}7867}{\sqrt{n}} \times 100 \quad \text{,,}$$

It is evident that the determination of w or h, from $\mu = \sqrt{\dfrac{\Sigma f^2}{n}}$, is the most accurate.

Suppose for example, from the sum of the squares of the deviations that μ has been calculated $= 30$ cm from $n = 10$ shots: then

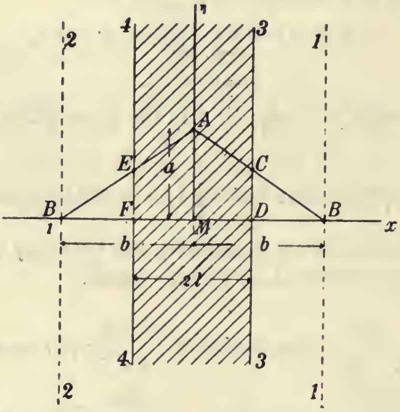
$$\mu = 30\left(1 \pm \frac{0{\cdot}4769}{\sqrt{n}}\right) = 30\,(1 \pm 0{\cdot}15) = 30 \pm 4{\cdot}5;$$

the probable limits of μ are $\pm 4{\cdot}5$ cm, while the probability is that μ is greater than $25{\cdot}5$, and less than $34{\cdot}5$ cm. It is near enough then to say that μ is about 30 cm, with a probable error of 15 °/ₒ.

Remark 3. The Gauss curve $\eta=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$ has two points of inflexion. The abscissae of the inflexions are given from $\eta''=0$, or $1-2h^2x^2=0$, $x_1=\pm\frac{1}{h\sqrt{2}}$. Therefore $x_1=\pm\mu$, and the abscissa of a point of inflexion is the mean quadratic deviation μ. It can be shown too that μ is the radius of gyration of half the area; and further that the mean deviation $E=\frac{1}{h\sqrt{\pi}}$ is the abscissa of the centre of gravity of half the area.

Remark 4. In place of the Gauss function $\eta=\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$, many other suggestions have been made, such as $\eta=\frac{a}{1+x^2}$, $y=a\left(1-\frac{x^2}{b^2}\right)$, etc. Hélie and Simpson excluded very large errors, and took two straight lines AB, AB_1, symmetrical to the vertical through M, such that $\eta=a\mp\frac{a}{b}x$.

Then $BB_1=2b$ is the breadth of the vertical strip of the target, symmetrical to M (1 1, 2 2) which contains all the shots.



The number of hits in the strip (1 1, 2 2) is

$$2n\int_{x=0}^{x=b}\left(a-\frac{a}{b}x\right)dx=n,$$

and thence

$$a=\frac{1}{b};\ \eta=\frac{1}{b}\left(1-\frac{x}{b}\right).$$

Then the number of hits in the strip of the target (3 3, 4 4) symmetrical to M, of breadth $2l$, is

$$2n\int_0^l\frac{1}{b}\left(1-\frac{x}{b}\right)dx$$

$=n$ times the area $ACDFEA=n\frac{l}{b}\left(2-\frac{l}{b}\right)$.

Put the probable deviation w in place of b, taking $l=w$; so that the number of hits $=\frac{1}{2}n$; then $\frac{w}{b}\left(2-\frac{w}{b}\right)=\frac{1}{2}$, $w=b\left(1-\frac{1}{\sqrt{2}}\right)=\frac{b}{3{\cdot}073}$.

Then the number of hits falling on the strip (33, 44) of breadth $2l$, expressed by the probability factor $\frac{l}{w}$ instead of $\frac{l}{b}$, is given by

$$N=\frac{nl}{3{\cdot}073w}\left(2-\frac{l}{3{\cdot}073w}\right)=0{\cdot}325\kappa\,(2-0{\cdot}325\kappa),$$

where $\kappa=\frac{l}{w}=\frac{2l}{2w}=\frac{\text{breadth of the target strip}}{\text{width of 50 °/₀ zone}}$.

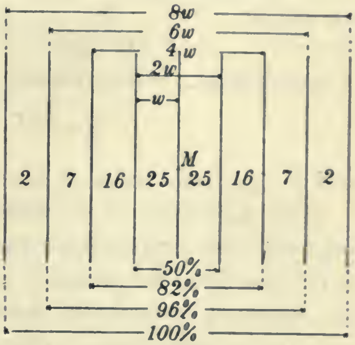
But the law of Gauss has always been considered to be better than the others.

Remark 5. Based on the law of Gauss the number of hits on a strip of the target symmetrical to the mean point of impact of breadth $2l$ is

$$N=\frac{2nh}{\sqrt{\pi}}\int_0^l e^{-h^2x^2}\,dx=\frac{2n}{\sqrt{\pi}}\int_0^{hl} e^{-t^2}\,dt=n\phi\,(hl)=n\phi\left(0{\cdot}4769\,\frac{l}{w}\right),$$

(consult Vol. IV, Table 14). It is probably better to apply the table of $\psi\left(\frac{l}{w}\right)$ in Table 15, giving immediately $N=n\psi\left(\frac{l}{w}\right)$ as a function of the so-called "probability factor" $\frac{l}{w}$.

With $n=100$, then $100\,\psi\left(\frac{l}{w}\right)$ gives the percentage of hits in the strip of breadth $2l$, when the 50 °/₀ error is w.

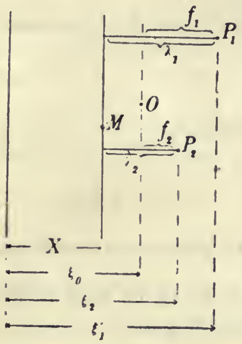


A brief extract from the table for $\psi\left(\frac{l}{w}\right)$ is given in the adjacent figure, and is of great use in practical ballistics, giving a diagram of the target strips about M of breadth $2w$, $4w$, $6w$, $8w$ $\left(\frac{2l}{2w}=1, 2, 3, 4,\right)$ and the corresponding percentage of hits.

§ 61. True and apparent deviations. Indirect measurement of ballistic values.

It has hitherto been assumed that the true position of the mean point of impact is known. If we speak of deviations to right and left, this assumes that we know X, the abscissa of the mean point, M, about which the hits are arranged: the deflections, f_1, f_2, f_3 of the various points of impact, P_1, P_2, P_3 are then measured from this mean point. If ξ_1, ξ_2, ξ_3 are the abscissae of the points of impact, then $\xi_1-X=f_1$, $\xi_2-X=f_2$, etc.

In point of fact, M is not known: instead of it, we take the probable mean point, O, the position of which $(\xi_0 \eta_0)$ is the mean of the points of impact, $(\xi_1 \eta_1)$, $(\xi_2 \eta_2)$, etc. Therefore ξ_0 is the arithmetic mean of the various abscissae, and is equal to $\frac{\xi_1 + \xi_2 + \xi_3 + \dots}{n}$, and the ordinate is calculated in a similar manner. If we consider equal weights to be distributed at the points of impact, then O is their centre of gravity. And O will be the most probable point of central impact, if the proposition of least squares can be considered to be experimentally true. This can be stated as follows. If n observations, $\xi_1, \xi_2, \dots$ are made to determine the value of a certain magnitude, then the most probable value, α, is such that the sum of the squares of $\alpha - \xi_1$, $\alpha - \xi_2$, etc., is a minimum. So that in this case, the abscissa is that value of α for which



$$(\alpha - \xi_1)^2 + (\alpha - \xi_2)^2 + (\alpha - \xi_3)^2 + \dots$$

is a minimum. Differentiating, we get

$$\alpha = \frac{\xi_1 + \xi_2 + \xi_3 + \dots}{n} = \xi_0:$$

and the probable ordinate is obtained in the same way.

The deviations of the different observations from the true value are called the true deviations and will be denoted by $f_1, f_2, f_3, \dots$; the deviations from the probable value or the arithmetic mean are called the apparent or probable deviations, and may be denoted by λ_1, λ_2, etc. In the present case, when reckoning, the abscissae, $\lambda_1, \lambda_2, \dots$ are the distances of the various points of impact from the vertical through O. These apparent deviations can be found by experiment, whereas the true deviations cannot be so found: usually in ballistical work, we are consequently concerned with the apparent deviations. We have to devise a method of determining the accuracy of fire from the values of these apparent deviations.

The difference between the true and apparent deviations is perhaps more clearly shown by the following example. A square, with sides exactly 16 cm long, is very carefully drawn, and the planimeter is passed over it ten times. The true value of the area is $X = 256{\cdot}0$ sq cm: the differences of the various readings were found to be f_1, f_2, etc., and the accuracy was given by

$$\mu = \sqrt{\frac{\Sigma f^2}{10}} = \sqrt{\frac{0{\cdot}961}{10}} = 0{\cdot}31 \text{ sq cm} = 0{\cdot}12\,{}^\circ\!/_\circ.$$

The arithmetic mean of the ten measurements was $\xi_0 = 256{\cdot}1$ sq cm. The differences with regard to the latter value, λ_1, λ_2, λ_3, etc. gave $\Sigma\lambda^2 = 0{\cdot}8650$. It is now a question of determining the mean quadratic error, μ, from $\Sigma\lambda^2$.

Obviously

$$f_1 = \xi_1 - X, \quad \lambda_1 = \xi_1 - \xi_0,$$
$$f_2 = \xi_2 - X, \quad \lambda_2 = \xi_2 - \xi_0,$$

etc.: therefore

$$\left.\begin{array}{l} f_1 - \lambda_1 = \xi_0 - X \\ f_2 - \lambda_2 = \xi_0 - X \\ f_3 - \lambda_3 = \xi_0 - X \end{array}\right\}, \quad \text{.........................(1)}$$

where

$$n\xi_0 = \xi_1 + \xi_2 + \xi_3 + \ldots.$$

Therefore

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots = 0.$$

By adding equations (1), we get

$$f_1 + f_2 + f_3 + \ldots = n(\xi_0 - X),$$

and therefore

$$f_1 - \lambda_1 = \frac{f_1 + f_2 + f_3 + \ldots}{n} = f_2 - \lambda_2 = \ldots,$$

or

$$\left.\begin{array}{l} \lambda_1 = f_1 - \dfrac{f_1 + f_2 + \ldots}{n} = \dfrac{n-1}{n} f_1 - \dfrac{f_2}{n} - \dfrac{f_3}{n} - \ldots \\ \lambda_2 = f_2 - \dfrac{f_1 + f_2 + \ldots}{n} = \dfrac{n-1}{n} f_2 - \dfrac{f_1}{n} - \dfrac{f_3}{n} - \ldots \\ \lambda_3 = f_3 - \dfrac{f_1 + f_2 + \ldots}{n} = \dfrac{n-1}{n} f_3 - \dfrac{f_1}{n} - \dfrac{f_2}{n} - \ldots \end{array}\right\}. \quad \text{......(2)}$$

These equations connect f_1, f_2, f_3, with $\lambda_1, \lambda_2, \lambda_3$; and we have $\Sigma\lambda = 0$, while Σf is nearly equal to zero.

In the theory of errors, the following proposition is often useful. Let y be given by the equation $y = f(x_1, x_2, x_3, \ldots)$, where y is not capable of direct measurement, while $x_1, x_2, x_3, \ldots$ can be experimentally observed. The value of y is thus found indirectly. (A case in point is the determination of the time of flight from the initial and residual charges on a condenser.) Let the errors in the determination of x_1, x_2, etc. be $\pm\, dx_1$, $\pm\, dx_2, \ldots$: the error in y is $\pm\, dy$.

Then

$$\pm\, dy = \pm \frac{\partial f}{\partial x_1} dx_1 \pm \frac{\partial f}{\partial x_2} dx_2 \pm \frac{\partial f}{\partial x_3} dx_3 + \ldots.$$

The maximum error (m') in the value of y then arises when the maximum errors, $m_1, m_2, \ldots$ arise in $x_1, x_2, \ldots$ and when these errors are all added together: this is obviously the most unfavourable case. Therefore

$$m' = + \frac{\partial f}{\partial x_1} m_1 + \frac{\partial f}{\partial x_2} m_2 + \ldots . \quad \ldots\ldots\ldots\ldots(3)$$

Generally speaking, the errors are not all in one direction, and some therefore counterbalance one another. Then the accuracy of the measurements of $x_1, x_2, x_3, \ldots$ is given by the mean quadratic errors, $\mu_1, \mu_2, \mu_3, \ldots$. Therefore the mean quadratic error in y is given by

$$\mu'^2 = \left(\frac{\partial f}{\partial x_1} \mu_1\right)^2 + \left(\frac{\partial f}{\partial x_2} \mu_2\right)^2 + \ldots . \quad \ldots\ldots\ldots\ldots(4)$$

The same is true for the probable and average errors, since $w = \mu\rho\sqrt{2}$, and $E = \mu\sqrt{\frac{2}{\pi}}$.

The validity of equation (4) can be shown from the following considerations. μ' is determined from the sum of all the squares of dy; μ_1 is found from the sum of all the squares of dx_1 and so on. But if the equation for dy is squared, we have

$$(dy)^2 = \left(\frac{\partial f}{\partial x_1} dx_1\right)^2 + \left(\frac{\partial f}{\partial x_2} dx_2\right)^2 + \ldots,$$

since the quantities

$$2\frac{\partial f}{\partial x_1} dx_1 \frac{\partial f}{\partial x_2} dx_2, \quad 2\frac{\partial f}{\partial x_1} dx_1 \frac{\partial f}{\partial x_3} dx_3, \ldots$$

cancel one another, if the number of observations is sufficiently great. If the equation for $(dy)^2$ is written out for all observations, and if all these equations are added together, we get (4) as the result. A strict proof of the formula (4) is given by Czuber in his *Theory of Probabilities*, Leipzig, 1903, No. 126.

In the special case in which y is a linear function of $x_1, x_2, x_3, \ldots$, we have $y = a_1x_1 + a_2x_2 + a_3x_3 \ldots$. Then

$$\frac{\partial f}{\partial x_1} = a_1, \quad \frac{\partial f}{\partial x_2} = a_2, \text{ etc.,}$$

and we therefore have

$$\mu^2 = (a_1\mu_1)^2 + (a_2\mu_2)^2 + \ldots \quad \ldots\ldots\ldots\ldots(5)$$

If y is the algebraic sum of $x_1, x_2, x_3, \ldots$ then

$$\left.\begin{aligned} \mu'^2 &= \mu_1^2 + \mu_2^2 + \mu_3^2 + \ldots \\ \text{and} \quad w'^2 &= w_1^2 + w_2^2 + w_3^2 + \ldots \end{aligned}\right\} . \quad \ldots\ldots\ldots\ldots(6)$$

Let us suppose that a variety of independent causes contribute to deflect the projectile: e.g., error of aim, variations in the vibration of the bore, small changes in the velocity of the wind. Then the resultant mean deflection will be the square root of the sum of the squares of the mean separate deflections. It is not equal to the sum of the separate deflections, since they partially counterbalance one another. In ballistics, the last equation (6) is generally called Didion's law.

Let us use equation (5) in conjunction with (2). The true errors, $f_1, f_2, \ldots$ have the same accuracy, given by the mean quadratic error μ: i.e., $\mu_1 = \mu_2 = \mu_3 = \ldots = \mu$. The coefficients $a_1, a_2, a_3, \ldots$ are here $\frac{n-1}{n}, -\frac{1}{n}, -\frac{1}{n}, \ldots$; the mean quadratic error in λ is μ', and is found by the same law as that for μ. Therefore we have $\mu' = \sqrt{\frac{\Sigma(\lambda^2)}{n}}$: and

$$\mu'^2 = \frac{\Sigma(\lambda^2)}{n} = \left(\frac{n-1}{n}\mu\right)^2 + \left(-\frac{\mu}{n}\right)^2 + \left(-\frac{\mu}{n}\right)^2 + \ldots$$

$$= \left\{\left(\frac{n-1}{n}\right)^2 + \frac{1}{n^2} + \frac{1}{n^2} + \ldots\right\}\mu^2$$

$$= \left\{\left(\frac{n-1}{n}\right)^2 + \frac{n-1}{n^2}\right\}\mu^2 = \frac{n-1}{n}\mu^2:$$

therefore

$$\mu = \sqrt{\frac{n}{n-1}}\,\mu' = \sqrt{\frac{\Sigma\lambda^2}{n-1}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

According to this rule, the mean quadratic error μ (and therefore w), is found by taking the apparent or probable deviations, $\lambda_1, \lambda_2, \lambda_3, \ldots$ as the basis of the calculations, instead of the arithmetic mean. This is the method which is always used in ballistics.

Further the average deflection, E, is not found from the equation $E = \frac{\Sigma(\lambda)}{n}$, but more accurately from the equation

$$E = \frac{\Sigma(\lambda)}{\sqrt{n(n-1)}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

For if $E' = \frac{\Sigma(\lambda)}{n}$, then $\mu' = E'\sqrt{\frac{\pi}{2}}$, just as previously we had

$$\mu = E\sqrt{\frac{\pi}{2}}.$$

Therefore

$$\frac{E}{E'} = \frac{\mu}{\mu'} = \sqrt{\frac{n}{n-1}}.$$

Consequently

$$E = E'\sqrt{\frac{n}{n-1}} = \frac{\Sigma(\lambda)}{n}\sqrt{\frac{n}{n-1}}$$

$$= \frac{\Sigma(\lambda)}{\sqrt{n(n-1)}}.$$

Also $w = w_g'\sqrt{\frac{n}{n-1}}$, where w_g' corresponds to w_g for the apparent deflections.

The accuracy of the values for μ, E, and w is given by the probable limits. They are here stated without proof, in accordance with Helmert's methods.

The probable limits for μ are

$$\sqrt{\frac{\Sigma\lambda^2}{n-1}}\left\{1 \pm \sqrt{2}\,\rho\sqrt{2 - \frac{\Gamma\left(\frac{n}{2}\right)\sqrt{\frac{8}{n-1}}}{\Gamma\left(\frac{n-1}{2}\right)}}\right\}:$$

n is the number of experiments: $\rho = 0{\cdot}476936$: for the value of Γ, see Vol. IV, Table No. 18: or approximately for values of n greater than 10 the limits are

$$\sqrt{\frac{\Sigma\lambda^2}{n-1}}\left[1 \pm \frac{\rho}{\sqrt{n-1}}\right].$$

Probable limits of E are

$$\frac{\Sigma\lambda}{\sqrt{n(n-1)}}\left\{1 \pm \sqrt{2}\,\rho\sqrt{\frac{1}{n}\left(\frac{\pi}{2} + \sqrt{n(n-2)} - n + \sin^{-1}\frac{1}{n-1}\right)}\right\},$$

or approximately for large values of n,

$$\frac{\Sigma\lambda}{\sqrt{n(n-1)}}\left\{1 \pm \rho\sqrt{\frac{\pi-2}{n-1}}\right\};$$

probable limits of w_g' are

$$w_g'\sqrt{\frac{n}{n-1}}\left\{1 \pm \frac{0{\cdot}7867}{\sqrt{n-1}}\right\}.$$

The terms, which follow the $\pm$ sign, if multiplied by 100, give the probable accuracy, expressed as a percentage.

The values of $\Sigma\lambda^2$, and μ can be obtained from the results of observation, e.g., from the values of ξ_1, ξ_2, ...; as can be seen from the first part of § 61, we have

$$\Sigma(\lambda^2) = \Sigma(\xi^2) - \frac{1}{n}(\Sigma\xi)^2,$$

which is Jordan's formula. The differences of these direct observations, d, are discussed in § 62, and

$$\Sigma(\lambda^2) = \Sigma(d^2) - \frac{1}{n}(\Sigma d)^2,$$

which is Wellisch's formula. Kozàk's method is referred to in the notes to § 58 to § 70: it depends on the so-called observation-residues.

§ 62. Successive Differences.

For the purposes of ballistics, there is another measure of accuracy, which is very important. One starts from the original observations: for instance, when it is a question of the deviation of the projectiles to right and left, the distances ($\xi_1, \xi_2, \xi_3, \ldots$) of the separate points of impact from the left edge of the target form the starting points: or when it is a case of deviation in respect of range, we start from the ranges, which are observed. Thus starting from $\xi_1, \xi_2, \xi_3, \ldots$ we take the successive differences, $\xi_1 - \xi_2 = d_1$, $\xi_2 - \xi_3 = d_2$, etc., without any regard to their signs, and reckon the average value of these differences, the number of which is s. Thus $E_d = \frac{\Sigma(d)}{s}$. If the observations are taken in their normal order, we have $s = n-1$, and we have $n-1$ independent differences from n observations. If nothing is known as to the order of the observations, then they are all used, and $s = \frac{n(n-1)}{2}$. Then the average deviation E is calculated from the equation

$$E = \frac{1}{\sqrt{2}} \frac{\Sigma(d)}{s} = \frac{E_d}{\sqrt{2}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Therefore $w = E\rho\sqrt{\pi} = 0{\cdot}5978 \frac{\Sigma(d)}{s}$.

In order to explain this, let us start with equation (5), and suppose that y is a linear function of two other quantities, x_1 and x_2. Therefore $y = a_1x_1 + a_2x_2$. Let the mean quadratic error for x_1 be μ_1 and for x_2 be μ_2. (Or the average error may be E_1 for x_1 and E_2 for x_2.) Then the mean quadratic error for y will be μ', or the average error will be E', where $\mu'^2 = a_1^2\mu_1^2 + a_2^2\mu_2^2$, or $E'^2 = a_1^2E_1^2 + a_2^2E_2^2$, since $E = \mu\sqrt{\frac{2}{\pi}}$. In the present case, it is a question of finding the differences $d_1 = \xi_1 - \xi_2$, etc. Let all the points of impact be measured with equal accuracy from the left edge of the disc. Therefore $E_1 = E_2 = E$. And in this case, $a_1 = 1$, and $a_2 = -1$. The average distance is $E' = \frac{\Sigma(d)}{s}$, and therefore

$$\frac{\Sigma(d)}{s} = \sqrt{(+E)^2 + (-E)^2} = \sqrt{2}\,E, \text{ or } E = \frac{\Sigma(d)}{\sqrt{2}\,s}.$$

According to Helmert the accuracy is given by the following probable limits, viz.

$$\frac{\Sigma(d)}{s}\left\{1 \pm \sqrt{2}\,\rho\sqrt{\frac{\frac{\pi}{3}(n+1) + 2\sqrt{3}(n-2) - 4n + 6}{n(n-1)}}\right\},$$

where n is the number of observations and $\rho = 0{\cdot}4769$.

If $n = 10$, then the probable deviation, w, (or $2w$, the so-called 50 per cent zone), is found from the mean quadratic deviation, μ, or from the average value of E, or from the differences in the observations, to degrees of accuracy which are in the following respective proportions, viz., 0·159; 0·170; 0·163. Thus the method of determining from μ is the most accurate, then that from $\frac{\Sigma(d)}{s}$, and finally that from the average deviation E.

The advantage of calculating the accuracy from the successive differences lies not only in the greater exactness of the results, as compared with the calculation from the average deviation, and in the greater ease with which the results can be obtained. The main point is that there are fewer cases in which it fails to give results. For instance, it might happen that during the tests some variable disturbance is at work, causing the mean point of impact to alter its position, or effecting a continuous change of the arithmetic mean. Thus the temperature might rise, or the bore become heated, or the velocity of the wind might change. In such a case the method of calculating from the arithmetic mean would give no results, and unless the whole series of observations is to be abandoned, recourse must be had to the method of successive differences. Vallier has pointed out that the calculation of the probable deviation from the successive differences is independent of any change in position of the mean point of impact. R. von Eberhard has given the general proof of this fact.

It is possible to determine whether there is any disturbing cause in a series of observations by the following process. The probable error is determined both from the average deviation and by the method of successive differences. The two values of w, found by these methods, must be approximately in agreement: that is, the ratio of $\frac{\Sigma(d)}{\sqrt{2}\,s}$ to $\frac{\Sigma(\lambda)}{\sqrt{n(n-1)}}$ must be nearly equal to unity. If the value of this ratio differs by more than 20 per cent from unity, i.e., if the value does not lie between 0·8 and 1·2, then it is probable that the mean point of impact changes its position, and that there is some disturbing cause at work.

By the method of successive differences we calculate h from the deviations with respect to the arithmetic mean. But if the arithmetic mean is determined from a few observations, there is still a considerable likelihood of error: the question is as to the probable accuracy of the mean value.

Suppose the quantities $\xi_1, \xi_2, \xi_3, \ldots$ are observed directly, and the mean is $\xi_0 = \frac{\xi_1 + \xi_2 + \ldots}{n}$.

The quantities $\xi_1, \xi_2, \xi_3, \ldots$ are all supposed to be measured with the same accuracy, with the mean quadratic error μ.

If equation (5) is applied to this case, and it is noticed that $\xi_0 = \frac{1}{n}\xi_1 + \frac{1}{n}\xi_2 + \ldots$, so that here $a_1 = a_2 = \ldots = \frac{1}{n}$, then the mean quadratic error of ξ_0 is given by

$$M^2 = \left(\frac{1}{n}\mu\right)^2 + \left(\frac{1}{n}\mu\right)^2 + \ldots = \frac{\mu^2}{n},$$

and so the mean quadratic error M of the arithmetic mean is

$$M = \frac{\mu}{\sqrt{n}}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Therefore also the probable error is $W = \frac{w}{\sqrt{n}}$, supposing μ and w to be the corresponding errors of the individual measurements.

While μ and w are independent of the number of observations, if these are numerous enough for the fundamental laws of the theory of probability to be applied, M and W diminish as the reciprocal of the square root of the number of observations, and for $n = 1, 2, 10, 20, 100, \ldots$, the values of $\frac{1}{\sqrt{n}}$ are respectively 1, 0·71, 0·32, 0·22, 0·1, ...

The accuracy of the arithmetic mean increases at first very rapidly with n, but later the gain is slower. On that ground, not more than 10 to 15 observations are employed.

Examples. 1. In the Ballistic Laboratory a measurement of the same time interval t by means of the Condenser-Chronoscope was made 10 times,

$$t = \frac{WC}{10^6}(\log \sin \tfrac{1}{2}a_0 - \log \sin \tfrac{1}{2}a),$$

W is the resistance of the circuit of discharge, $\log W = 2{\cdot}69110$; C is the capacity of the condenser, $\log C = 0{\cdot}69897$; a_0 is the galvanometer deviation before the experiment, a after the breaking of both circuits.

In 10 measurements the mean of $\sin \frac{1}{2}a_0$ was found to be 0·039886, with a mean quadratic error $\mu_1^2 = 0{\cdot}\overset{(9)}{0}8546$; and the mean of $\sin \frac{1}{2}a = 0{\cdot}01214$, and $\mu_2^2 = 0{\cdot}\overset{(8)}{0}4423$.

The mean quadratic error μ for a single determination of the time interval t, as given by (4), is

$$\mu = \frac{WC}{10^6}\sqrt{\left[\left(\frac{\mu_1}{\sin \frac{1}{2}a_0}\right)^2 + \left(\frac{\mu_2}{\sin \frac{1}{2}a}\right)^2\right]}.$$

Therefore $\mu = 0{\cdot}0000014$ sec, or $w = 0{\cdot}00000093$ sec.

The same value of μ must be obtained from the separate measurements of t.

The mean of the 10 experiments was found to give $t=0{\cdot}0002975$; and from the differences of the single calculated values of t from the mean value, the value of μ was obtained $=0{\cdot}0000014$ sec or 0·47 °/ₒ of the measured time t, which was about the time of flight of the S bullet over a distance of 25 cm.

2. A time-difference of about 0·016 second was measured repeatedly by the Boulengé apparatus. The reading of the time-measuring rod was made by vernier and magnifying glass, so that it could be obtained within 0·01 mm.

The greatest possible error of reading of the disjunction-mark amounted to 0·05 mm, corresponding to a time-difference of $\pm 0{\cdot}000034$ sec. In the measurement of the time-mark the error of reading might reach a maximum of 0·05 mm, corresponding to a time-difference of $\pm 0{\cdot}000031$ sec.

Thus the greatest possible error in the measurement of the time will be reached when both parts of the errors add together; and then it is

$$+0{\cdot}000034+0{\cdot}000031=0{\cdot}000065 \text{ sec}:$$

while the mean quadratic error gave 0·000057 sec.

3. The resultant dispersion in range a in shrapnel fire is a combination of the longitudinal dispersion b due to the burst and of the longitudinal dispersion c of the points of impact. Then if a, b, c represent 50 °/ₒ zones, $a=\sqrt{(b^2+c^2)}$.

§ 63. Recapitulation of results.

If n single observations are carried out, the arithmetic mean of the observations is taken, and the deviation of each individual observation from this mean is also found.

Denote these deviations, regardless of sign, by $\lambda_1, \lambda_2, \lambda_3, \ldots$.

(*a*) The so-called *average deviation*, which is called the *mean deviation* in the German Army, is

$$E=\frac{\lambda_1+\lambda_2+\lambda_3+\ldots}{\sqrt{(n \,.\, n-1)}}.$$

(*b*) The so-called *mean quadratic deviation* of a single observation is

$$\mu=\sqrt{\frac{\lambda_1{}^2+\lambda_2{}^2+\ldots}{n-1}}.$$

(*c*) In the case where the observations have been recorded in their correct order, the successive differences are written down; there are $n-1$ of these differences, denoted in order by d_1, d_2, d_3, ... irrespective of sign; denote their arithmetic mean by

$$D=\frac{d_1+d_2+d_3+\ldots}{n-1}.$$

(*d*) Then the probable or 50 °/ₒ deviation w of a single observation is given

most exactly by $w=0{\cdot}6745\mu$,
less exactly by $w=0{\cdot}5978D$,
still less accurately by $w=0{\cdot}8453E$.

(*e*) The probable or 50°/₀ zone s_{50} (called the *mean zone* in the German Army) is the double of w, and is given

most exactly by $s_{50} = 1{\cdot}3490\mu$,
less exactly by $s_{50} = 1{\cdot}1956D$,
still less accurately by $s_{50} = 1{\cdot}6906E$.

(*f*) The probable error of the arithmetic mean is

$$W = \frac{w}{\sqrt{n}}.$$

Example. The velocity v_{25} was measured by a Boulengé apparatus over a measured distance of 50 cm from the muzzle, for 10 shots.

(*a*) Average deviation

$$E = \frac{22}{\sqrt{90}} = \pm 2{\cdot}3 \text{ m/sec} = \pm 0{\cdot}27\ \%.$$

(*b*) Mean quadratic deviation

$$\mu = \sqrt{\frac{61.24}{10-1}} = \pm 2{\cdot}6 \text{ m/sec} = \pm 3\ \%.$$

Number of test	Velocity v_{25} (m/sec)	Difference from the mean value λ	Square of difference λ^2	Successive differences d
1	863·0	3·7	13·69	
				5·1
2	857·9	1·4	1·96	
				1·5
3	859·4	0·1	0·01	
				2·2
4	857·2	2·1	4·41	
				2·1
5	855·1	4·2	17·64	
				2·5
6	857·6	1·7	2·89	
				3·9
7	861·5	2·2	4·84	
				0·8
8	862·3	3·0	9·00	
				0·8
9	861·5	2·2	4·84	
				3·6
10	857·9	1·4	1·96	
	mean =859·3	sum =22·0	sum =61·24	sum =22·5

(*c*) $D = \frac{22{\cdot}5}{9} = 2{\cdot}51.$

(*d*) 50 °/₀ deviation w

from μ, $w = 0{\cdot}6745 \times 2{\cdot}6 = 1{\cdot}7$ m/sec.
" D, $w = 0{\cdot}5978 \times 2{\cdot}51 = 1{\cdot}5$ "
" E, $w = 0{\cdot}8453 \times 2{\cdot}3 = 1{\cdot}9$ "

(e) 50 °/ₒ deviation

from μ, $s_{50}=3\cdot5$ m/sec.
" D, $s_{50}=3\cdot0$ "
" E, $s_{50}=3\cdot9$ "

§ 64. Calculation of the arithmetic mean in the case of grouped observations.

Suppose for example a definite time-interval is measured by several Boulengé instruments $A, B, C, \ldots$, of different accuracies; 10 times by A, 15 times by B, 9 times by $C, \ldots$; and suppose the mean values obtained by each apparatus are $x_1, x_2, x_3, \ldots$.

Then it is evident from the preceding that it is not permissible to take the most likely value of x as $\frac{x_1+x_2+\ldots}{n}$, because the accuracy of the single mean values x_1, x_2, x_3 is not the same, in consequence of differences in the apparatus, and also in the number of experiments.

The individual arithmetic means $x_1, x_2, x_3, \ldots$ must be multiplied by certain factors $p_1, p_2, p_3, \ldots$, before being added, and then divided by the sum of these factors; so that

$$x=\frac{x_1 p_1+x_2 p_2+\ldots}{p_1+p_2+\ldots}. \quad \ldots\ldots\ldots\ldots(1)$$

The same holds for any ballistic measurement whatever, such as gas pressure, velocity, mean point of impact on the target, and so forth; and the question arises of determining the values of $p_1, p_2, p_3, \ldots$.

(a) The accuracy of each measurement being assumed to be the same in all groups, with a mean quadratic error μ for each measurement, suppose only the number $n_1, n_2, n_3, \ldots$ of experiments to be different, from which the mean values $x_1, x_2, x_3, \ldots$ are determined.

Moreover suppose all the measurements independent of each other. In this case the mean x of the whole is

$$x=\frac{n_1 x_1+n_2 x_2+\ldots}{n_1+n_2+\ldots}. \quad \ldots\ldots\ldots\ldots(2)$$

Therefore the factors are $p_1=n_1$, $p_2=n_2, \ldots$.

The mean quadratic error for x is $\frac{\mu}{\sqrt{(p_1+p_2+\ldots)}}$.

(b) Suppose the number of experiments $n_1, n_2, n_3, \ldots$ in each group to be the same, $n_1=n_2=\ldots$, but the accuracies to be different; so that μ_1 is the mean quadratic error in the first group, μ_2 in the second, and so on.

Then the mean x of the whole is to be taken as

$$x = \frac{p_1 x_1 + p_2 x_2 + \dots}{p_1 + p_2 + \dots}, \quad \dots\dots\dots\dots\dots\dots\dots(3)$$

where $p_1 = \frac{1}{\mu_1^2}$, $p_2 = \frac{1}{\mu_2^2}$, ...

The same procedure arises when $x_1, x_2, x_3, \dots$ are not separate mean values, but the results of individual observations made with different degrees of accuracy. Then the mean quadratic error μ for x is given by

$$\frac{1}{\mu^2} = \frac{1}{\mu_1^2} + \frac{1}{\mu_2^2} + \dots, \quad \dots\dots\dots\dots\dots\dots\dots\dots(4)$$

or if $\frac{1}{\mu^2}$ denotes the factor p of the mean x, then $p = p_1 + p_2 + \dots$.

(*c*) Suppose the accuracy in the various groups to be different, with mean quadratic error $\mu_1, \mu_2, \mu_3, \dots$, as well as the number of experiments $n_1, n_2, n_3, \dots$, from which the means $x_1, x_2, x_3, \dots$ are determined, then in (*b*), $\mu_1, \mu_2, \mu_3, \dots$ must be replaced by $\frac{\mu_1}{\sqrt{n_1}}, \frac{\mu_2}{\sqrt{n_2}}, \dots$, and the mean x is given by

$$x = \frac{p_1 x_1 + p_2 x_2 + \dots}{p_1 + p_2 + \dots}, \quad \dots\dots\dots\dots\dots\dots\dots(5)$$

and

$$p_1 = \frac{n_1}{\mu_1^2}, \quad p_2 = \frac{n_2}{\mu_2^2}, \quad \dots.$$

Proof of these propositions.

(*a*) Suppose x_1 to be the mean of two equally accurate observations y_1 and y_2, so that $y_1 + y_2 = 2x_1$; suppose moreover x_2 to be the mean of three observations, equally accurate with y_1 and y_2, so that $y_3 + y_4 + y_5 = 3x_2$; then the resultant mean

$$x = \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5} = \frac{2x_1 + 3x_2}{2 + 3},$$

and the proposition is evident. Then if μ is the constant mean error of a single measurement, the error of the resultant mean x is μ divided by the square root of the number of measurements, as in § 63; so that here it

$$= \frac{\mu}{\sqrt{5}} = \frac{\mu}{\sqrt{(p_1 + p_2)}}.$$

(*b*) Given the three means x_1, x_2, x_3, derived from n measurements; let x_1 be found with the mean error μ_1 of a single measurement, x_2 with μ_2, and x_3 with μ_3. The mean error of the mean x_1 is then $\frac{\mu_1}{\sqrt{n}}$.

On the other hand, suppose the observations, as in (*a*), to be made with equal accuracy and the same mean error μ, and suppose the number of observations in the groups to be different, p_1 for x_1, p_2 for x_2, and so on.

Then according to (a) $x=(p_1x_1+p_2x_2+p_3x_3):(p_1+p_2+p_3)$. The mean error of x_1 in this case is clearly $\frac{\mu}{\sqrt{p_1}}$, of x_2 is $\frac{\mu}{\sqrt{p_2}}, \ldots$; so that $\frac{\mu_1}{\sqrt{n}}=\frac{\mu}{\sqrt{p_1}}$, or $p_1=\frac{n\mu^2}{\mu_1^2}$; $p_2=\frac{n\mu^2}{\mu_2^2}, \ldots$; and thus

$$x=\left(\frac{n\mu^2}{\mu_1^2}x_1+\frac{n\mu^2}{\mu_2^2}x_2+\frac{n\mu^2}{\mu_3^2}x_3\right):\left(\frac{n\mu^2}{\mu_1^2}+\frac{n\mu^2}{\mu_2^2}+\frac{n\mu^2}{\mu_3^2}\right)$$

$$=\frac{\frac{1}{\mu_1^2}x_1+\overbrace{\frac{1}{\mu_2^2}x_2+\frac{1}{\mu_3^2}x_3}}{\frac{1}{\mu_1^2}+\underbrace{\frac{1}{\mu_2^2}+\frac{1}{\mu_3^2}}}$$

which is the same expression as given above.

Since n does not occur here, it is evident that the result holds for $n=1$, and for the case where $x_1, x_2, x_3, \ldots$ are not mean values, but individual cases, measured with the various accuracies expressed by $\mu_1, \mu_2, \mu_3, \ldots$.

Let us imagine again the observation x_2, to be made with mean error μ_2, and x_3 with μ_3, and their combined mean x' to be found, as shown by the brackets in the equation. Let this mean have the error μ' and the factor p'.

Then the resultant mean is

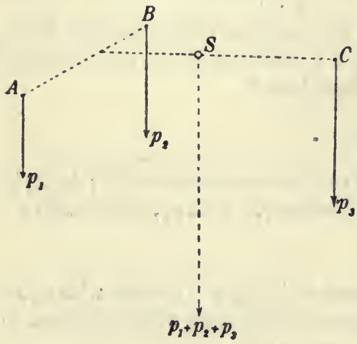
$$x=\frac{\frac{1}{\mu_1^2}x_1+\frac{1}{\mu'^2}x'}{\frac{1}{\mu_1^2}+\frac{1}{\mu'^2}},$$

so that

$$\frac{1}{\mu'^2}x'=\frac{1}{\mu_2^2}x_2+\frac{1}{\mu_3^2}x_3, \text{ and } p'=p_2+p_3, \text{ or } \frac{1}{\mu'^2}=\frac{1}{\mu_2^2}+\frac{1}{\mu_3^2},$$

$$x'=\left(\frac{1}{\mu_2^2}x_2+\frac{1}{\mu_3^2}x_3\right):\left(\frac{1}{\mu_2^2}+\frac{1}{\mu_3^2}\right).$$

This gives equation (4).



The proposition may be compared with an analogy in Mechanics.

The equation

$$x=(p_1x_1+p_2x_2+p_3x_3):(p_1+p_2+p_3)$$

is similar to the equation of moments for the calculation of the abscissa of the centre of gravity S of three particles A, B, C with abscissae x_1, x_2, x_3, and weights p_1, p_2, p_3 which may be concentrated as a single weight $p_1+p_2+p_3$ at the centre of gravity S.

Example. The same time-element of about 0·016 second was measured 50 times by six different Chronographs; and the mean value and the mean quadratic error μ with respect to the mean were measured in each group. It was found that in the

Condenser chronograph,	the mean value of	$x_1 = 0{\cdot}016315$,	$\mu_1 = 0{\cdot}000016$;
Spark chronograph,	„ „	$x_2 = 0{\cdot}016480$,	$\mu_2 = 0{\cdot}000128$;
Tuning fork chronograph,	„ „	$x_3 = 0{\cdot}016552$,	$\mu_3 = 0{\cdot}000134$;
Boulengé apparatus *A*	„ „	$x' = 0{\cdot}016339$,	$\mu' = 0{\cdot}000055$;
„ „ *B*	„ „	$x'' = 0{\cdot}016398$,	$\mu'' = 0{\cdot}000057$;
„ „ *C*	„ „	$x''' = 0{\cdot}016575$,	$\mu''' = 0{\cdot}000143$.

As the three Boulengé apparatus are not essentially different, they are treated as a single apparatus, so that finally the resultant mean is to be calculated from four groups only.

The mean x_4 of x', x'', x''' and the value of its μ_4 are thus first to be calculated. According to the above

$$x_4 = \frac{\dfrac{0{\cdot}016339}{(0{\cdot}000055)^2} + \dfrac{0{\cdot}016398}{(0{\cdot}000057)^2} + \dfrac{0{\cdot}016575}{(0{\cdot}000143)^2}}{\dfrac{1}{(0{\cdot}000055)^2} + \dfrac{1}{(0{\cdot}000057)^2} + \dfrac{1}{(0{\cdot}000143)^2}} = 0{\cdot}016382 \text{ sec};$$

$$\frac{1}{\mu_4^2} = \frac{1}{\mu'^2} + \frac{1}{\mu''^2} + \frac{1}{\mu'''^2}, \quad \mu_4 = 0{\cdot}0000381 \text{ sec.}$$

Then to calculate the resultant mean x with the use of $x_4 = 0{\cdot}016382$, and $\mu_4 = 0{\cdot}0000381$, the mean of the six groups would be found, treating the three Boulengés as separate instruments; but they are to be treated as a single apparatus, with $\mu_4 = 0{\cdot}0000381 \times \sqrt{3} = 0{\cdot}000066$ sec.

The resultant mean is therefore

$$x = \frac{\dfrac{0{\cdot}016315}{16^2} + \dfrac{0{\cdot}016480}{128^2} + \dfrac{0{\cdot}016552}{134^2} + \dfrac{0{\cdot}016382}{66^2}}{\dfrac{1}{16^2} + \dfrac{1}{128^2} + \dfrac{1}{134^2} + \dfrac{1}{66^2}} = 0{\cdot}01632 \text{ sec.}$$

(see Vol. IV, § 144).

§ 65. Investigation of a series of observations. Axes of symmetry of a target-diagram.

In a definite series of ballistic observations, the deviations $\lambda_1, \lambda_2, \lambda_3, \ldots$ from the mean value must be divided into groups, in order to see how many lie, for instance, between 0 and 2 cm, between 2 and 4, and so on; and then μ or w must be calculated; thence we know the number that should lie in each interval, according to the law of Probable Errors.

If the two error-curves are in sufficient agreement, it can be inferred that no disturbance is present.

Usually however the number of experiments is too small for this procedure to be possible; in such a case the following criterion is generally employed.

(*a*) After the numbers of the experiments have been recorded in the order in which the tests were made, and the sequence of the different deviations has been determined, the number of positive and negative deviations should prove nearly equal, as well as the number of sequences of change of sign, $+-$ or $-+$, and $++$, $--$.

(*b*) It is more important that the calculation of the probable error w from the mean quadratic μ, average E, and successive differences d should give about the same value from the equations

$$w = 0{\cdot}6745\sqrt{\frac{\Sigma\lambda^2}{n-1}} = 0{\cdot}8453\,\frac{\Sigma|\lambda|}{\sqrt{(n\,.\,n-1)}} = 0{\cdot}5978\,\frac{\Sigma|d|}{s}.$$

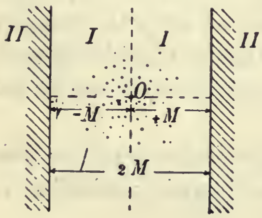
If this is not the case, and if the two last determinations of w are so related that their ratio differs from 1 by more than 20 %, the suspicion arises that a disturbing cause is present.

As stated already, w should be determined in this case from the successive differences, by $w = 0{\cdot}5978\,\frac{\Sigma|d|}{s}$, unless it is preferred to reject the whole series of observations.

(*c*) The question is of particular importance as to whether an observation with a numerical value of λ of extraordinary amount should be rejected.

Gauss's law only excludes deviations of infinite amount, and it is therefore to be expected that in practice certain arbitrary limitations will be necessary. Many men object to any system which allows an observation to be rejected for any cause. Airy, Bessel, and Faye took this view. Others allow the exclusion of a reading if there was anything suspicious in connection with the circumstances under which the observation was taken. But in connection with firing tests, it seems necessary to reserve the right to exclude any figure which deviates from the mean by an excessive amount.

Many rules have been proposed; in particular by Bertrand, B. Pierce (with Tables by Gould and Chauvenet), by Chauvenet himself, Stone, Vallier, Heydenreich, and Mazzuoli.



The rule of Chauvenet refers to the calculation of the maximum deviation M of a series of observations. Suppose w to be the probable error or $2w$ the 50 % zone, $\pm M$ the greatest error occurring, or $2M$ the extreme dispersion. Then the probability that a shot falls in the region I, I, is given by

$$\frac{h}{\sqrt{\pi}}\int_{x=-M}^{x=+M} e^{-h^2x^2}\,dx = \frac{2}{\sqrt{\pi}}\int_{t=0}^{t=Mh} e^{-t^2}\,dt = \phi(Mh) = \phi\left(0{\cdot}4769\frac{M}{w}\right) = \psi\left(\frac{M}{w}\right),$$

where ψ is found in Vol. IV, Table 15.

Thus $1-\psi\left(\frac{M}{w}\right)$ is the probability that a shot falls outside these limits, that is in the shaded regions II, II; for n shots this is $n\left[1-\psi\left(\frac{M}{w}\right)\right]$. Then if the condition is assumed that this last expression $=1$, the equation $n\left[1-\psi\left(\frac{M}{w}\right)\right]=1$ determines the maximum deviation or error.

To obtain a law, Chauvenet argues as follows. If this number is less than $\frac{1}{2}$, an error of magnitude M is on the whole unlikely.

The equation

$$n\left[1-\psi\left(\frac{M}{w}\right)\right]=\frac{1}{2}, \text{ or } \psi\left(\frac{M}{w}\right)=\frac{2n-1}{2n} \quad \text{.........(1)}$$

decides the limits.

Suppose for example the number of trials was 10, and the 50 °/$_\circ$ zone $2w=4$ cm, $w=2$, then M is determined from

$$\psi\left(\frac{1}{2}M\right)=\frac{20-1}{20}=0{\cdot}95\,;$$

according to Table 15, $\frac{M}{2}=2{\cdot}92$, $M=5{\cdot}84$ cm; this means that if among the deviations of a single observation from the arithmetic mean an error is found greater than 5·84 cm, this shot must be rejected.

Vallier replaces the equation (1) by

$$\psi\left(\frac{M}{w}\right)=\frac{n^2-1}{n^2}, \quad \text{........................(2)}$$

and agrees with Chauvenet only for $n=4$ and 5. He rejects less than Chauvenet when n has higher values than 6.

Mazzuoli has lately employed the equation

$$\psi\left(\frac{M}{w}\right)=\frac{n-1}{n}, \quad \text{..........................(3)}$$

for the direct construction of rules of rejection, and says that it gives good results.

B. Pierce, on the basis of theoretical considerations, arrives at rules very similar to those of Chauvenet, but they differ as to whether 1, or 2, or 3,... extreme deviations are to be excluded.

The theoretical discussions of Stone lead to the conclusion that different rules must hold, according to the nature of the observations and the observer in question.

Heydenreich treats a shot as rejected when its deviation is greater than would be expected once in $2(n-1)$ shots; the condition is then

$$2(n-1)\left[1-\psi\left(\frac{M}{w}\right)\right]=1, \quad \psi\left(\frac{M}{w}\right)=\frac{2n-3}{2n-2}. \quad \ldots\ldots(4)$$

The different rules are collected together in the following. An observation is rejected when the deviation from the arithmetic mean is greater than κ times the probable error w, where w = half the 50 °/ₒ zone.

Number of shots n	1. Chauvenet κ	2. Pierce (for 1 rejection) κ	3. Vallier κ	4. Heydenreich κ	5. Mazzuoli κ
3	—	1·80	—	—	1·46
4	2·27	2·05	as in 1 { 2·27	—	1·73
5	2·43	2·24	2·43	2·76	1·91
6	2·56	2·39	3·25	2·91	2·05
7	2·66	2·51	3·45	3·03	2·18
8	2·77	2·61	3·60	3·12	2·28
9	2·83	2·70	3·69	3·20	2·36
10	2·92	2·78	3·84	3·27	2·44
12	3·02	2·92	4·00	3·37	2·58
20	3·33	3·27	4·49	3·64	2·91

These figures differ very materially.

The numbers of Chauvenet appear to the author to be the best, but it would be better to test the figures by comparing the results obtained from a variety of ballistical tests. In this way it would be possible to arrive at some sort of final judgment. See also § 66.

Numerical example. Measurement of the velocity of a gun, at 40 m from the muzzle. Twelve observations were made.

Mean quadratic error of a single measurement

$$\mu=\sqrt{\frac{\Sigma\lambda^2}{n-1}}=\sqrt{\frac{17\cdot65}{12-1}}=1\cdot26=1\cdot26\left(1\pm\frac{0\cdot4769}{\sqrt{11}}\right)$$

$$=1\cdot26\pm0\cdot18=1\cdot08 \text{ to } 1\cdot44$$

$$= \text{about } 1\cdot3 \text{ m/sec, or } 0\cdot29\ \%_0 \text{ of } v.$$

v m/sec	Deviation from mean λ		Square of deviations λ^2		Successive differences d
439·1		−2·8		7·84	
442·9	+1·0		1·0		3·8
442·2	+0·3		0·09		0·7
442·3	+0·4		0·16		0·1
442·1	+0·2		0·04		0·2
442·4	+0·5		0·25		0·3
441·5		−0·4		0·16	0·9
442·2	+0·3		0·09		0·7
441·5		−0·4		0·16	0·7
442·0	+0·1		0·01		0·5
444·2	+2·3		5·29		2·2
440·3		−1·6		2·56	3·9
Mean $v=441·9$	$\Sigma\|\lambda\|=10·3$		$\Sigma\lambda^2=17·65$		$\Sigma\|d\|=14·0$

Mean quadratic error of the result, i.e., of the mean v,

$$M=\frac{\mu}{\sqrt{n}}=\frac{1·26}{\sqrt{12}}=0·36 \text{ m/sec}=0·081\ ^\circ/_\circ \text{ of } v.$$

Average error of a single measurement

$$E=\frac{\Sigma|\lambda|}{\sqrt{(n.n-1)}}=\frac{10·3}{\sqrt{(11.12)}}=0·89 \text{ m/sec.}$$

$$\frac{\Sigma|d|}{s}=\frac{14·0}{11}=1·276; \qquad 0·5978\frac{\Sigma|d|}{s}=0·76.$$

So that,

as derived from μ, $w=0·6745\mu=0·85$;

" " E, $w=0·8453E=0·75$;

" " $\frac{\Sigma|d|}{s}$, $w=0·5978\frac{\Sigma|d|}{s}=0·76$.

The ratio of the last two determinations, $\frac{w \text{ from } (d)}{w \text{ from } E}=\frac{0·76}{0·75}$ is very nearly 1, and is between 0·8 and 1·2; and therefore there is no cause for assuming that any disturbing cause is at work.

Moreover there are five sequences of sign, and 6 changes in the values of λ. On the other hand there are 8 positive values of λ against 4 negative, and 5 positive in immediate succession—(+1·0, +0·3, +0·4, +0·2, +0·5). This arises because the first measurement 439·1 was rather small compared with the rest. As this often occurs, many are in the habit of omitting this first measurement on principle. But the question arises whether 439·1 is to be rejected or not. Since $n=12$, $w=0·85$, from the determination of w from μ, and this shot according to Chauvenet should be rejected; because

$$2·8>3·02\times 0·85 \text{ or } >2·56.$$

According to Pierce this shot is to be rejected, since

$$2{\cdot}8 > 2{\cdot}9 \times 0{\cdot}85 > 2{\cdot}48.$$

According to Vallier, the shot is not to be excluded, since

$$2{\cdot}8 \text{ is not} > 4{\cdot}00 \times 0{\cdot}85 > 3{\cdot}4.$$

According to Heydenreich this shot is just excluded, since

$$2{\cdot}8 = \sim 3{\cdot}37 \times 0{\cdot}85.$$

According to Mazzuoli, the shot should be rejected, since

$$2{\cdot}8 > 2{\cdot}58 \times 0{\cdot}85 > 2{\cdot}2.$$

On general grounds, the shot in question should obviously be rejected.

The calculations should then be repeated, and based on the other 11 measurements.

Remark. H. Rohne employs the equation $n\left[1-\psi\left(\frac{M}{w}\right)\right]=1$ to obtain the 50 °/₀ zone from the total dispersion $2M$. By a combination of theory and experiment he obtains the result: In 5, 10, 15, 20, 25, 30, 40, 50 shots the ratio of the total dispersion to the 50 °/₀ zone was respectively 1·95, 2·40, 2·59, 2·76, 2·90, 3·02, 3·12, 3·20.

Another method has been investigated by H. Rohne; he obtains the 50 °/₀ zone by excluding the less accurate half of the shots. This has been employed also by A. v. Burgsdorff and others.

§ 66. The axes of the target-diagram.

When the distribution of errors with respect to the mean value is treated in more than one dimension, as for instance in the grouping of the hits in the plane of a vertical target about the mean point of impact, or in the distribution of the bursts in space with respect to the mean point of burst, the question must be considered whether the causes of deviation in relation to the coordinate axes are independent of one another or not.

For the sake of simplicity the preceding treatment was connected with the distribution of the hits on a vertical target, and moreover the deviations along the horizontal axis Ox were the only ones that were considered.

In a similar manner the deviations may be considered, in the direction of the vertical axis.

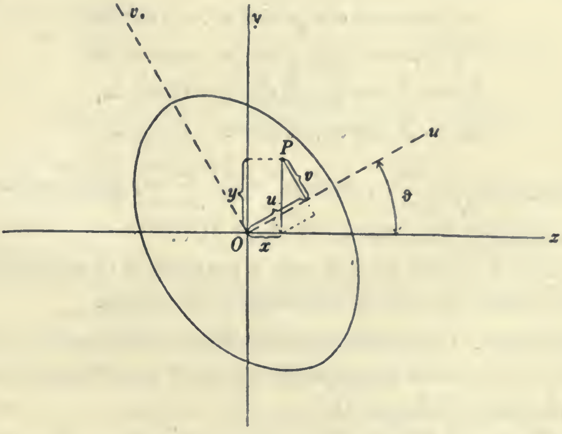
Suppose the deviations in the x direction to be denoted by x_1, x_2, $x_3, \ldots$, and in the y direction by $y_1, y_2, y_3, \ldots$, the mean quadratic error in the x direction is $\mu_1 = \sqrt{\frac{\Sigma x^2}{n-1}}$, and in the y direction is $\mu_2 = \sqrt{\frac{\Sigma y^2}{n-1}}$.

But the assumption here is that the errors in the directions of the

x axis and of the y axis may be calculated independently; and this is the case when the target-diagram is symmetrical in relation to both axes.

When this is the case, in a sufficiently large number of shots there will always be points $(+4, +5)$, $(-4, +5)$, $(+4, -5)$, and $(-4, -5)$. In consequence of this symmetry $\Sigma xy = 0$.

Conversely, the numerical value of Σxy will determine whether symmetry exists with respect to a system of coordinate axes, chosen arbitrarily. If the value of Σxy is distinctly different from zero, then



in strictness the coordinate system must be turned so as to make the new coordinate system of u and v possess this symmetry, and so have $\Sigma uv = 0$.

The question as to the angle α, through which the (xy) coordinate system must be turned, has now to be considered.

The considerations are the same as those arising in mechanics and analytical geometry, when it is a question of placing the coordinate axes along the principal axes of a body, or in the direction of the axes of a conic section.

With the horizontal direction for x and the vertical for y, let $\Sigma x^2 = A$, $\Sigma y^2 = B$, $\Sigma xy = C$.

If we turn the axes through the angle θ,

$$u = x\cos\theta + y\sin\theta, \quad v = -x\sin\theta + y\cos\theta,$$

$$uv = -\sin\theta\cos\theta\,(x^2 - y^2) + (\cos^2\theta - \sin^2\theta)\,xy,$$

and

$$\Sigma uv = -\tfrac{1}{2}(A - B)\sin 2\theta + C\cos 2\theta.$$

When the u, v axes are the axes of symmetry, $\Sigma uv = 0$; the value of θ determined from this condition is denoted by α, so that

$$0 = -\tfrac{1}{2}(A - B)\sin 2\alpha + C\cos 2\alpha, \quad \tan 2\alpha = \frac{2C}{A - B}.$$

After the coordinate system has been turned through the angle α, all deviations are referred to the new (u, v) axes.

But it is not necessary to repeat the calculations when it is only required to obtain the mean quadratic errors μ' and μ'' with respect to the new axes; because

$$u^2 = x^2\cos^2\alpha + y^2\sin^2\alpha + xy\sin 2\alpha,$$
$$v^2 = x^2\sin^2\alpha + y^2\cos^2\alpha - xy\sin 2\alpha;$$
$$\Sigma u^2 = A\cos^2\alpha + B\sin^2\alpha + C\sin 2\alpha,$$
$$\Sigma v^2 = A\sin^2\alpha + B\cos^2\alpha - C\sin 2\alpha,$$

so that since $\mu' = \sqrt{\dfrac{\Sigma u^2}{n-1}}$, and $\mu'' = \sqrt{\dfrac{\Sigma v^2}{n-1}}$, their values are obtained immediately from those of A, B, C, α.

The values of μ' and μ'' are now a maximum or minimum, a relation equally useful for the determination of $\tan 2\alpha$.

Recapitulation. Consider the deviations with respect to horizontal and vertical axes, drawn through the point of mean impact, and calculate $\Sigma x^2 = A$, $\Sigma y^2 = B$, $\Sigma xy = C$.

If C is distinctly different from zero, this shows that the axes of symmetry are inclined to the horizontal at an angle given by

$$\tan 2\alpha = \frac{2C}{A - B}.$$

The mean quadratic deviations, μ' and μ'', with respect to the true axes are calculated from

$$(n-1)\mu'^2 = A\cos^2\alpha + B\sin^2\alpha + C\sin 2\alpha,$$
$$(n-1)\mu''^2 = A\sin^2\alpha + B\cos^2\alpha - C\sin 2\alpha.$$

Examples. Target practice of a rifle, 6 mm calibre, at a range of 1500 m.

The 20 hits were measured from the vertical left-hand edge of the target to the right $(+\xi)$, and from the horizontal lower edge upward $(+\eta)$. Mean point of impact, $\xi_0\eta_0$. The deviations with respect to this were denoted by $\pm x$, right and left, and $\pm y$, up and down.

ξ=515	645	658	622	627	592	696	572	615	596	733	662
η=218	265	274	281	293	304	309	316	352	352	374	371
ξ=591	565	730	654	626	604	672	726 cm; mean ξ_0=635·05 cm.				
η=375	459	526	541	573	583	636	665 cm; mean η_0=403·75 cm.				

Thence, in the horizontal direction	in the vertical direction
$\Sigma\|x\|=921,\ \Sigma x^2=63969$	$\Sigma\|y\|=2315{\cdot}5,\ \Sigma y^2=352034$
average deviation $E_1=47{\cdot}2$ cm	$E_2=119$ cm
probable deviation $w_1=39$ cm	$w_2=92$ cm
mean quadratic deviation $\mu_1=58$ cm	$\mu_2=136$ cm
probable error $w_1=40$ cm	$w_2=92$ cm
probable error of the mean $W_1=9$ cm	$W_2=21$ cm

$$\Sigma xy=60820,$$

$$\tan 2a=\frac{2C}{A-B}=\frac{2\times 60820}{63969-352034};\ a=-11^\circ\ 26'{\cdot}8.$$

For the symmetrical axes, (u, v)

$\Sigma u^2=51655$	$\Sigma v^2=364354$
mean quadratic error $\mu'=52$ cm	$\mu''=138$ cm
probable error $w'=35$ cm	$w''=93$ cm

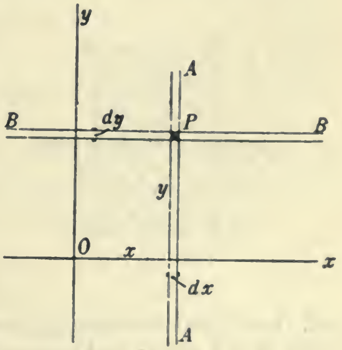
In addition for 100 shots from a 7·65 mm rifle at a vertical target at a range of 250 m, it was found that $a=+7^\circ\ 32'$.

Bertrand investigated 1000 rifle shots, and found $a=-19^\circ 47'$; Mayevski from a series of 44 shots from a 10·5 cm gun, found $a=+0^\circ\ 47'$.

No laws, connecting the distance of the target with the position of the axes of symmetry, are known, in spite of the fact that it has been sometimes stated that the two things are interdependent. But an investigation of the matter might lead to some useful results.

§ 67. The probability of hitting a given area.

Let us suppose a system of rectangular coordinates to be drawn on the target, the origin being the mean point of impact, and let the axes be arranged symmetrically with respect to the target-diagram, as explained in § 66.



The mean quadratic errors in the direction of the axes are found from

$$\mu_1=\sqrt{\frac{\Sigma x^2}{n-1}};\ \mu_2=\sqrt{\frac{\Sigma y^2}{n-1}};$$

thence we find the probable errors, w_1 and w_2, and the widths of the 50 °/₀ zones given by $s_1 = 2w_1$, and $s_2 = 2w_2$. The chance of hitting the point P is the same as that of hitting an indefinitely small rectangle, dx, dy, at P. This is naturally infinitesimally small: it may be considered as the product of two probabilities, which are (1) the chance of hitting a vertical strip at P of width dx, and (2) the chance of hitting a horizontal strip at P, of width dy. These chances are respectively

$$\frac{h_1}{\sqrt{\pi}} e^{-h_1^2 x^2} dx, \text{ and } \frac{h_2}{\sqrt{\pi}} e^{-h_2^2 y^2} dy.$$

Now $$h_1 = \frac{1}{\mu_1 \sqrt{2}} = \frac{0{\cdot}4769}{w_1}, \quad h_2 = \frac{1}{\mu_2 \sqrt{2}} = \frac{0{\cdot}4769}{w_2}.$$

Then in n shots, the number falling on the elementary rectangle is

$$n \frac{h_1 h_2}{\pi} e^{-(h_1^2 x^2 + h_2^2 y^2)} dx\,dy.$$

Suppose now a rectangular target $ABCD$ is taken, with O at the centre, breadth $2l_1$, height $2l_2$. Integrating over this rectangle, the number of hits on the rectangular target is

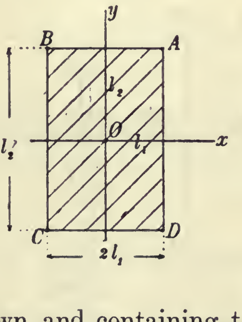
$$t = \frac{n h_1 h_2}{\pi} \int_{x=-l_1}^{x=+l_1} \int_{y=-l_2}^{y=+l_2} e^{-(h_1^2 x^2 + h_2^2 y^2)} dx\,dy$$

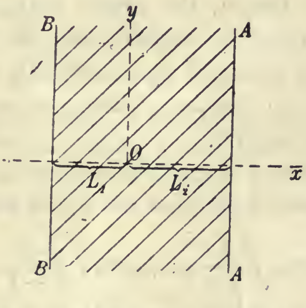
$$= n \frac{h_1}{\sqrt{\pi}} \int_{-l_1}^{l_1} e^{-h_1^2 x^2} dx \times \frac{h_2}{\sqrt{\pi}} \int_{-l_2}^{l_2} e^{-h_2^2 y^2} dy = n\phi(h_1 l_1)\,\phi(h_2 l_2)$$

$$= n\phi\left(\frac{l_1}{w_1} . 0{\cdot}4769\right) \phi\left(\frac{l_2}{w_2} . 0{\cdot}4769\right) = n\psi\left(\frac{l_1}{w_1}\right) \psi\left(\frac{l_2}{w_2}\right), \quad \text{......(1)}$$

and ψ is given in Vol. IV, Table 15.

Next suppose a strip $AABB$ is taken, extending indefinitely up





and down, and containing the origin O, at distances L_1, L_2 from the sides of the strip; and suppose the probable lateral error to be $w_1 = w$.

Of n shots, the number falling in the strip is

$$t = \frac{nh}{\sqrt{\pi}} \int_{x=-L_1}^{x=+L_2} e^{-h^2x^2}\,dx, \text{ where } h = \frac{0{\cdot}4769}{w};$$

$$t = \frac{nh}{\sqrt{\pi}} \int_{x=0}^{x=+L_1} e^{-h^2x^2}\,dx + \frac{nh}{\sqrt{\pi}} \int_{0}^{x=+L_2} e^{-h^2x^2}\,dx;$$

and since $$\frac{h}{\sqrt{\pi}} \int_{-l}^{l} e^{-h^2x^2}\,dx = \phi(hl),$$

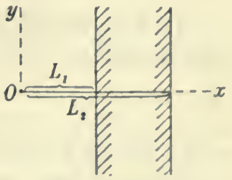
we have $$t = \frac{1}{2} n \left[\psi\left(\frac{L_1}{w}\right) + \psi\left(\frac{L_2}{w}\right)\right]. \quad \text{.................(2)}$$

On the other hand, on a strip as in the figure not containing the origin O,

$$t = \frac{nh}{\sqrt{\pi}} \int_{x=+L_1}^{x=+L_2} e^{-h^2x^2}\,dx = \frac{1}{2} n \left[\psi\left(\frac{L_2}{w}\right) - \psi\left(\frac{L_1}{w}\right)\right]. \quad \text{...(3)}$$

Let side AA in the first strip recede to infinity, and $L_2 = \infty$, and so also in the second strip; then

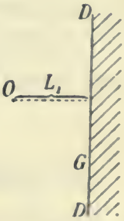
$$\psi\left(\frac{L_2}{w}\right) = \psi(\infty) = 1.$$



Then the probability of making a hit in the first or second case becomes

$$\frac{1}{2}\left[\psi\left(\frac{L_1}{w}\right) + 1\right] \text{ or } \frac{1}{2}\left[1 - \psi\left(\frac{L_1}{w}\right)\right].$$

The result is therefore as follows: If a straight line G is at a distance L_1 from the mean point of impact O, and if w is the probable error in a direction perpendicular to the line, then in n shots



$\frac{1}{2} n \left[1 + \psi\left(\frac{L_1}{w}\right)\right]$ fall on same side of G as O;

$\frac{1}{2} n \left[1 - \psi\left(\frac{L_1}{w}\right)\right]$ fall on the opposite side of G to O.

Recapitulation. (*a*) The probability of hitting a strip on the target bounded by two parallel lines, perpendicular or parallel to the plane of fire, equidistant from the point of mean impact O is

$$\phi\left(0{\cdot}4769\,\frac{2l}{s_{50}}\right) \quad \text{........................(I)}$$

$$= \psi\left(\frac{2l}{s_{50}}\right), \quad \text{...........................(I } a\text{)}$$

where $2l$ is the breadth of the strip, and s_{50} that of the 50 °/。 zone; consult Tables 14, 15, Vol. IV for ϕ and ψ; $\frac{2l}{s_{50}}$ is called the probability factor, and 100ψ is the percentage of hits.

(*b*) The probability of hitting a rectangle, of which O is the centre, is

$$=\psi\left(\frac{2l_1}{s_1}\right)\psi\left(\frac{2l_2}{s_2}\right), \quad \text{.....................(II)}$$

$2l_1$, $2l_2$ are the lengths of the sides of the rectangle, taken parallel and perpendicular to the plane of fire; s_1, and s_2 of the 50 °/。 zones parallel to the sides of the rectangle.

(*c*) The probability of hitting an infinite strip with parallel sides, where O has an unsymmetrical position within the strip, at distances l_1 and l_2 from the sides, is

$$\frac{1}{2}\psi\left(\frac{2l_2}{s_{50}}\right)+\frac{1}{2}\psi\left(\frac{2l_1}{s_{50}}\right). \quad \text{..................(III)}$$

(*d*) The probability of hitting the strip outside O, is with the same notation

$$\frac{1}{2}\psi\left(\frac{2l_2}{s_{50}}\right)-\frac{1}{2}\psi\left(\frac{2l_1}{s_{50}}\right). \quad \text{.....................(IV)}$$

(*e*) Given a line on the target, drawn obliquely to the plane of fire, at a distance l from O; the probability that a hit lies on the same side of the line as O is

$$\frac{1}{2}+\frac{1}{2}\psi\left(\frac{2l}{s_{50}}\right). \quad \text{.......................(V)}$$

A table for these functions is given by Sabudski and v. Eberhard; the calculation can also be made from Table 15, Vol. IV.

(*f*) So also the probability that the hit is on the side of the line away from O is

$$\frac{1}{2}-\frac{1}{2}\psi\left(\frac{2l}{s_{50}}\right). \quad \text{.......................(VI)}$$

The quantities l, l_1, l_2, s_{50} must all be expressed in the same unit.

Example 1. A strip on the target stretches in the direction of fire, of breadth $2l=6$ m; the breadth of 50 °/。 zone is $s_{50}=4$ m. The gun is aimed at the middle line of this strip. What is the percentage of hits on the strip?

(*a*) From Table 14 in Vol. IV,

$$\frac{2l}{s_{50}}=\frac{6}{4}=1{\cdot}5,$$

so that from (I) the probability required is

$$\phi\,(0{\cdot}4769\times1{\cdot}5)=\phi\,(0{\cdot}7153)=0{\cdot}688.$$

Therefore 68·8 °/。 are hits.

(*b*) From Table 15, Vol. IV,

$$\psi(1{\cdot}5)=0{\cdot}688, \text{ and thus } 68{\cdot}8\ \% \text{ are hits.}$$

Example 2. A horizontal strip on the target is drawn obliquely to the plane of fire; the gun is fired at the horizontal middle line of the strip. Given that the breadth $2l$ of the strip is one-third of the 25 % error, calculate the percentage of hits on the strip.

Consider another strip with the same middle line but of breadth $2l_{25}$. On this strip the probability of hitting is 0·25; then according to (I *a*)

$$\psi\left(\frac{2l_{25}}{s_{50}}\right)=0{\cdot}25, \text{ and } \frac{2l_{25}}{s_{50}}=0{\cdot}472 \text{ from Table 15.}$$

On the other hand $2l_{25}=3\times 2l$, so that $0{\cdot}472\, s_{50}=3\times 2l$, or

$$\frac{2l}{s_{50}}=\frac{0{\cdot}472}{3}=0{\cdot}157;$$

and from Table 15, $\psi(0{\cdot}157)=0{\cdot}082$; hence 8·2 % of hits are to be expected.

Example 3. The height of a strip, extending right and left, is to be so adjusted that when a rifle is fired at the middle line of the strip, 41 % of the hits strike the strip, the 50 % zone for height being 3·5 m.

Let the height be $2l$; then $\psi\left(\frac{2l}{3{\cdot}5}\right)=0{\cdot}41$, and $\frac{2l}{3{\cdot}5}=0{\cdot}80$, from Table 15: and $2l=2{\cdot}8$ m.

Example 4. A strip extending indefinitely right and left has a height of 1·9 m. The 50 % zone for height is 3·5 m. The mean point of impact is on the upper boundary line of the strip. To calculate the percentage of hits on the strip.

In formulae (III) or (IV), $s_{50}=3{\cdot}5$, $l_1=0$, $l_2=1{\cdot}9$; then the percentage of hits is

$$100\times\frac{1}{2}\left[\psi\left(\frac{2\times 1{\cdot}9}{3{\cdot}5}\right)\pm 0\right]=50\,\psi(1{\cdot}08)=50\times 0{\cdot}5337=27\ \%.$$

Example 5. Firing with a fixed elevation, in 10 shots 7 were observed on the near side of an upright mound, and consequently were recorded as short. The 50 % length zone was given as 20 m. Calculate the probable distance of the mean point of impact from the mound, and the amount of correction required.

Denote the distance of the mean point from the mound by l. This is given by formula (V) from the equation

$$\frac{7}{10}=\frac{1}{2}+\frac{1}{2}\psi\left(\frac{2l}{20}\right),\quad \psi\left(\frac{l}{10}\right)=0{\cdot}4, \text{ and } \frac{l}{10}=0{\cdot}778, \text{ from Table 15,}$$

$l=7{\cdot}8$ m. Consequently the mean point of impact is about 7·8 m on this side of the mound; and this is at the same time the most probable correction.

To determine the probable limits of this result for l, the most probable value being 7·8, or 8 m, the rule of Bayes in § 59 may be used; and on this it is to be assumed that it is an even chance that the ratio of the shorts to the number of all the shots will lie between the limits

$$\frac{7}{10}\pm 0{\cdot}4769\sqrt{\frac{2\times 7\,(10-7)}{10^3}}=0{\cdot}7\pm 0{\cdot}0974,$$

that is between 0·7974 and 0·6026.

The probable limits of l_1 and l_2 are calculated from l from the equations

$$\frac{1}{2}+\frac{1}{2}\psi\left(\frac{2l_1}{20}\right)=0{\cdot}7974,\quad \frac{1}{2}+\frac{1}{2}\psi\left(\frac{2l_2}{20}\right)=0{\cdot}6026,$$

and then

$$\frac{2l_1}{20}=1{\cdot}23,\quad l_1=12{\cdot}3\text{ m};$$

$$\frac{2l_2}{20}=0{\cdot}38,\quad l_2=3{\cdot}8\text{ m};$$

these are the probable limits for the determination of l.

The probable correction is thence 8 ± 4 m, in round numbers; and ±4 m is the probable error of the trial.

Example 6. Calculations with percussion and time fuzes.

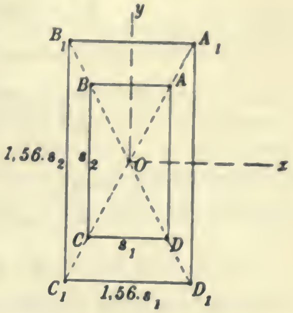
(*a*) Let us suppose that percussion fuzes are used with a given target, the charge and the angle of elevation being kept constant. Then the final portions of the trajectories may be considered to be a number of parallel lines, inclined to the horizontal at an angle ω, where ω is the acute angle of descent. One of these is actually the mean trajectory of the group. As in § 63, the deviations from the mean trajectory, l_{50} and b_{50}, can be calculated from the horizontal target-diagram for level ground on the muzzle-horizon. Here l_{50} is the 50 per cent zone for length, and is measured on the horizontal line in the plane of fire, while b_{50} is the 50 per cent zone for breadth, and is measured on the horizontal line at right angles to the plane of fire. It will be seen that b_{50} has the same value for level and for rising ground, while l_{50} has obviously a variable value, depending on the angle of slope. On a vertical target, if h_{50} is the 50 per cent zone for height, $h_{50}=l_{50}\tan\omega$. For a horizontal target, which is at a higher or lower level than the muzzle-horizon, graphical methods can be employed to obtain the value of l_{50}, but the experimental method is better.

(*b*) With time-fuzes, the same sort of thing takes place. This is the first reason why shells do not always burst at the same point, even though the fuze-setting and all the other conditions are the same. Besides the difference in the trajectories, the variability of the action of the time-fuzes is a second contributing cause: this is due to various unavoidable differences in the uniformity of the composition of the fuzes. The position of the point of burst depends partly on the individual trajectory and partly on the rate of action of the time-fuze. The result is that the 50 per cent breadth-zones are the same as for percussion fuzes, and they need not be further considered.

The 50 per cent height-zones and length-zones are referred to a vertical plane, containing the mean trajectory, and with time-fuzes the equation $h_{50}=l_{50}\tan\omega$ does not hold. In this case the values of h_{50} and l_{50} may be either greater or less than they would be in the case of simple trajectories: and in practice they are calculated, independently of one another, from the moments of burst, as found by experiment. Let us suppose h_{50} to be known, i.e., the 50 per cent height-zone of the point of burst: then for a given mean height of burst, H, the probability w can be calculated from formula (V), and is given by the equation

$$w=\frac{1}{2}\left[1+\psi\left(\frac{2H}{h_{50}}\right)\right].$$

Example 7. The point of mean impact, O, is at the centre of the rectangle $ABCD$, the sides of which represent the 50 °/。 zones, and are respectively $s_1=2w_1$, and $s_2=2w_2$. This rectangle will therefore receive 25 per cent of the hits, i.e., 50 per cent of 50 per cent. It is required to construct a rectangle $A_1\,B_1\,C_1\,D_1$ with sides λs_1 and λs_2, so that this rectangle shall include half the shots.



We have

$$\psi\left(\frac{\lambda s_1}{s_1}\right)\cdot\psi\left(\frac{\lambda s_2}{s_2}\right)=1,\ \text{whence}\ \psi(\lambda)=\frac{1}{\sqrt{2}},$$

and $\lambda=1{\cdot}56$.

Example 8. The works of Krause, Rohne, Ch. Minarelli-Fitzgerald, Zedlitz, and Heydenreich, to which reference is made in the notes, may be consulted as to the significance of target-diagrams in firing practice. Thus with company fire, it is possible to see whether the sighting is correct for the whole company, or whether it should to some extent be altered. Krause gives a good deal of information about rifle firing.

Let us take the case of firing at a range of 800 m: the rifle is sighted at the lower edge of the target A. The mean trajectory is calculated in the usual way. The ordinates BB_0, CC_0, ... are given at ranges of 775 m, 750 m, ...

Moreover the 50 °/。 height-zones are supposed to be known at the corresponding ranges $s_{50}=2w$. It is therefore assumed that at the ranges

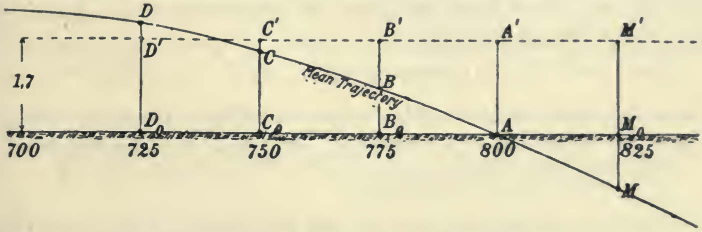
800 | 825 | 850 | 875 | 900 | ... | 775 | 750 | 725 | 700 ... m,

the mean ordinate y is measured as

$y=0\,|-0{\cdot}89\,|-1{\cdot}85\,|-2{\cdot}90\,|-4{\cdot}04\,|\,...\,|+0{\cdot}80\,|+1{\cdot}52\,|+2{\cdot}16\,|+2{\cdot}74$ ... m,

and that the 50 °/。 height-zones are

2·26 | 2·36 | 2·45 | 2·54 | 2·64 | ... | 2·18 | 2·09 | 2·00 | 1·91 ... m.



The probability of hitting the target AA' at 800 m is

$$\frac{1}{2}\psi\left(\frac{1{\cdot}70}{1{\cdot}13}\right)=0{\cdot}344=34\ {}^{\circ}/_{\circ}.$$

The probability of hitting the target B_0B' at 775 m is

$$\frac{1}{2}\psi\left(\frac{0{\cdot}80}{1{\cdot}09}\right)+\frac{1}{2}\psi\left(\frac{1{\cdot}70-0{\cdot}80}{1{\cdot}09}\right)=40\ {}^{\circ}/_{\circ}.$$

The probability of hitting the target C_0C' at 750 m is

$$\frac{1}{2}\psi\left(\frac{1\cdot52}{1\cdot045}\right)+\frac{1}{2}\psi\left(\frac{1\cdot70-1\cdot52}{1\cdot045}\right)=38\cdot2\ \%.$$

The probability of hitting the target M_0M' at 825 m is

$$\frac{1}{2}\psi\left(\frac{0\cdot89+1\cdot70}{1\cdot18}\right)-\frac{1}{2}\psi\left(\frac{0\cdot89}{1\cdot18}\right)=23\cdot7\ \%$$

and so on.

Thus for the ranges

800	825	850	875	900	925	950	...	775	750	725	700	675	650	625	600	575	...

the series of the percentage of hits is as follows:

34·4	23·7	12·9	5·4	1·8	0·4	0	...	40·0	38·2	30·0	20·5	12·0	5·8	2·4	0·9	0	...

If the percentage is set out as a function of the range, we obtain a curve, corresponding to the sighting for 800 m.

As for artillery, the probability that a shot lies on the same side as O was given as $\frac{1}{2}+\frac{1}{2}\psi\left(\frac{L_1}{w}\right)$.

Denote by ξ, the unknown range of the target from the gun as given in the range table, a the range at which the firing is to be carried out. These distances as well as the distance L_1 of the mean point of impact O from the target are to be measured in a positive direction away from the gun.

Since $\xi-a=L_1$ and $\psi(-y)=-\psi(+y)$, the probability of a short shot is given in all cases by $\frac{1}{2}+\frac{1}{2}\psi\left(\frac{\xi-a}{w}\right)$.

Putting $\frac{\xi}{w}=x$, $\frac{a}{w}=a$, as ξ, a, w are all measured in the same unit, viz., the metre, $\frac{1}{2}+\frac{1}{2}\psi(x-a)$, or $F(x-a)$, is the probability of a short shot; and consequently $F(a-x)$ or $1-F(x-a)$ of a long shot. A table of the function F is given in the works of Sabudski and v. Eberhard, as well as of Kozàk.

When $s=m+n$ shots have been fired, the probability that in the s shots, m are short and n long, is given by

$$[F(x-a)]^m\,[F(a-x)]^n.$$

The most probable distance of the target, according to the range table, is that for which this expression is a maximum; and so x is to be calculated from $F(x-a)=\frac{m}{s}$, as in the previous example 5.

More generally, let $s=m+n$ shots be fired with different elevations; let short shots be observed at the ranges $a_1, a_2, \ldots a_m$, and long shots at ranges $b_1, b_2, \ldots b_n$, in a definite order. The unknown range x is then obtained according to Mangon by the following procedure. The probability of this event is

$$\eta=F(x-a_1)\,F(x-a_2)\ldots F(x-a_m)\,F(b_1-x)\,F(b_2-x)\ldots F(b_n-x).\ \ldots(a)$$

By logarithmic differentiation and equating to zero, the condition of a maximum is obtained, and thence the most probable range x.

The condition will be

$$f(x-a_1)+\ldots+f(x-a_m)-f(b_1-x)-\ldots-f(b_n-x)=0, \quad \ldots\ldots\ldots(b)$$

where $f(y)$ denotes $\frac{F'(y)}{F(y)}$, which is found in the tables of Sabudski and Kozàk. This equation (b) is to be solved by trial.

For example in shooting with a gun the following observations were made, as given by Sabudski and v. Eberhard. At an elevation of 50 mm on the sight, corresponding to the range A_1, a $-$ was obtained; with 51 mm at A_2, $+, -, -$; with 52 mm at A_3, $+, +$; (a long shot is denoted by $+$, short by $-$); 1 mm on the sight alters the range about 26 m, and the probable error in range $w=9{\cdot}4$ m. Required the most probable range of the target.

Condition (b) leads here to

$$f(x-A_1)+2f(x-A_2)-f(A_2-x)-2f(A_3-x)=0. \quad \ldots\ldots\ldots\ldots(c)$$

To solve this equation, $x=A_2$ is first tried; and then the left-hand side of (c) is

$$f(A_2-A_1)+2f(0)-f(0)-2f(A_3-A_2)=0.$$

The range difference A_2-A_1 corresponds to the difference of $51-50$ mm; but since 1 mm on the sight alters the range 26 m,

$$A_2-A_1=(51-50)\,26\text{ m}=2{\cdot}77\,w.$$

So also $A_3-A_2=2{\cdot}77$; and thus $f(2{\cdot}77)+2f(0)-f(0)-2f(2{\cdot}77)$ is obtained.

From the table, the left-hand side is thus

$$0{\cdot}11-2\times1{\cdot}18-1{\cdot}18-2\times0{\cdot}11=+1{\cdot}07,$$

which is positive.

Calculating again x as the range corresponding to the elevation 51·36 mm, the left-hand side works out $=-1{\cdot}26$; and finally $-0{\cdot}21$ for 51·2 mm. Consequently the most probable elevation for the range is between 51 and 51·2 mm, and nearer to 51·2.

This method of Mangon, first employed in Germany by H. Rohne, has been extended by him to the case where in one shot or in several the distance of the point of impact from the target is to be measured.

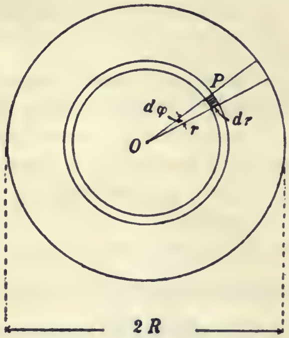
Concerning this and other details, see Ritter v. Eberhard's German translation of N. Sabudski's work, in which the questions are treated theoretically.

§ 68. Probability of hitting a given circular area.

Suppose a circle of radius R to be described on a vertical target; the gun is fired at the centre O so that O is the centre of mean impact; and the zones in the plane of the target are assumed to be the same in all directions about O.

About a point P let a small sector df be drawn with central angle $d\phi$, and bounded by a thin ring of inner radius r and outer $r+dr$.

On Gauss's Law the probability of hitting df is $a^2 e^{-b^2 r^2} df$, where a and b are two constants, to be determined later, and $df = r\,d\phi\,dr$.



Integrate with respect to ϕ from 0 to 2π; then out of n shots, $2n\pi a^2 e^{-b^2r^2} r\,dr$ will fall on the thin ring.

Thus the number of hits in the complete circle of radius R is

$$t = 2n\pi a^2 \int_{r=0}^{r=R} e^{-b^2r^2} r\,dr = n\pi \frac{a^2}{b^2}(1 - e^{-b^2R^2}). \quad \text{.........(1)}$$

The constant a is then determined from the condition that the area of an infinite circle is certain to be struck, and then $t = n$, for $r = \infty$; this gives $a = \dfrac{b}{\sqrt{\pi}}$.

The constant b determines the measure for radial deviations, reckoned from O.

Of these radial errors, denote the mean quadratic error by μ_r, the average error by E_r, the probable or 50 °/ₒ error (half the 50 °/ₒ zone) by w_r or R_{50}.

The relations between these errors are different from those given previously with respect to parallel deviations.

(*a*) Mean quadratic radial error μ_r.

Denoting the single radial errors of the shots by $\epsilon_1, \epsilon_2, \epsilon_3, \ldots$, then according to the definition,

$$\mu_r = \sqrt{\frac{\epsilon_1^2 + \epsilon_2^2 + \ldots}{n}} = \sqrt{\frac{\Sigma\epsilon^2}{n}}.$$

Therefore $n\mu_r^2 = \Sigma\epsilon^2 = \Sigma$ (r^2 times the number of hits made on the thin ring between radii r and $r + dr$) $= \Sigma\,(r^2\, 2n\pi a^2 e^{-b^2r^2} r\,dr)$, the sum being taken over the whole plane; so that

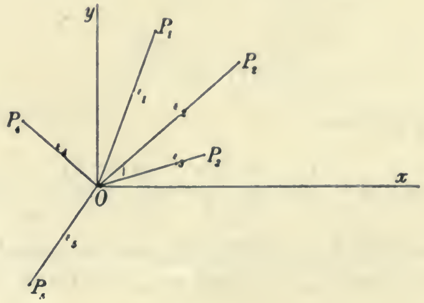
$$n\mu_r^2 = 2n\pi \frac{b^2}{\pi} \int_{r=0}^{r=\infty} e^{-b^2r^2} r^3\,dr;$$

and substituting $b^2r^2 = t$, and by partial integration, we get

$$\mu_r^2 = \frac{1}{b^2}, \quad b = \pm\frac{1}{\mu_r}, \quad \text{......................(2)}$$

and thus from (1), the number of hits

$$t = n\,(1 - e^{-\frac{R^2}{\mu_r^2}}).$$



Thus for $R = \mu_r$, $t = n\left(1 - \frac{1}{e}\right) = 0{\cdot}631n$ or 63 °/₀ of the shots.

When the mean quadratic errors μ_1 and μ_2 in the direction of the two axes of symmetry x and y are known, $\mu_r^2 = \mu_1^2 + \mu_2^2$, since $r^2 = x^2 + y^2$, and $\Sigma r^2 = \Sigma x^2 + \Sigma y^2$; and if $\mu_1 = \mu_2 = \mu$, then $\mu_r = \mu\sqrt{2}$.

(*b*) Probable radial error or 50 °/₀ zone w_r or R_{50}.

Here R_{50} is the radius of the circle round O, which contains half the shots, so that $t = \frac{1}{2}n$; and

$$\tfrac{1}{2} = 1 - e^{-\frac{R_{50}^2}{\mu_r^2}}, \quad R_{50} = w_r = \mu_r\sqrt{(\log_e 2)} = 0{\cdot}83255\mu_r. \quad \text{......(3)}$$

$$t = n\,(1 - e^{-\frac{R^2\log 2}{R_{50}^2}}) = n\,[1 - (e^{-\log 2})^{\frac{R^2}{R_{50}^2}}] = n\,[1 - (0{\cdot}5)^{\frac{R^2}{R_{50}^2}}].$$

(*c*) Average radial error E_r.

This is the arithmetic mean of all the absolute values of the radial errors.

$E_r = \frac{\Sigma\epsilon}{n}$, $nE_r = \Sigma$ (r times the number of deviations of magnitude r), extended over the whole area, assuming that Gauss's Law can hold for deviations of unlimited extent; so that

$$nE_r = \Sigma\,(rna^2e^{-b^2r^2}\,2\pi r dr) = 2n\pi a^2\int_0^\infty e^{-b^2r^2}r^2 dr;$$

and putting $br = t$, we get

$$E_r = 2\pi a^2 \frac{1}{b^3} \int_0^\infty e^{-t^2} t^2 dt = 2\pi a^2 \frac{1}{b^3} \frac{\sqrt{\pi}}{4},$$

or, since $a = \frac{b}{\sqrt{\pi}}$,

$$E_r = \frac{1}{b} \frac{\sqrt{\pi}}{2} = \frac{0{\cdot}8862}{b}. \quad \text{.....................(4)}$$

Therefore

$$t = n\,(1 - e^{-\frac{\pi R^2}{4E_r^2}}).$$

Consequently we have

$$b = \frac{1}{\mu_r} = \frac{1}{E_r} \frac{\sqrt{\pi}}{2} = \frac{\sqrt{(\log 2)}}{w_r \text{ or } R_{50}};$$

and w_r or $R_{50} = 0{\cdot}83255\mu_r = 0{\cdot}9395E_r$, for the radial errors about O; compared with the previous $w = 0{\cdot}6745\mu = 0{\cdot}8453E$, for parallel deviations.

Recapitulation. The percentage of hits in shooting at a circular target of radius R with centre at O, the mean point of impact, is

$$100\,(1 - e^{-\frac{R^2}{\mu_r^2}}) = 100\,[1 - (0{\cdot}5)^{\frac{R^2}{R_{50}^2}}] = 100\,(1 - e^{-\frac{\pi R^2}{4E_r^2}}).$$

A table for $100\,(1 - 0{\cdot}5^{\frac{R^2}{R_{50}^2}})$ as a function of $R : R_{50}$ is as follows:

Radius of circle R / 50 °/₀ zone radius R_{50}	=	0·1	0·2	0·3	0·4	0·5	0·6
Percentage of hits	=	0·69	2·73	6·04	10·50	15·91	22·08

0·7	0·8	0·9	1	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8
28·80	35·82	42·96	50·00	56·77	63·14	69·01	74·30	78·98	83·04	86·51	89·42
1·9	2	2·1	2·2	2·3	2·4	2·5	2·6	2·7	2·8	2·9	3
91·81	93·75	95·29	96·51	97·44	98·15	98·69	99·07	99·37	99·56	99·71	99·80

Numerical example. A pistol is fired accurately at a ring target; the 50 °/₀ circle had a diameter of 0·5 m.

The rings were numbered 12, 11, 10, 9, 8, 7, 6 and were marked out by circles of radii 5, 10, 15, 20, 25, 30, 35 cm.

In 1000 shots it is required to determine the number of hits to be expected in each ring.

Here $2R_{50}=50$ cm, $R_{50}=25$ cm.

Circular area up to ring	12,	$\frac{R}{R_{50}}=\frac{5}{25}=0\cdot2$,	percentage	2·73,
" "	11,	" $=\frac{10}{25}=0\cdot4$,	"	10·50,
" "	10,	" $=\frac{15}{25}=0\cdot6$,	"	22·08,
" "	9,	" $=\frac{20}{25}=0\cdot8$,	"	35·82,
" "	8,	" $=\frac{25}{25}=1\cdot0$,	"	50,
" "	7,	" $=\frac{30}{25}=1\cdot2$,	"	63·14,
" "	6,	" $=\frac{35}{25}=1\cdot4$,	"	74·30.

So that in the area of the ring

Number	12	11	10	9	8	7	6
Hits	27	105 − 27 = 78	221 − 105 = 116	358 − 221 = 137	500 − 358 = 142	631 − 500 = 131	743 − 631 = 112

Thus 257 shots fell outside the outermost ring.

§ 69. Probability of hitting a given elliptical target.

1. *Elliptical Target.*

The coordinate axes are taken as the axes of symmetry of the target-diagram.

The mean quadratic errors in these directions are denoted by μ_1 and μ_2. Thus there is an infinite series of similar ellipses, for which the axes are symmetrical.

Among these ellipses select CDC_1D_1 with semi-axes $OC=\lambda\mu_1$, $OD=\lambda\mu_2$; and the question is to determine the percentage of hits on this ellipse.

For this purpose the given elliptical area is divided up into elements of area df, and the number of hits on this area df is calculated, and then an integration gives the number on the given ellipse CDC_1D_1.

The most convenient way is to take the element of area df in the form of a very thin elliptic ring ABA_1B_1, enclosed by ellipses similar to the boundary ellipse. Then it can be shown that round such an ellipse the probability of a hit is constant.

The probability of hitting an element $dxdy$ round a point P, is found to be

$$\frac{h_1}{\sqrt{\pi}}e^{-h_1^2x^2}dx\,\frac{h_2}{\sqrt{\pi}}e^{-h_2^2y^2}dy, \text{ where } h_1=\frac{1}{\mu_1\sqrt{2}}, \text{ and } h_2=\frac{1}{\mu_2\sqrt{2}},$$

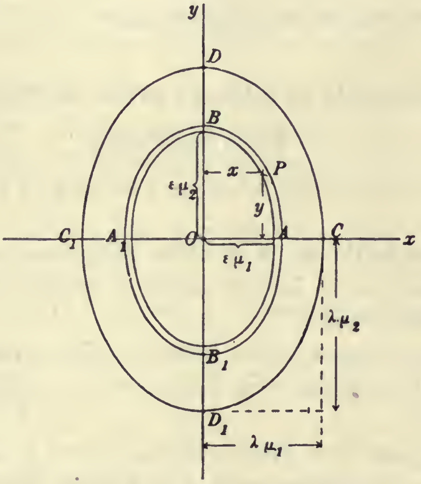
and so it is equal to

$$\frac{1}{2\pi\mu_1\mu_2} e^{-\frac{1}{2}\left(\frac{x^2}{\mu_1^2}+\frac{y^2}{\mu_2^2}\right)} dxdy.$$

If the point $P(xy)$, and with it the element $dxdy$, moves round the plane, so that P remains on the ellipse ABA_1B_1, with semi-axes $\epsilon\mu_1$ and $\epsilon\mu_2$; then $\frac{x^2}{(\epsilon\mu_1)^2}+\frac{y^2}{(\epsilon\mu_2)^2}=1$, or $\frac{x^2}{\mu_1^2}+\frac{y^2}{\mu_2^2}=\epsilon^2$, and the probability remains constant, because ϵ is constant round the ellipse; and ϵ, μ_1, μ_2 are given.

Integrating round the ring ABA_1B_1, the probability of hitting this thin ring, of area df, is then equal to

$$\frac{1}{2\pi\mu_1\mu_2} e^{-\frac{1}{2}\epsilon^2} df.$$



The value of df is found as follows. The inner ellipse of the ring has semi-axes $\epsilon\mu_1$ and $\epsilon\mu_2$, and the area

$$f = \epsilon\mu_1 . \epsilon\mu_2 . \pi = \pi\mu_1\mu_2 . \epsilon^2.$$

Therefore

$$df = 2\pi\mu_1\mu_2\epsilon\, d\epsilon.$$

Thence the probability of hitting the thin ring is

$$\frac{1}{2\pi\mu_1\mu_2} e^{-\frac{1}{2}\epsilon^2} 2\pi\mu_1\mu_2\epsilon\, d\epsilon = e^{-\frac{1}{2}\epsilon^2}\epsilon\, d\epsilon,$$

that is, in n shots, the number falling on the ring is $ne^{-\frac{1}{2}\epsilon^2}\epsilon\, d\epsilon$.

Integrating this over the whole extent of the ellipse CDC_1D_1, from $\epsilon = 0$ to $\epsilon = \lambda$, the number of hits on this ellipse is

$$n\int_{\epsilon=0}^{\epsilon=\lambda} e^{-\frac{1}{2}\epsilon^2}\epsilon\, d\epsilon = n\,(1 - e^{-\frac{1}{2}\lambda^2}).$$

Recapitulation. The ellipse with semi-axes

$$OC = \lambda\mu_1 = 1{\cdot}483\lambda w_1, \quad OD = \lambda\mu_2 = 1{\cdot}483\lambda w_2,$$

in the direction of the axes of symmetry of the target-diagram, and centre O at the mean point of impact, receives $100\,(1 - e^{-\frac{1}{2}\lambda^2})$ per cent of the hits. Outside the ellipse the percentage of hits is $100e^{-\frac{1}{2}\lambda^2}$.

If the ellipse CDC_1D_1 is to represent the probable error ellipse, receiving 50 °/ₒ of the hits, then

$$\tfrac{1}{2}n = n\,(1 - e^{-\frac{1}{2}\lambda^2}), \quad \lambda = \sqrt{(2\log_e 2)} = 1{\cdot}1774\,;$$

that is, the ellipse with semi-axes $1{\cdot}177\mu_1$, $1{\cdot}177\mu_2$ will receive half of the shots.

In this way the semi-axes of the ellipse may be calculated for any percentage of hits.

In this group of similar ellipses that one is included which is the boundary of the target, since the maximum deviations M_1 (about $3w_1$), and M_2 (about $3w_2$), are in the same ratio. Consequently these ellipses constitute the smallest targets, which contain a given percentage of hits.

The following is a table for $100\,(1 - e^{-\frac{1}{2}\lambda^2})$, as a function of $\dfrac{\lambda}{\sqrt{2}}$:

$\frac{\lambda}{\sqrt{2}}$ =	0·00	0·10	0·20	0·30	0·40		
$100\,(1 - e^{-\frac{1}{2}\lambda^2})$ = °/ₒ	0·00	0·99	3·92	8·61	14·79		
0·50	0·60	0·70	0·80	0·90	1·0	1·2	1·4
22·12	30·23	38·74	47·27	55·51	63·21	76·31	85·91
1·6	1·8	2·0	2·2	2·4	2·6	2·8	3·0
92·27	96·08	98·17	99·21	99·68	99·88	99·96	99·99

Conversely:

$100\,(1 - e^{-\frac{1}{2}\lambda^2})$ = °/ₒ	10	20	30	40
$\frac{1}{2}\lambda^2$ =	0·10537	0·22315	0·35668	0·51083
50	60	70	80	90
0·69315	0·91630	1·20398	1·60944	2·30259

In this way elliptic targets may be constructed for a given rifle and for any range, of which the separate rings correspond to the actual height and breadth errors, with an area to suit any definite percentage of hits.

2. *A target with any arbitrary bounding line.*

The probability of hitting a target was given in § 67 by the following double integral, extended over the boundary line of the target:

$$\frac{h_1 h_2}{\pi} \iint e^{-(h_1^2 x^2 + h_2^2 y^2)} \, dx \, dy.$$

As shown in §§ 67—69, this double integral can be worked out, not only for rectangles, but also for circular and elliptical boundaries, provided the mean point of impact lies at the centre of the circle or ellipse.

But let us now suppose that the bounding line of the target is wholly irregular, as is the case with the ordinary targets of the battle-field. In such circumstances it is impossible to evaluate the double integral in finite form. The usual method of approximation is to suppose that the mean point of impact coincides with the centre of gravity, and the target is then replaced by a circle or rectangle of equal area, described about the centre of gravity.

Rothe has lately described a general method for dealing with cases of this kind: it is partly graphical and partly mechanical. It deals with a target bounded by any kind of curve, with any known errors of deviation, or with any position of the axes, or with any assumption as to the position of the mean point of impact. He is able to find the percentage of hits on the target with sufficient accuracy.

For this purpose, other variables are introduced instead of $h_1 x$ and $h_2 y$, and the double integral becomes $\frac{1}{\pi}\iint e^{-(x^2+y^2)} \, dx \, dy$. Then Rothe's general procedure is to evaluate the double integral $\iint f(x, y) \, dx \, dy$ over a given bounding line with the aid of the planimeter.

The process is briefly as follows. A plan of the surface $\zeta = f(x, y)$ is made by means of contour lines of section, along which z=constant. He draws the bounding line of the integral, and determines with the planimeter, for a sufficient number of constant values of ζ, the sections of the cylinder erected on the bounding line. These sections are plotted as ordinates, corresponding to ζ, in a new diagram, and a curve is constructed in this way. He determines by the planimeter the area bounded by this curve, the two extreme ordinates, and the axis of abscissae.

R. Rothe has employed this procedure for many examples, for instance on an infantry breast-plate of 1283 cm^2 surface; there he found the probability of hitting was 0·662, while the assumption of a circular target of equal area gave the value 0·754.

Rothe's planimetric method can also be extended to a target in three dimensions.

Finally he has shown how to obtain the point of mean impact, and how the target must be placed so as to obtain the greatest probability. Further details cannot be given here.

§ 70. Application of Gauss's method of Least Squares.

1. The matter may best be illustrated by means of a simple example.

Suppose at the ranges $X = 1, 2, 3, 4, 5, 6$ km, the times of flight are $T = 2·70, 6·20, 10·30, 15·20, 21·40, 30·30$ seconds.

A relation between X and T is required, and let us suppose that

$$T = AX + BX^2.$$

For the determination of A and B, there are six equations, namely,

$$2·7 = 1 \,.\, A + 1^2 \,.\, B, \quad 6·2 = 2A + 2^2B, \quad 10·3 = 3A + 3^2B,$$
$$15·2 = 4A + 4^2B, \dots \text{ and so on.}$$

Considering only the first two equations, we find $A = 2·3$, $B = 0·4$; so that the relation is

$$T = 2·3X + 0·4X^2. \quad \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

Then the values of T are as follows:

measured,	$T =$	2·7	6·2	10·3	15·2	21·4	30·3,
calculated,	$T =$	2·7	6·2	10·5	15·6	21·5	28·2,
errors,	$f =$	± 0	± 0	$\pm 0·2$	$+ 0·4$	$+ 0·1$	$- 2·1$,

$$\Sigma f^2 = 0·2^2 + 0·4^2 + 0·1^2 + 2·1^2 = 4·6_2.$$

It is required to find the most convenient way of using all the observations, or of distributing the errors over the whole series. This consists in the method of Least Squares. The coefficients A, B, must be so determined that the sum of the squares of the errors is made as small as possible. The procedure is analogous to the graphical method of drawing a smooth curve through a series of observed points.

The errors are the differences between the calculated and the measured times of flight; so that the sum of the squares of the errors is

$$(1 . A+1^2 . B-2{\cdot}7)^2+(2 . A+2^2 . B-6{\cdot}2)^2+(3 . A+3^2 . B-10{\cdot}3)^2+\ldots.$$

If this is to be a minimum, we have by partial differentiation

$$\left.\begin{aligned}(A+B-2{\cdot}7)+2(2A+4B-6{\cdot}2)+3(3A+9B-10{\cdot}3)+\ldots=0,\\(A+B-2{\cdot}7)+4(2A+4B-6{\cdot}2)+9(3A+9B-10{\cdot}3)+\ldots=0,\end{aligned}\right\}$$

and

$$\left.\begin{aligned}A(1+4+9+\ldots)+B(1+8+27+\ldots)=2{\cdot}7+2\times 6{\cdot}2+3\times 10{\cdot}3+\ldots,\\A(1+4.2+9.3+\ldots)+B(1^2+4^2+9^2+\ldots)=2{\cdot}7+4\times 6{\cdot}2+9\times 10{\cdot}3+\ldots.\end{aligned}\right\}$$

$$91A+441B=395{\cdot}6, \quad 441A+2275B=1989{\cdot}2,$$

$$A=1{\cdot}81388, \quad B=0{\cdot}52276;$$

$$T=1{\cdot}81388X+0{\cdot}52276X^2. \qquad\qquad (2)$$

Equation (2) gives the relation between T and X for general purposes, as deduced from the six observations.

As to the mean error of any individual determination of the time of flight, the following is the procedure to employ: Calculate from (2) the times of flight for $X=1, 2, 3, \ldots$; then to the observed times of flight

2·7 6·2 10·3 15·2 21·4 30·3

correspond the calculated times,

2·33664 5·71880 10·14648 15·61968 22·13840 29·70264,

so the errors are

0·36336 0·48120 0·15352 0·41968 0·73840 0·59736,

and the squares of the errors are

0·13203 0·23156 0·02357 0·17613 0·54523 0·35685.

Thus $\Sigma f^2=1{\cdot}46537$, which is smaller than for any other values of A and B; for instance the determination from (1) gave $\Sigma f^2=4{\cdot}62$.

Further, the mean quadratic error $\mu=\sqrt{\dfrac{\Sigma f^2}{n-m}}$, where n is the number of observations and m is the number of coefficients to be determined; so that here $\mu=\sqrt{\dfrac{1{\cdot}46537}{6-2}}=0{\cdot}605$, and this error μ, and also the probable error w, have thereby been reduced to a minimum, on the assumption of a function of the form

$$AX+BX^2.$$

2. The problem may also be treated in the following manner: We know that $T=\phi(X, X^2, A, B)$; and for the unknown coefficients A and B, some approximate values are obtained; denote them by $\bar{A}$ and $\bar{B}$. They may for instance be determined from the two first observations, $X=1$, $T=2·7$ and $X=2$, $T=6·2$; and then $\bar{A}=2·3$, $\bar{B}=0·4$.

Therefore we suppose that $A=\bar{A}+\alpha$, $B=\bar{B}+\beta$; so that

$$T=\phi(\bar{A}+\alpha,\ \bar{B}+\beta);$$

and expanding by Taylor's theorem we get

$$T=\phi(\bar{A},\bar{B})+\alpha\frac{\partial\phi}{\partial A}+\beta\frac{\partial\phi}{\partial B}.$$

Here $\phi(\bar{A},\bar{B})$ denotes the approximate value of T, denoted by $\bar{T}$, obtained from $\bar{A}$ and $\bar{B}$; and so

$$T-\bar{T}=\alpha\frac{\partial\phi}{\partial A}+\beta\frac{\partial\phi}{\partial B}.$$

But in the preceding case

$$\phi=AX+BX^2,\ \frac{\partial\phi}{\partial A}=X,\ \frac{\partial\phi}{\partial B}=X^2.$$

Therefore

$$T-\bar{T}=\alpha X+\beta X^2.$$

It follows from the above that the values of the error $T-\bar{T}=\kappa$ for the various ranges X are as follows:

$T-\bar{T}=\kappa$	X
0	1
0	2
− 0·2	3
− 0·4	4
− 0·1	5
+ 2·1	6

The method of Least Squares must be applied again to the equation $\kappa=\alpha X+\beta X^2$ to obtain the coefficients α and β; so that

$$(\alpha.1+\beta.1^2-0)^2+(\alpha.2+\beta.2^2-0)^2$$
$$+(\alpha.3+\beta.3^2+0·2)^2+\ldots \text{ is a minimum.}$$

Proceeding by partial differentiation, we get

$$91\alpha+441\beta=9·9,\quad 441\alpha+2275\beta=64·9,$$
$$\alpha=-0·4862,\quad \beta=+0·1228.$$

Consequently the values of A and B are

$$A = \bar{A} + \alpha = 2\cdot3 - 0\cdot4862 = 1\cdot8138,$$

$$B = \bar{B} + \beta = 0\cdot4 + 0\cdot1228 = 0\cdot5228,$$

and

$$T = 1\cdot81X + 0\cdot52X^2, \text{ as before.}$$

The accuracy of this procedure can be shown without the use of Taylor's theorem, as follows:

Suppose the approximate values, $\bar{A}$ and $\bar{B}$, to be found, and

$$\bar{T} = \bar{A}X + \bar{B}X^2, \quad T = AX + BX^2.$$

Then

$$T - \bar{T} = (A - \bar{A})X + (B - \bar{B})X^2, \text{ or } \kappa = \alpha X + \beta X^2.$$

3. The method described in 2 must be applied when equations are treated which cannot be solved by an approximation process.

Suppose, for example, it is required to obtain a relation between y and x in the form

$$y = Ax^B + C.$$

Suppose the values $y_1, y_2, \ldots$ to correspond to the values $x_1, x_2, \ldots$. Then A, B, C are determined from the condition

$$(Ax_1^B + C - y_1)^2 + (Ax_2^B + C - y_2)^2 + \ldots = \text{a minimum};$$

and so we have the three equations

$$x_1^B(Ax_1^B + C - y_1) + x_2^B(Ax_2^B + C - y_2) + \ldots = 0,$$

$$x_1^B \log x_1 (Ax_1^B + C - y_1) + x_2^B \log x_2 (Ax_2^B + C - y_2) + \ldots = 0,$$

$$Ax_1^B + C - y_1 + Ax_2^B + C - y_2 + \ldots = 0.$$

But these equations can only be solved approximately; and approximate values, $\bar{A}$, $\bar{B}$, $\bar{C}$, must first be determined.

This can be done in the following manner. The observed values, $x_1, x_2, \ldots$, corresponding to the observed values $y_1, y_2, \ldots$ are plotted on a diagram, and a curve is drawn through the points. The slope of the tangent to this curve is measured at the first two points, and let p be the tangent of the angle made with the axis of x. Then $\frac{dy}{dx} = p = ABx^{B-1}$, and if p_1 and p_2 are the values of p at the first two points on the curve, we have $p_1 = ABx_1^{B-1}$, and $p_2 = ABx_2^{B-1}$.

These equations give the values of A and B; C is found from the equation $y_1 = Ax_1^B + C$. This gives us the first approximations,

$\bar{A}$, $\bar{B}$, $\bar{C}$: using these values, we get an approximation for y from the equation $\bar{y} = \bar{A}x^{\bar{B}} + \bar{C}$.

Further approximations, α, β, γ are then introduced, so that

$$y = \phi(\bar{A} + \alpha, \bar{B} + \beta, \bar{C} + \gamma):$$

therefore

$$y - \bar{y} = \alpha \frac{\partial \phi}{\partial \bar{A}} + \beta \frac{\partial \phi}{\partial \bar{B}} + \gamma \frac{\partial \phi}{\partial \bar{C}}.$$

But

$$\frac{\partial \phi}{\partial \bar{A}} = x^{\bar{B}}; \ \frac{\partial \phi}{\partial \bar{B}} = \bar{A}x^{\bar{B}} \log x; \ \frac{\partial \phi}{\partial \bar{C}} = 1.$$

Therefore

$$y - \bar{y} = \alpha x^{\bar{B}} + \beta \bar{A} x^{\bar{B}} \log x + \gamma.$$

Here only the first powers of α, β, and γ occur; so they can be calculated by method (1).

This, therefore, gives us further approximations, viz.,

$$\bar{A} + \alpha, \ \bar{B} + \beta, \ \bar{C} + \gamma:$$

in some cases it may be necessary to continue the process of approximation to a further stage.

4. The best method of expending ammunition on the preparation of a range table must now be considered.

With guns of heavy calibre, the question is important, having regard to the available ammunition. Suppose it is required to prepare a range table, and more particularly to ascertain the angle of departure for several ranges. Then we must consider whether it would be better to fire a few shells at a large number of ranges, or a large number of shells at a few ranges. Vallier prefers the latter plan of firing a number of shells at a few ranges. On the theory of probabilities, he arrives at the following conclusions for guns of medium calibre.

If less than 16 rounds are available, he recommends firing at a single range, taking measurements of the initial velocity, v_0, and of the error in the angle of departure.

If between 16 and 24 rounds are available, he recommends firing at two ranges, with the same measurements as before.

If between 24 and 40 rounds are available, then three ranges should be chosen, with measurements as before.

With heavy calibres, the number of rounds can be decreased by about 33 per cent.

The useful ranges x, at which the practice should be carried out, are given in the following table, in which W denotes the longest range that comes into consideration.

Number of rounds employed	Range x to be chosen					
1	0	v_0 measurement				
2	0	0·828 W				
3	0	0·464 W	0·928 W			
4	0	0·281 W	0·679 W	0·960 W		
5	0	0·186 W	0·4875 W	0·790 W	0·975 W	
6	0	0·131 W	0·359 W	0·622 W	0·849 W	0·981 W

This holds when the initial velocity v_0 is known with the same accuracy as that to which each range x can be measured.

On the other hand, if the accuracy of the v_0 measurement is higher than that of the range:

Number of rounds employed	Range x to be chosen			
1	0	v_0 measurement		
2	0	0·894 W		
3	0	0·603 W	0·952 W	
4	0	0·415 W	0·730 W	0·996 W

In some cases the supply of ammunition may be such that the measurement of v_0 depends on a measurement at the short range, x'. In this case the table is as follows:

Number of rounds employed	Range x to be chosen			
1	x'			
2	x'	$0{\cdot}1057\,x' + 0{\cdot}894\,W$		
3	x'	$0{\cdot}397\,x' + 0{\cdot}603\,W$	$0{\cdot}048\,x' + 0{\cdot}952\,W$	
4	x'	$0{\cdot}585\,x' + 0{\cdot}415\,W$	$0{\cdot}270\,x' + 0{\cdot}730\,W$	$0{\cdot}004\,x' + 0{\cdot}996\,W$

These rules hold as well for any ballistic function.

Suppose, for instance, with an infantry rifle the hits on a vertical target are to be examined, and that as regards the measurement of v_0 the first assumption holds and that 2000 m is the greatest range in the range table; then if there are to be 6 ranges, the targets should be placed at about the following ranges, 0 (measurement of v_0 between 0 and 50 m), 260 m, 720 m, 1240 m, 1700 m, 1960 m.

These rules of Vallier cannot always be used in practice; nevertheless they are useful in the preparation of range tables.

CHAPTER XII

On the effects produced by the explosion of shells

§ 71. The penetration of bullets and unexploded shells into rigid bodies. Calculation of the depth of penetration.

The passage of a shell through the air tends to accelerate the air-particles, which it sets in motion: this causes waves and eddies in the atmosphere. Waves of condensation are also formed when the projectile penetrates a fluid or solid body, but these waves generally precede the shell or bullet, since their speed is high. Thus the velocity of sound in air is about 340 m/sec, in wood about 1440 m/sec, and in steel 5000 m/sec. The effect of these waves in a block of stone or metal is sometimes seen in the fact that small pieces break off on the opposite side to that on which the shell impinges.

The effect of cohesion must also be overcome. The projectile loses energy owing to two causes: in the first place, it accelerates the particles of the body with which it comes in contact, and the amount of this acceleration depends on the ease with which the particles can be displaced: and in the second place, work is done against the forces of cohesion. Let W be the resistance opposed to the motion of the shell by the body, and let the shell be moving with velocity v and have a cross-section $R^2\pi$. Then Euler supposes that $W = \pi a R^2$: Poncelet assumes that $W = \pi R^2 (a + bv^2)$; and Résal supposes that $W = \pi R^2 (av + bv^2)$. On the other hand T. Levi-Civita assumes that $W = \pi R^2 (a + bv^2)(1 + kv_0)$. Here v_0 denotes the velocity at the moment of impact, a and b are constants, which depend on the nature of the body into which the projectile penetrates, and k is an empirical constant.

Taking Poncelet's formula, the calculations can be made as follows. The projectile penetrates over a short distance into the body: this short path can be considered as a straight line, seeing that the effects of gravity may be neglected. The resistance, which is equal to $\pi R^2 i (a + bv^2)$, is then the only force to be considered, and i is a coefficient depending on the shape of the shell-head. Let the projectile penetrate into the body through a distance x in a time t, reckoned from the instant of impact. Its initial velocity at the moment of impact may be supposed to have been v_0, and at the end of the time t it is v. Let $\frac{P}{g}$ be the mass of the projectile. Then we have

$$\frac{P}{g}\frac{dv}{dt} = \frac{Pv}{g}\frac{dv}{dx} = -\pi R^2 i\,(a + bv^2).$$

Integrating we have

$$x = \frac{P}{2\pi bgR^2 i}\log\frac{a + bv_0^2}{a + bv^2}, \quad \text{..................(1)}$$

since when $t = 0$, $x = 0$, and $v = v_0$, and

$$t = \frac{P}{\pi R^2 ig\sqrt{ab}}\left\{\tan^{-1} v_0\sqrt{\frac{b}{a}} - \tan^{-1} v\sqrt{\frac{b}{a}}\right\}. \quad \text{......(2)}$$

If a body, of a thickness of x metres, is penetrated by a projectile, then equation (1) gives the velocity which the projectile has at the moment of escape from the body, and equation (2) gives the time taken to effect the penetration. If, however, the body is too thick for complete penetration, the projectile will eventually come to rest within the body, in which case $v = 0$. Then the total depth of penetration will be

$$X = \frac{P}{2\pi R^2 bgi}\log\left(1 + \frac{bv_0^2}{a}\right), \quad \text{..................(3)}$$

and the total time of penetration is

$$T = \frac{P}{\pi R^2 gi\sqrt{ab}}\tan^{-1}\left(v_0\sqrt{\frac{b}{a}}\right). \quad \text{...............(4)}$$

The value of i will be discussed later. According to the tests carried out by Didion-Morin-Piobert in 1839—1840, the coefficients a and b have the following values, viz.,

For limestone	$a = 12{,}000{,}000$:	$\frac{10^6 b}{a} = 15.$
Strong masonry	$a = 5{,}520{,}000$:	$\frac{10^6 b}{a} = 15.$
Ordinary masonry	$a = 4{,}400{,}000$:	$\frac{10^6 b}{a} = 15.$
Brickwork	$a = 3{,}160{,}000$:	$\frac{10^6 b}{a} = 15.$
Sand and gravel	$a = 435{,}000$:	$\frac{10^6 b}{a} = 200.$
Clay	$a = 1{,}045{,}000$:	$\frac{10^6 b}{a} = 35.$

Loose earth: covered with grass ...	$a=$ 700,000:	$\frac{10^6 b}{a} =$ 60.
Loose earth: half clay, half sand ...	$a=$ 461,000:	$\frac{10^6 b}{a} =$ 60.
Damp clay	$a=$ 266,000:	$\frac{10^6 b}{a} =$ 80.
Oak, beech, ash	$a=$ 2,085,000:	$\frac{10^6 b}{a} =$ 20.
Elm	$a=$ 1,600,000:	$\frac{10^6 b}{a} =$ 20.
Pine and birch	$a=$ 1,160,000:	$\frac{10^6 b}{a} =$ 20.
Poplar	$a=$ 1,090,000:	$\frac{10^6 b}{a} =$ 20.

Vallier in 1913 came to the conclusion that for earth, wood, and masonry, the value of $\frac{10^6 b}{a}$ is 50, and the value of bi is determined experimentally, so that equation (3) becomes

$$X = \frac{\lambda P}{\pi R^2} \log \left(1 + \frac{1}{2} \frac{v_0^2}{10^4}\right), \qquad \text{.................(5)}$$

where λ depends on the nature of the body into which the projectile penetrates, and must also be found by experiment.

Pétry in 1910 gives the following formula, viz.,

$$X = \frac{P\kappa f(v_0)}{4R^2} \qquad \text{.........................(6)}$$

where X is the depth of penetration in metres, P is the weight of the projectile in kilogrammes, and $2R$ is the calibre in centimetres. Here κ is the only coefficient which depends on the nature of the body. Its value for concrete masonry is 0·64: for good stone masonry, is 0·94: for good brickwork, 1·63: for sandy earth, 2·94: for ploughed earth, 3·86: for clay soil, 5·87: $f(v_0)$ is a function of the velocity of impact, and its value is given in the following table.

$v_0 =$	40	60	80	100	120	140	160	180	200	220	240	260
$f(v_0) =$	0·33	0·72	1·21	1·76	2·36	2·97	3·58	4·18	4·77	5·34	5·89	6·41
$v_0 =$	280	300	320	340	360	380	400	420	440	460	480	500
$f(v_0) =$	6·92	7·40	7·87	8·31	8·74	9·15	9·54	9·92	10·29	10·64	10·98	11·30

The projectile may, of course, glance off the surface of the body: Pétry thinks that this happens with the surface of the earth if the angle of impact, i.e., the angle between the direction of motion and the normal to the surface of the ground, exceeds 75°: this angle is said to be 60° in the case of masonry or brickwork. Sometimes the depth of penetration is used for forming an estimate of the velocity of impact, and in this connection Journée's empirical formula is often useful. He considers that if a bullet penetrates a fir tree,

$$X = 0{\cdot}000093\, dv_0^2, \qquad (7)$$

where X is the depth of penetration in centimetres, d is the diameter of the bullet in centimetres, and v_0 is the velocity of impact.

Example. A shrapnel bullet, weighing 10 grammes, and having a diameter of 1·22 cm, is to strike a man with such force as to put him out of action. For this purpose 8 m-kg is sufficient. This velocity is to be roughly estimated from firing into a fir tree. What should be the depth of penetration? We have

$$8 = \frac{10\, v_0^2}{2 \times 9{\cdot}81 \times 1000},$$

and thus $v_0 = 125$ m/sec.

Thus the depth of penetration is

$$X = 0{\cdot}000093 \times 1{\cdot}22 \times 125^2 = 1{\cdot}8 \text{ cm}.$$

The formulae of Poncelet, Euler, and Résal rest, of course, on assumptions, which cannot be expected to give very exact results. The tables, giving the values of a and b in Poncelet's formula, do not pretend to any great accuracy, more especially as it is not possible to give any exact definition of good masonry and the like. Moreover the experiments, on which these values are based, were carried out with much lower velocities than can be obtained with the modern rifle.

The coefficient i is said to be equal to unity for spherical projectiles, and for the ordinary shell it is said to be $\frac{2}{3}$. In the formula

$$W = \pi R^2 ai \left(1 + \frac{bv^2}{a}\right),$$

it is better to determine the constants ai and $\frac{b}{a}$ by experiment. For this purpose, it is best to find X' and X'' for two different velocities of impact, v' and v_0'. Then by the use of equation (3), we have two equations for the determination of ai and $\frac{b}{a}$.

§ 72. Depth of penetration of a projectile.

1. The formula of Levi-Civita was suggested by the fact that with the modern rifle the greatest depth of penetration takes place at a considerable distance from the muzzle. For instance, the following table was drawn up in 1900 for the French bullet.

At a distance of	Depth of penetration of the bullet in			
	Sand	Loose earth	Pinewood	Oak
10 m	11 cm	25 cm	90 cm	20 cm
40 ,,	18 ,,	39 ,,	82 ,,	19 ,,
100 ,,	32 ,,	62 ,,	70 ,,	18 ,,
200 ,,	45 ,,	75 ,,	60 ,,	18 ,,
300 ,,	46 ,,	77 ,,	56 ,,	17 ,,
400 ,,	44 ,,	73 ,,	53 ,,	16 ,,
500 ,,	40 ,,	67 ,,	50 ,,	15 ,,
600 ,,	38 ,,	63 ,,	49 ,,	15 ,,

The fact is usually explained by the supposed compression of the bullet. If the velocity of the bullet is very great, the compression causes the sectional area of the bullet to be considerably increased. The effect of the change of πR^2 is more important than that due to $a + bv^2$; in consequence the resistance is very considerably increased with the result that the depth of penetration may be greater at a lower velocity. In order to allow for this, we ought to multiply πR^2 by $1 + kv_0$, where k is determined by experiment. Of course, this throws no light on the true nature of the resistance or its relation to the other variables. The most that can be said is that these formulae provide a method of mathematically examining a certain limited number of cases.

Some part of the energy of the bullet is undoubtedly expended in its compression and deformation. Various experiments were made in the author's laboratory and in the testing station at Halensee, near Berlin: the results are shown in the appended table, the ordinary S-bullet being fired into beechwood and sand. The table includes the velocity of impact, the ranges at the end of which the final velocities with a normal charge are equal to the velocities of impact, the depths of penetration, and remarks on the nature of deformation of the projectile. The lower velocities were produced by decreasing the charge.

2. N. von Wuich proposes the following indirect method of determining the law of penetration for a given material. The depths X, X', X'', corresponding to impact-velocities v_0, v_0', v_0'', are noted. Let X

Velocity of impact m/sec	Range in metres	Firing into sand		Firing into beech	
		Depth of penetration in cm	Nature of deformation of projectile	Depth of penetration in cm	Nature of deformation of projectile
98	2500	19·5	No deformation: the surface slightly roughened	4	No deformation
330	874	24·7	Ditto	12·7	Bullet compressed: section becomes oval, with the long axis in direction of fibres
473	560	28·6	Ditto	26·5	Ditto, but more marked deformation
579	398	31·4	Bullet compressed	41·6	Ditto, but greater deformation
710	218	22·4	A portion of the lead is pressed out at the back	65	Ditto
735	186	19·7	The outside completely broken	70·7	Ditto
762	153	18·2	Ditto, but more markedly	76·7	Lead begins to be extruded at the back
788	121	17	Ditto, shape not recognizable	40·7	Marked deformation. External surface broken: the lead partially extruded
815	90	15·8	Separate fragments of the bullet	34·7	Ditto: bullet bent: tip uninjured
870	2	13·8	Bullet broken into small fragments	29·7	Maximum deformation. The parts still cohere. Tip almost uninjured
The first deformation was noticed with an impact-velocity of 559 m/sec. Maximum depth of penetration, 33 cm. Range, 314 m.				Maximum depth of penetration, 76·7 cm. Range, 153 m.	

be the greatest of these depths. It is then assumed that after penetrating through a depth of $X—X'$ the bullet still has the velocity v_0', and after penetrating through $X—X''$, it has a velocity v_0''. In this way the velocity v is obtained as a function of the distance through which the bullet has travelled, and therefore

$$W = \frac{P}{g}\frac{dv}{dt} = \frac{Pv}{g}\frac{dv}{dx}.$$

This assumes that the rate of travel is independent of the deformation already produced. This hypothesis is not generally true, as it is impossible to suppose that the compression of the bullet, and the lateral friction of the material through which it is passing, produce no effect.

3. Assuming that a shell-hole is of the nature of a surface of revolution, Poncelet endeavoured to determine its shape in the following manner. He supposed that the volume of the material excavated by the penetration of the shell was proportional to the loss of *vis viva*. Therefore $\int_0^x \pi y^2 dx$ is proportional to $\frac{P}{2g}(v_0^2 - v^2)$, where y is the ordinate of the excavated curve, corresponding to x, the distance of penetration. Therefore $\frac{P}{g} v\,dv = -\lambda \pi y^2 dx$. This gives the relation between x, y, and v; and the use of (1) in § 71 gives us that between x and y. Thus we have the equation for the contour-line of the curve of excavation. But the results, obtained by this method, are not in agreement with the facts, as found by experiment. This is not surprising seeing that the process of penetration is a complicated phenomenon. The particles of the material are by no means so arranged as to be directly across the path of the projectile: a certain portion of the *vis viva* is used to overcome the forces of cohesion, and the percentage, which is so applied, is very variable.

4. There are many empirical and theoretical formulae for determining the thickness of armour plate, which can be penetrated under given conditions. G. Ronca quotes no less than 36 such formulae. For present purposes it may be sufficient to give the one suggested by the firm of Krupp in 1880, and also a formula which has been largely used in France.

Let z denote the *vis viva* of the projectile in m-kg per sq cm of cross section. Therefore $z = \frac{Pv^2}{2\pi R^2 g}$. Let e denote this *vis viva* per cc of the volume of a sphere, having a diameter equal to the calibre of the shell.

Then $e = \frac{3Pv^2}{8\pi R^3 g}$. Therefore the formula is

$$z = 100S\sqrt[3]{\frac{S}{D}}, \text{ or } e = 150\left(\frac{S}{D}\right)^{\frac{4}{3}},$$

where S is the thickness of the armour plate, in centimetres, $D = 2R$ = calibre in cm, P is the weight of the shell in kilogrammes, v is the velocity of impact in m/sec, and $g = 9{\cdot}81$. This formula presupposes that the direction of impact is at right angles to the surface, and that Krupp's wrought iron armour plate is used without any support in the rear. If the impact takes place at an angle α with the surface of the plate, then $z = \frac{100}{\sin^2\alpha} S\sqrt[3]{\frac{S}{D}}$. It is perhaps better to use a coefficient λ instead of 100, and to determine it by experiment with a similar shell and a similar plate, seeing that the natures of the shell and of the plate undoubtedly affect the matter.

Example. Calibre $D = 2R = 26$ cm, $S = 38$ cm, $P = 205$ kg. It is then found that $v = 468$ m/sec.

Jacob de Marre gives another formula, which is

$$v = \frac{AS^{0{\cdot}7}(2R)^{0{\cdot}75}}{P^{0{\cdot}5}},$$

where S and R are measured in decimetres. This supposes that the direction of impact is normal to the surface: but if this direction makes an angle α with the normal, then S must be multiplied by $\left(\cos\frac{3\alpha}{2}\right)^{1{\cdot}43}$. For ordinary steel A is said to have the value 1·53, and for hardened steel de Marre gives the formula

$$v = \sqrt{1{\cdot}885 - 0{\cdot}0014S} \times \frac{1{\cdot}53\,(2R)^{0{\cdot}75}S^{0{\cdot}7}}{P^{0{\cdot}5}}.$$

Shells, which are used for firing at armour plate, frequently have caps of wrought iron or mild steel, as suggested by Makarow in Russia: this causes them to penetrate further into the plate: Pétry states that the depth of penetration is increased in the ratio of 245 to 190. This phenomenon is usually explained by supposing that the cap acts as a sort of lubricating material. It is more probable that the shell slides through the expanded cap, and that the latter acts as a kind of band, protecting the tip of the shell. The tip of the projectile is thus preserved from injury. Instantaneous photography might perhaps throw some light on the matter. B. A. Mimey has propounded a theory on the subject, to which reference is made in the notes: he treats the matter theoretically, and considers it to be a particular case of the deformation of a solid body by impact.

The following phenomena are observed when a modern steel-clad bullet is fired at a plate of mild steel. The steel covering breaks away at the tip: the lead core, which forms the larger part of the projectile, is then driven forward and excavates a portion of the plate. The steel portion gradually falls to the rear, and the effect of friction is to break it to pieces.

If one of the modern type of steel-clad bullets is fired at a hardened steel plate of sufficient thickness, it flies to pieces without penetrating. The fragments have considerable velocities, mostly in a lateral direction across the surface of the plate: boards of wood in the immediate neighbourhood are more or less sawn through by the flying particles, very few of which fly back along the line of fire. It would have seemed probable that the fragments would mostly tend to fly back: the inertia of the hinder portion of the bullet is the most likely explanation of the facts.

5. It is a question as to the precise amount of energy, which a bullet must possess, in order to put a man or a horse out of action. The French suppose that 4 m-kg is sufficient for a man, and 19 m-kg for a horse, whereas in Germany it is thought that 8 m-kg is required for a man. The estimates are necessarily rough: something depends on the calibre of the projectile, and something also on the position of the point of impact.

J. Pangher states the following facts for projectiles having a calibre between 6 and 11 mm. If the *vis viva* per square centimetre of cross section of the projectile falls below a certain minimum, contusions are the only effect: this minimum is 2 m-kg per sq cm for a man, and about 10 m-kg per sq cm for a horse. The depth of flesh wounds is proportional to the *vis viva* per unit area. The destructive effect of bullets in the neighbourhood of bones is proportional to the total energy. The least impact-energy that is sufficient to break a man's bones is 5 m-kg: in order to produce an absolutely certain result, 16 m-kg are required; with a horse the corresponding figures are 17 and 35.

6. If a projectile strikes a body without complete penetration, the total heat that is produced is not always equal to the calories in the energy of impact, because it frequently happens that portions of the projectile are ejected laterally or in a backward direction. Thus a portion of the energy is wasted externally. The energy of impact is

$\frac{Pv_0^2}{854g}$ calories, and this is equal to the heat produced, if the body remains at rest during the time of impact. If, however, the body, together with the projectile, moves forward, then the heat produced is only $\frac{PP_1}{854g(P+P_1)}$, where P_1 is the weight of the body struck by the projectile. The velocity of their common motion is $u = \frac{vP}{P+P_1}$. Before impact, the *vis viva* was $\frac{Pv^2}{2g}$, and after impact it is $\frac{u^2(P+P_1)}{2g}$: the difference is therefore the amount of energy which has been dissipated as heat.

7. Let us consider a bullet, weighing 14·7 grammes; its calibre is 0·79 cm, its velocity is 444 m/sec, and its *vis viva* is 145 m-kg. Therefore an average resistance of 145 kg over a space of 1 metre can be overcome or one of 1450 kg over 10 cm, or one of 20,700 kg over 0·7 cm. Such a bullet can penetrate an iron plate of a thickness of 0·7 cm (see Wille, *Waffenlehre*, vol. I, p. 215, 1905). The force required to punch a hole of this size in the plate is

$$0{\cdot}79 \times 0{\cdot}7 \times 0{\cdot}8 \times 3500\pi \doteq 5000 \text{ kg, approximately,}$$

if we suppose the mechanical strength of the plate to be 3500 kg per sq cm. It therefore seems that calculations of this kind lead to wholly inaccurate conclusions. If the question is treated on statical principles, we must suppose that the punch is not permanently deformed, and that the velocity is negligible. But in this case neither of these assumptions is true. Let us take a copper wire, 15 cm long and 0·5 cm in diameter, and clamp it at the ends: let us suppose a gradually increasing pressure to be applied to the middle of the wire. Then the tension in the wire will eventually reach a maximum. After this, the tension falls quickly and the wire breaks. In the first part of this process, work is done in extending the wire, and in the second, work is done in breaking the particles asunder: there is no question of any noticeable *vis viva*. If the wire is broken by the modern type of bullet, instantaneous photography gives no trace of any extension of the wire. The wire seems to be instantaneously fractured at the moment of impact. When the bullet travelled further over one or two lengths the wires seemed to be slightly coiled upwards and downwards in the neighbourhood of the point of fracture: the rest of the wire seemed to be at rest. At a later stage, the wires are seen to be slightly bent along their whole length. In this case the work expended on stretching the wire is probably small: the main part of the energy is spent in breaking the wire, while the *vis viva* and the resistance due to inertia are considerable. The coiling of the wires, upwards and downwards, near the point of impact takes place very quickly: the other portions of the wire are subsequently set in motion: consequently in comparison with the small mass of the bullet, a not inconsiderable mass of the wire is set in motion. Consequently the accelerations and the resistances, due to inertia, are seen to be of great importance in the case of firing a bullet through a plate, whereas if the hole is slowly and steadily punched, the strength of the material is the only consideration.

As for the effects on the bullet, we find that in some cases it is shattered, and in other cases it is quite uninjured, though the conditions are such that in neither case is the result expected. Suppose that a steel-clad bullet, weighing 10 grammes, is fired into a large volume of water with a velocity of 900 m/sec. It undergoes considerable compression and is frequently broken to pieces. On the other hand, a candle can be shot through a thin board, and afterwards fairly large pieces can be picked up. A rod of soft wood can be fired through a board of harder wood without being much bent or deformed.

The time, which is taken by the penetration, is a factor of the greatest importance. In the case of the wooden rod, there is not sufficient time for bending or compression. Generally speaking, the deformation is small, if the time-interval is short, always supposing that the forces are equally great in all cases. A skater can pass rapidly over a thin sheet of ice, which would give way, if he were to stand still. The barrel of a rifle seems capable of withstanding a greater gas-pressure without exceeding the elastic limits than would be expected from a statical examination of the problem. The copper cylinder in a crusher apparatus is under certain circumstances less compressed by the gas-pressure than it would be by a similar pressure in a lever press. When the wooden rod hits the board, there is a pressure at the front end, and consequently a retardation. This difference of pressure and the retardation are transmitted with considerable speed through the rod: this speed is equal to that of the longitudinal waves of sound through the wood. The board is pierced before great differences of pressure can be set up, and before there is any great relative retardation. There is thus little tendency to bend or compress.

If a large glass plate is suspended by two pieces of string, and a bullet is fired at it from a modern rifle, a hole with sharp edges is formed in the plate: its diameter is about equal to that of the bullet. The glass plate seems scarcely to move, and the velocity of the bullet is hardly diminished. The maximum force of resistance is certainly considerable, but its time-integral is very small. The glass plate bends transversely in all directions, radiating from the point of impact. This transverse vibration is shown by instantaneous photography: the negatives indicate waves of condensation and rarefaction in the air on either side, which can be seen for a short distance from the point of impact. But before any notable bending of the plate can take place, the bullet has passed completely through the plate.

The time, which a bullet takes to penetrate into a large volume of water, is much longer than this. The force of resistance, due to the water, is proportional to the square of the velocity, and this explains why a steel-clad projectile is sometimes compressed to pieces. Suppose a cylindrical bullet, with a length of 5 cm and a cross-section of 1 sq cm, to be moving in the direction of its longitudinal axis with a velocity of 800 m/sec, and to strike a considerable mass of water at right angles. The dynamic resistance of the water is usually supposed to be $\frac{0{\cdot}7\, F\gamma v^2}{9{\cdot}81}$, where F is the cross-section of the body in sq m, and γ is the weight of a cubic metre of water in kilogrammes. Let us take this value as correct: then the resistance to motion is about 4500 kg, or at any rate of this order of magnitude. A force of this kind is sufficient to break up the projectile, if, as in the present case, it acts on it for a sufficient length of time.

As for the case of the candle, shot against the board, or the relative motion of the water with respect to the projectile, these things are included in those cases of motion in which a soft substance in rapid movement appears harder than if it were at rest or were moving at a low speed. For instance, a piece of metal can be polished by a rapidly rotating paper disc, and a blast of air from a compressor feels like a solid substance. In these cases, the resistance due to inertia plays an important part.

§ 73. The explosive effects of shell bursts.

1. Conical angle, resulting from the explosion of shrapnel and shells.

If a shell explodes, the centre of gravity of the fragments continues to travel over the same trajectory as before. The fragments continue to rotate in the same way as they previously rotated in the coherent shell about the longitudinal axis. Their axes of rotation about their respective centres of gravity immediately after the explosion will in general be parallel to the axis of the shell before the explosion. The angular velocities of the fragments about their centres of gravity must be the same as the angular velocity of the shell immediately before the explosion. The fragments move within a cone, the axis of which coincides with the position of the tangent at the moment of explosion: the angle of this cone is the conical angle, the magnitude of which must be ascertained. Certain fragments, amounting to about 15 per cent of the whole, disperse in an irregular fashion, and these are usually neglected in the methodical investigation of the problem.

Calculation of the Conical Angle.

The velocity of a flying fragment may result from four different causes.

Firstly, some part of this velocity is in the direction of the tangent to the trajectory at the moment of explosion: therefore in the direction of the axis of the cone, it has a velocity v_1 equal to that of the shell before the explosion: this can be calculated by the methods explained in Chapter VIII. When the point of burst is near the muzzle-horizon, this velocity can generally be taken as that with which the shell would strike the ground.

Secondly, a part of the velocity is at right angles to the tangent and is due to the force of the explosion. This velocity is fairly high in the case of shells, and may amount to anything between 400 and 2000 m/sec. With shrapnel, this velocity, which we will call v_s, is much smaller: it varies considerably with the nature of the construction of the shrapnel. In any case there is a certain velocity in the direction of the normal, the problem being similar to that of a shot-gun. In this latter case, the shot escape from the muzzle together with the exploded gases. There are thus gas pressures between the different shot, which tend on the whole to drive the shot outwards in the direction in which the gases naturally expand. (Pétry reports the case of Belgian shrapnel, in which he estimates v_s at 15 m/sec.)

J. de la Llave gives the following formula for the calculation of v_s in m/sec, in the case of shrapnel, having a chamber in the middle. According to this,

$$v_s = \frac{3000 p_2 L^{0.6}}{P p_1^{0.4}}, \qquad (1)$$

where L is the weight of the charge in kg; P is the weight of the shell; p_1 is the weight of a single bullet in kg; and p_2 is the total weight of the bullets.

Thirdly, in consequence of the rotation of the shell about its longitudinal axis, each of the bullets has a velocity v_d in a direction at right angles to the axis: before the explosion the bullet describes a circle about the axis of the shell, its linear velocity being the product of the angular velocity and the distance from the axis. Therefore after the explosion, the bullet tends to fly along the tangent to this circle, its velocity depending on its distance from the axis. Suppose that all the bullets were on the cylindrical surface of the shell: then their circumferential velocities at the beginning of the trajectory would have been $v_0 \tan \Delta$, where v_0 is the muzzle-velocity, and Δ is the final angle of the rifling. The angular velocity of the shell, however, falls off slightly, and moreover the bullets are at different distances from the axis. (See Vol. III, § 184.) Therefore A. Noble has suggested the following formula, viz.,

$$v_d = 0{\cdot}555\,(v_0 + v) \tan \Delta, \qquad (2)$$

while Pétry supposes $v_d = \lambda v_0 \tan \Delta$, where $\lambda = 0{\cdot}75$ for the shrapnel of the Belgian siege guns. (λ should be experimentally determined.) Finally Heydenreich gives the same formula, stating that λ lies between 0·67 and 0·8.

Fourthly, a bullet may receive from the force of the explosion an additional velocity, v_z, in the direction of the tangent to the trajectory. With shrapnel, having a chamber in the head of the shell, this velocity is negative: Pétry states that in a French gun it was – 25 m/sec. If the chamber is in the middle, $v_z = 0$, and if it is in the base, v_z lies between 20 and 80 m/sec. Heydenreich thinks the average value of v_z is 50 m/sec, in the case in which the charge is contained in the base, and weighs $\frac{1}{40}$ of the whole: on the other hand $v_z = 80$ m/sec, roughly speaking, if the ratio is $\frac{1}{25}$. J. de la Llave gives the following empirical formula for shrapnel with a base-chamber, viz.,

$$v_z = \frac{620 L^{0.6}}{p_2^{0.4}}, \qquad (3)$$

where L is the weight of the charge in kg, and p_2 is the total weight of the bullets.

These four velocities refer to the fragments which fly along the surface of the cone. The two velocities v_s and v_d are at right angles to one another and to the tangent: they can be compounded into a resultant velocity, V_s, which is $=\sqrt{v_s^2+v_d^2}$: the velocities v and v_z can be added together algebraically. Therefore the half of the angle at the apex of the cone is given by

$$\tan\frac{\gamma}{2}=\frac{\sqrt{v_d^2+v_s^2}}{v+v_z}=\frac{V_s}{v+v_z}. \qquad \text{.................(4)}$$

Along the trajectory, v_s and v_z scarcely vary: v_d decreases slowly, while v decreases considerably, increasing somewhat finally. Consequently the angle of the cone increases at first, and under certain circumstances it decreases at a later stage.

Therefore we have the following results, viz.,

(*a*) with the old type of shrapnel with the loading chamber in the head, v_s is small as compared with v_d, and v_z is negative, and

$$\tan\frac{\gamma}{2}=\frac{V_s}{v-v_z}; \qquad \text{..........................(5)}$$

(*b*) with the old type with the loading chamber in the middle, or with the tube-shrapnel, we have $v_z=0$, and

$$\tan\frac{\gamma}{2}=\frac{V_s}{v}; \qquad \text{..........................(6)}$$

(*c*) with the loading chamber in the base, we have

$$\tan\frac{\gamma}{2}=\frac{V_s}{v+v_z}. \qquad \text{..........................(7)}$$

For instance, with the French F.K. 97, the values of $\frac{\gamma}{2}$ are 7° 38′; 10°; 12° 9′ respectively at distances of 1000, 3000, and 6000 metres. The values of v at the moments of explosion are respectively 422, 300, and 230 m/sec: number of bullets, 291: weight of a single bullet, 12 grammes: total weight of bullets, 3·48 kg: weight of charge, 40 grammes: initial velocity, $v_0=529$ m/sec: angle of rifling $\Delta=7°$.

Using Noble's formula for a range of 6000 m we have

$$v_d=(529+230)\times 0{\cdot}555\times\tan 7°$$
$$=52 \text{ m/sec.}$$

The velocity v_z according to de la Llave's formula is

$$v_z=\frac{620\times 0{\cdot}04^{0{\cdot}6}}{3{\cdot}48^{0{\cdot}4}}=55 \text{ m/sec.}$$

If v_s is neglected, the value of $\frac{\gamma}{2}$ for 6000 m is given by

$$\tan\frac{\gamma}{2}=\frac{52}{230+55} \text{ and } \frac{\gamma}{2}=10° 21'.$$

Comparing this with the value 12° 9′, we see that if the formulae are to be used for practical purposes, v_s cannot be neglected in comparison with v_d, and that the value of v_s is about 9 m/sec.

(*d*) With shells, $v_z = 0$, and

$$\tan\frac{\gamma}{2} = \frac{V_s}{v}. \quad \text{..........................(8)}$$

The velocity v_s, imparted by the explosion with shells, is much greater than with shrapnel. With shells it is between 400 and 2000 m/sec: therefore the conical angle with shells is much greater. Here $\frac{\gamma}{2}$ lies between 50° and 90°: it increases somewhat along the trajectory, since v decreases. $V_s = \sqrt{v_d^2 + v_s^2}$: seeing that v_s is much greater than v_d, V_s is nearly independent of the range and the initial velocity, v_0, and therefore of the charge. The cone, along which the fragments of a shell disperse, is partially hollow on the inside, though this depends to some extent on the construction. Sometimes there is a second cone inside the first, the second having fewer fragments than the first.

Determination of the Conical Angle by Measurement.

It is better to determine the conical angle experimentally than to use the preceding equations. H. Rohne proposes to use the equation $\tan\frac{\gamma}{2} = \frac{V_s}{v + v_z}$ and to take v from the range table, or to calculate it in the usual way. He measures the conical angle for two different ranges, and thus he determines V_s and v_z from the two equations. For instance, let us take a range of 1000 m, for which $v = 421$ m/sec: $\frac{\gamma}{2}$ is found by measurement to be 7° 38′. Also for 4000 m, $v = 274$ and $\frac{\gamma}{2} = 10°\ 54'$. Then we have

$$\tan 7°\ 38' = \frac{V_s}{421 + v_z} \quad \text{and} \quad \tan 10°\ 54' = \frac{V_s}{274 + v_z}.$$

Therefore $V_s = 66$ m/sec, and $v_z = 72$ m/sec. Therefore $\tan\frac{\gamma}{2} = \frac{66}{v + 72}$. On this plan, V_s and v_z are considered as constants under the given conditions. With shrapnel this is only partially true, since v_d, which is a component of V_s, varies somewhat.

The measurement of the conical angle is usually carried out in the neighbourhood of the muzzle on the following plan. With shrapnel a vertical disc, on which the shell explodes, is arranged at a suitable distance. Behind this is the main target, which is divided into squares: it is erected in a vertical position at such a distance from the disc that it is struck by all the bullets. By photographic methods the position of the point of burst is determined accurately to 0·5 m. The length and breadth of the rectangle, which includes the main portion (i.e. 85 per cent) of the flying bullets, are determined. From the known data, we get two slightly different conical angles, γ_1 and γ_2: we therefore take $\gamma = \frac{1}{2}(\gamma_1 + \gamma_2)$. These measurements are generally made for two different ranges. The increase of velocity, v_z, due to the charge is generally found by a separate experiment: the shell is suspended without a fuse, the charge is ignited, and the velocity of the foremost bullet is measured by the Boulengé apparatus and the wire screens.

We may therefore be considered to have found v_z and two values of γ for two different ranges. For these ranges we know the value of v: therefore from the equation $\tan\frac{\gamma}{2} = \frac{V_s}{v + v_z}$ we get the values of V_s for the two ranges. The arithmetic mean of these two values is taken, and this is assumed to be the value of V_s for the purpose of any further calculations in which the equation is used.

With shells the arrangement is essentially the same, except for the fact that a box of sufficient size is used to intercept the flying fragments. Taking into account the direction of the tangent to the trajectory at the point of burst, the conical angle can be determined as before. From equation (8), we have $\tan\frac{\gamma}{2} = \frac{V_s}{v}$, in which we know γ, and v is known from the range table. Thus we know V_s, and the equation serves to determine γ under any conditions.

The value of γ is of importance for the theoretical discussion of the depth of penetration and the effect to be expected from an explosion under known conditions. The velocity of the shrapnel bullets or of the flying fragments of a shell on the surface of the cone is given by the equation

$$V = \sqrt{V_s^2 + (v + v_z)^2}.$$

If we regard this as the initial velocity, we can calculate the distance at which the highest and the lowest bullet will have sufficient *vis viva* to put a man or horse out of action. After the point of burst the uppermost bullet will move at first upwards or downwards, according as the acute angle of descent of the trajectory is less or greater than $\frac{\gamma}{2}$.

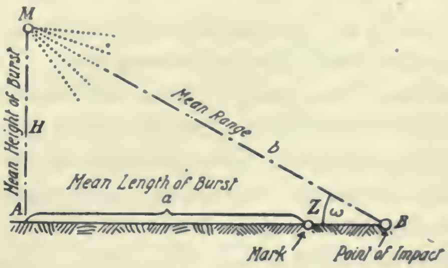
If the quadratic law of air-resistance is assumed, and if, as a first approximation, the path of the bullet is supposed to be rectilinear, the following formula may be employed. The range b, at the end of which the velocity of the bullet has fallen from V to v, is given by

$$b = \frac{p_1}{\kappa d^2 \delta} \log \frac{V}{v},$$

where p_1 is the weight of the bullet in kg, d its diameter in metres, δ is the density of the atmosphere, (e.g., $= 1 \cdot 22$ kg/cub m): $\kappa = 0 \cdot 367$, $0 \cdot 269$, $0 \cdot 166$, $0 \cdot 132$, for the following values of $\frac{V+v}{2}$ respectively, viz., 400, 300, 200, and 100 m/sec.

H. Rohne has made calculations for a given gun and a shrapnel bullet, weighing 10 grammes. If the bullet at the point of burst has a velocity of 400, 300, or 200 m/sec respectively, then after ranges of 300, 262, and 145 m, its *vis viva* will be 8 m-kg, and its velocity 125 m/sec.

It is possible in this way to determine whether the distance of the target from the point of burst is correct. Let us suppose that the shell bursts over level ground at the point M, the target being at Z. A is the point vertically below M, and B would be the mean point of impact, if the shell did not burst. Then MB is the distance along the trajectory from the point of burst, MA is the height of burst, AZ is the range of burst, and AB is the range from the point of burst to the hypothetical point of impact. If things are properly adjusted, Z and B will tend to coincide. If MB is a very small part of the whole of the trajectory above the muzzle-horizon, MAB can be regarded as a right-angled triangle, and on this assumption the following considerations are based.



The range and height of burst must be sufficient to enable the bullets to strike a number of men: in other words, there must be a sufficient distance over which the bullets can scatter. Obviously on the other hand this distance must not be too great. Experience, however, shows that it is better for this distance to be too great rather than too small.

Let $AZ = a$ and $MB = b$. The values of a and b with a given type of shrapnel and a given gun are usually kept constant, for a definite initial velocity: this value is usually about 60 metres. Then the height of burst is $a \tan \omega$, where ω is the acute angle of descent, and its value increases more rapidly than the range. Rohne states that in France and Austria, the apparent height of burst, which is the angle subtended by the height of burst at the muzzle, is kept constant. In this case, b decreases with an increase of range: the values of b with the French F.K. gun for ranges of 1000, 2000, and 4000 m are 122, 94, and 64 m respectively. The dispersion of the bullets varies correspondingly. With the old Krupp guns (up to 1905), a was kept constant for all ranges, and had a value of about 60 m. With the modern field guns, made by the same firm, Rohne says that a decreases slowly in proportion to the limiting value of the horizontal velocity, $v_e \cos \omega$.

Thus $$a = \tfrac{1}{4} v_e \cos \omega, \quad \text{.........................(9)}$$

and therefore the height of burst $= H = \dfrac{v_e \sin \omega}{4}$.

If for a given range a, H $(= a \tan \omega)$, and b $(= a \sec \omega)$ are known, together with the conical angle of dispersion, γ, and the number of bullets in the shell, N, then the bullet-density on the target can be reckoned as follows. Let us take a section through the cone of dispersion on a plane at right angles to the axis at a distance b from the point of burst. The section is a circle of radius ρ. Let the bullet density on this circle be D: that is to say, D bullets strike the circle on an area of 1 sq m. We usually neglect 15 per cent of the bullets, which are apt to disperse irregularly: the remainder, $0{\cdot}85N$, will strike on an area $= \pi\rho^2 = \dfrac{0{\cdot}85N}{D}$. Since $\tan \dfrac{\gamma}{2} = \dfrac{\rho}{b}$, we have

$$D = \frac{0{\cdot}85N}{\pi b^2 \tan^2 \dfrac{\gamma}{2}}.$$

If we take the instance of the French F.K. gun, with $N = 291$, $\dfrac{\gamma}{2} = 12^\circ\, 9'$, and $b = 50$ m, we have

$$D = \frac{0{\cdot}85 \times 291}{\pi\,(50 \tan 12^\circ\, 9')^2} = 1{\cdot}4.$$

Therefore 14 bullets will be found to be spread over an area of 10 sq m.

Heydenreich gives particulars as to the measurement of the depth of penetration of shrapnel: a reference is made to this in the notes at the end of this volume.

2. On the size of a shell-hole.

J. de la Llave, E. Vallier, and N. Sabudski give some purely empirical formulae relating to the size of shell-holes in earth and masonry: but they themselves point out that great caution is necessary in the employment of these formulae, as there may easily be errors amounting to 50 per cent. These methods can therefore serve only as rough approximations.

Let the size of the shell-hole be denoted by J cub m, any earth which falls back into the hole being supposed to be removed, and let the weight of the charge in the shell be L kg. Then the formulae are as follows.

(*a*) For earth $J = \mu m \lambda L$, where μ has the value 0·503, if the velocity of impact is less than 300 m/sec, and the value 0·816, if it is greater than 300 m/sec: m is a coefficient, which depends on the nature of the soil, and has the value 0·7 for hard pasture land, 0·85 for sandy soil, 1·0 for ordinary earth, and 1·2 for ploughed land. λ depends on the nature of the explosive: its value is unity for ordinary gunpowder or picric acid, when there is no time-fuse: it is equal to 2 for damp gun-cotton. With time-fuses Sabudski thinks that J is 1·4 times as great as the value given by the formula. As for the shape of the shell-hole, the depth is said to be about $\frac{1}{4}$ of the diameter, and $J = \frac{3\pi d^2 t}{16}$.

(*b*) With masonry, $J = 0{\cdot}194 X \lambda L$, where X is the depth of penetration in metres, as found in § 71 for the velocity-component in a direction normal to the surface of impact: L is the weight of the charge in kg. λ depends on the nature of the explosive: it is unity for gunpowder; from 2 to 2·1 for gun-cotton and from 2 to 2·2 for picric acid. If the diameter of the hole is d, then the depth is given by $J = \frac{\pi}{8} d^2 t$. For concrete, de la Llave uses the coefficient 0·035 instead of 0·194, if the explosion takes place in freshly made material: if the concrete has set a long time, this coefficient should be 0·014. Figures, which have been recently obtained, cannot at present be published.

§ 74. On the penetration of shells through water or mud. The effects of Dum-dum bullets.

When a bullet penetrates into a soft body, like that of a horse or a man, an explosive effect is produced within the body. If it strikes a pond of water, the spray dashes to all sides: or if it falls on a mass of mud, it excavates a volume many times its own size, the shape and dimensions of the excavation depending on the circumstances of the case. A similar effect on a smaller scale takes place with lead, but nothing of the kind happens with dry sand or wood. Details of phenomena of this kind can be found in the medical reports of the Prussian Ministry of War, to which reference is made in the notes.

The various explanatory theories may be summarised as follows:

1. The compression of the projectile produces a flattening at the tip, and a certain amount of extrusion at the sides. Therefore a certain portion of the mud is pushed sideways, as shown in the figure, and this undoubtedly explains something. The same thing happens with a bullet fired with the tip backwards except that in this position it is more easily compressed, while the effects produced with dum-dum bullets are well known. There is also the explosive effect with heavy shells, so that (1) is not the main cause.

2. The temperature of the projectile is high, and this might tend to volatilise the water or lead. Some attribute the explosion to the steam pressure. However it seems improbable that there is sufficient time to volatilise any considerable volume of water. Much experimental work has been done to determine the temperature of the projectile immediately after penetration. If the bullet is fired into sulphur, gunpowder or gun-cotton, these materials are not inflamed: this seems to give an upper limit for the temperature. The bullet has sometimes been extracted immediately and its temperature measured by the calorimeter: figures between 70° and 110° C. have been found in this way. An easily fusible metallic core has been inserted in a bullet: e.g. one made of Wood's metal, which fuses between 65° and 70°, or Rose's metal (95°), or Pb_3Bi_8 (125°). With the bullet M. 88, the maximum temperature was found to lie between 140° and 160°. So that the volatilisation-theory seems to fail.

3. Some people suggest that the projectile resembles a spinning top, so that powerful vibratory movements are set up within the mud.

But experiments in the ballistic laboratory showed that precisely similar explosive effects were produced by non-spinning projectiles, fired from a smooth bore. Medical reports seem to indicate that some of the effects on the human body may be partially due to a cause of this character, though this is not the main cause.

4. The pressure due to the air-waves is also very unlikely to be important in this connection. E. Mach has shown that it is not possible to suppose that a definite mass of air-particles is carried forward by the bullet: he has proved that it is only a vibratory movement in the atmosphere, which takes place from instant to instant. Neither have volumes of gas been observed to issue from a body after impact. Mach measured the air-pressures in different directions round a shell, and found them to be too small to explain any explosive effect. Neither do the tales appear likely to be true according to which men have been killed by the air-pressure, due to a shell flying past them.

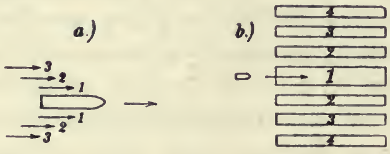
Lehmann has lately suggested the pressure of the head-wave as being a likely cause of the apparent explosive effect. P. A. Günther has therefore carried out certain experiments in conjunction with the author in order to ascertain the facts. Plates and balls of damp clay of different sizes were taken: holes were bored in them, slightly larger in diameter than the bullet, which was fired through them. If the head-wave theory were correct, there ought to be an explosive effect, but there was no trace of anything of the kind. In another experiment the bullet was fired immediately over the surface of a vessel that was full to the brim with water or mercury: here again no effect was observed. A ball of moist clay was put in a vacuum under the air pump, and a bullet was fired through it. It was found that the explosive effect took place in a vacuum in the usual way. The head-wave theory therefore seems to be unsupported by experimental evidence, and may consequently be dismissed.

5. It is known that hydraulic pressure is transmitted in all directions through the fluid, and it has been suggested that a bullet, flying through water, transmits such pressures like the cylinder of a hydraulic ram. Seeing that by Pascal's law the pressure is transmitted in every direction, this would seem to furnish a simple explanation of the surprising fact that the water is shot out violently on all sides, and even in the direction of the gun. But there are serious objections to this theory. Fluids are very slightly compressible. If there is a high pressure inside a water vessel, a slight crack in the vessel is sufficient to reduce the pressure at once to that of the atmosphere.

This is the reason why hydraulic pressure is not so dangerous as pneumatic pressure. If a vessel breaks under hydraulic pressure, the fragments have very small velocities, while the reverse is the case with gases under pressure.

In the present case, however, the fragments in certain cases are flung to considerable distances. In moist clay there is a shell-hole, more than 400 times the size of the projectile: and this seems to make the hydraulic theory unlikely. Moreover it fails entirely to explain why any effect is produced by firing a bullet into an open water vessel, or in cases where there can be no question of any hydraulic pressure. It is none the less probable that a good deal depends on the smallness of the forces of cohesion and the ease with which the particles can be displaced relatively to one another: and these facts are also of importance in connection with Pascal's law. The maximum explosive effect is produced in cases in which the friction of the particles is a minimum: under such conditions, they can be relatively displaced with great ease.

6. It might be thought that the viscosity of the fluid had something to do with the matter.



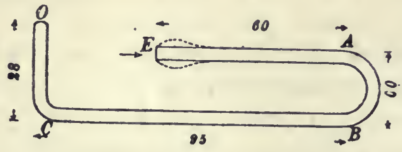
In fig. *a*, the bullet seems to drag the adjacent layers of air, while external layers are to some extent adherent to the internal ones: thus the whole mass of water takes part in the motion to a greater or less extent, just as it does in a vertical rotating cylinder, filled with water. The long interval which elapses between the moment of penetration and the moment of the explosion seems to be explained simply in this way. But the explosive effects in a lateral direction are not quite so easy to understand. Let us take a lump of clay, as in fig. *b*, that is bored with vertical holes, parallel to the direction of the bullet. Then if the viscosity-theory were true, we should expect channel 1, through which the bullet flies, to be fractured, but not 2—2, and still less 3—3. This, however, is not the case, since all the channels partake of the motion. Under given conditions the explosive effect would increase with the internal friction of a fluid or semi-fluid body on the viscosity-theory: it would be greater with mud or pitch than with water, and with lead or copper it would be still greater, these metals being to some extent of a semi-fluid nature. As a matter of fact, lead does exhibit explosive effect, but it is much less noticeable than in the case of water.

There are still two other theories, between which we must choose, and these will now be considered.

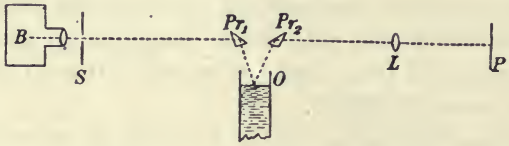
7. The pressure-wave theory was enunciated by Reger in 1884, and is somewhat as follows. If the projectile strikes a sheet of water at high velocity, the water receives a blow: a longitudinal sound-wave, consisting of condensations and rarefactions, travels in all directions in consequence of the elasticity of the fluid. The velocity of propagation is that of sound in water, and is about 1435 m/sec: possibly it is somewhat higher, since the velocity of sound, both in air and water, depends to some extent on the intensity of the blow. When the vibratory wave reaches the surface, where there is no external hydraulic pressure, the superficial layer, A, is shot off. When the next wave of condensation arrives at the surface, the next layer B is similarly shot off. There is a certain analogy with a thundering waterfall (Cagniard-Latour and Dvorak), from which a kind of mist seems to distil.

This explanation seems likely to help us to understand the phenomenon. The vibrations, due to the impact, must under any circumstances travel through the water in all directions. If we fire into very deep water, there is no real explosion in spite of the fact that the waves of sound extend in all directions. When mines are exploded, it has often happened that the shock has been observed. (See § 88.) If a block of stone is struck, and is not entirely penetrated, it sometimes happens that fragments break off on the rear side: this may be explained on the assumption of a longitudinal wave of impact. But the powerful explosive effect, which is seen from shooting bullets into water or masses of clay, is not produced by waves of elastic condensation. This is shown by the following experiments, which were carried out by K. R. Koch and the author in 1900.

(*a*) A lead tube has inner and external diameters of 4·6 and 5·5 cm respectively: it is closed at the end E. The portion, EA, is 60 cm long and lies in the direction of fire: the tube is bent, as shown, so that a length of 95 cm is horizontal, while the final portion of 28 cm is vertical: the tube is filled with water, and is open to the air at O. The vibrations on the surface of O are photographically recorded as a function of the time by the methods employed by the author in recording the vibrations of the barrel of a rifle in 1899.

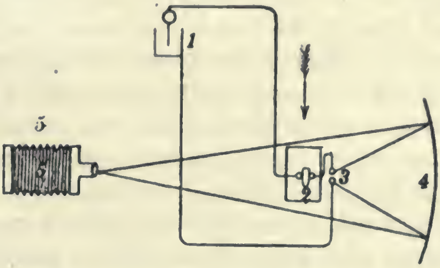


(See Vol. III, § 183.) The following arrangement, shown diagrammatically in the figure, was adopted. B is an arc lamp, the light from which passes through a stop S, and is totally reflected from the prism Pr_1 to the water-surface, O: from O it is reflected to the prism Pr_2, and thence by total reflection it passes through the lens L to the sensitised plate P. An image of the opening in the stop is produced on the plate. If the plate is rapidly pulled away in a direction



at right angles to the direction of the light, a narrow straight line is produced as long as the water is at rest: a very slight vibration on the surface of the water causes a marked rise or fall in the position of the line. The moment of firing is recorded by the image of a spark, a, while the time-intervals are given by the records of the vibrations of a tuning-fork. The results are given in Vol. IV, No. 24, where the photographic records are reproduced. CD shows the vibrations of the tuning-fork, AB is the line given by the surface of the water, using the first procedure: the perpendicular from a gives the instant of firing: the period of vibration of the tuning-fork was 0·0023 second. It will be seen that the first vibration, B, occurs much later than is to be expected from the sound-wave theory.

(*b*) Further experiments were made to examine the course of the explosion. A cylinder of sheet metal, 15 cm long and 12·5 cm in diameter, was suspended horizontally by cords: the cylinder was filled with water, the end, facing the bullet, being closed by parchment paper, and the other end by a thin sheet of rubber. The bullet measured 6 mm in diameter, and had a velocity of 750 m/sec. The cylinder was placed between a concave mirror, 4, and a camera, 5: the illuminating spark-gap is seen at 3. The bullet is seen to be flying in the direction of the arrow and to be passing through the glass-tipped spark-gap at 2. When the bullet reaches this position, the Leyden jar discharges through the path 1, 2, 3, 1. It is then



possible to observe the shape of the rubber-sheet, while the bullet is passing through the water. The position of the spark-gap, 2, can be varied, and it can also be placed outside the cylinder. The following conclusions were arrived at. As long as the bullet is inside the water, there is not the slightest distension of the rubber sheet: the whole vessel appears to be at rest. It is only after the bullet has completely penetrated through the vessel that the first distension of the rubber is perceptible, and this takes place when the base of the bullet is about 1 cm from the rubber sheet. At the point where the bullet penetrates through the parchment paper, the water soon spurts out towards the rifle: this goes on increasing, and the edges of the hole are gradually broken away. The rubber sheet is quickly distended in tubular form after the bullet has passed through the cylinder, though this distension does not begin till the bullet is clear by 1 cm: the water naturally streams out through the tube. A peculiar peaky distension takes place at the upper edge of the rubber and is gradually accentuated.

Another experiment was made by firing through a wrought-iron pipe, 1 m long; the external diameter was 13·8 cm, and the thickness of the wall, 8 mm. The ends were sealed, as before, with parchment paper and a sheet of rubber, the pipe being filled with water. On the upper side of the pipe, there was a longitudinal slit, 2 cm broad, in order to observe any lateral effects: the tube was prevented from bending by three stout rings. The longitudinal slit was closed by a wedge of wood, which was bound round by 40 turns of iron wire, 1·75 mm in diameter. As a matter of fact, the rubber membrane was always broken before the bullet reached it, but the records showed that it did not break instantly by any means. The rubber membrane bulges out and finally breaks. If it is very tightly stretched, it breaks round the edges: if it is kept in tension by support on the one side, it slits: if it is merely bound on without tension, a small round hole is formed, the portion, shot off, being recovered entire in the form of a circular piece, from 0·5 to 8 cm in diameter. The force of the explosive effect may be judged from the fact that the iron binding wires were broken and had to be replaced by hemp cords, 6 mm in diameter.

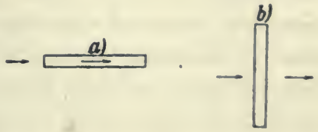
Similar results were obtained with lower velocities and smaller volumes of water. Shots from a Flobert pistol were fired through a lead pipe, 20 cm long, and 5·5 cm in external diameter, filled with water: on the upper side of the pipe there were 6 small circular holes. It was plainly seen that a definite volume of water was shot out in the form of a mushroom, and at the same time the rubber membrane was distended: water was also shot out at the same time through the upper holes, the outflow being of the same type. The same thing happened, when firing through a pig's bladder, filled with water. Firstly there was a cone of water emitted at the point of entry, and then on the side of exit: these streams gradually increased in volume. The bladder, as a whole, seemed for some time to be at rest. The bladder exploded after the bullet had travelled over a distance of 245 cm from the bladder, having meanwhile penetrated a board, 4·3 cm thick. The same sort of thing happened when firing through clay balls.

Tielmann made kinematographic records, when firing through a skull. His apparatus took 50 records in a second, and it was found that the roof of the skull was broken off in 0·04 sec. Bircher shot through a cylinder of sheet iron, having a broad soldered joint: from the shape of the broken vessel, it was clearly evident that the breakage took place after the bullet had passed through it. In 1903, Kranzfelder and Schwinning took instantaneous photographs by discharging 10 Leyden jars in succession by one bullet; thus they had a series of 10 records, derived from one bullet. Finally we may mention the kinematographic methods, referred to in Vol. III, § 188, which give from 5000 to 100,000 records per second: portions of several of these are reproduced in Vol. IV, Tables VII to IX.

8. The result of these experiments is that it is possible to form some idea of the processes accompanying the explosion. The energy of the projectile is wholly or partially transferred to the body which

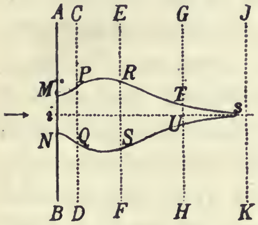
it strikes, this energy being at first communicated to the particles with which it comes into immediate contact, and gradually distributed throughout the mass. The particles of the body themselves become of the nature of projectiles, which have considerable velocities, though these are gradually dissipated by overcoming the resistance of the surroundings. Thus the various particles are set in motion with very considerable accelerations, and move in the directions of least resistance. The force of the explosion is most marked when all the particles can be easily displaced with regard to one another, as in the case of fluids. The explosive effect is not produced in bodies where there is great internal friction, in which case the energy of motion is dissipated in the form of heat. The process of explosion is very similar to that which would take place, if a charge were placed within the body and then ignited: the difference consists in the method by which the particles receive their accelerations, though the results are the same in both cases. In shell-holes, the craters are formed so that the axis is in the direction of least resistance.

If we fire into a plate of clay, the particles fly at first towards the rifle, because this is the direction of least resistance. Within the body there are pressure-zones round the line of fire: their diameters depend on the energy of the projectile and the nature of the material, i.e., on the internal friction and the force of cohesion between the particles. A given mass of water or clay can be so arranged that an explosive effect is produced, or it can be so arranged that nothing of the kind can be noticed. In the first case, the mass is arranged so that it is mostly in the line of fire, as at *a*, or it can be largely arranged in a plane, perpendicular to the line of fire as at *b*.

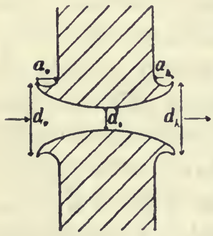


These facts agree with the violent effects produced on water: the explosion in thick mud is less violent, and much less so in the case of wood, rubber, and dry sand. On account of the friction between the sharp-edged particles of quartz, it is scarcely possible to explode a sand heap. A bullet, 6 mm in diameter, moving with a velocity of 800 m/sec, will only penetrate into dry sand to a depth of 15 cm: the sand feels hot, the covering of the bullet has a bluish colour, and nearly the whole of the energy is converted into heat. A large part of the projectile seems to volatilise. At the Mauser rifle factory in Oberndorf, more than a million bullets, each weighing 10 grammes, were fired into a sand heap, and only 500 kilogrammes were afterwards collected from the heap instead of 10,000 kg. If the sand is moist, there is a tendency to produce the explosive effect: the moisture acts as a kind of lubricator and decreases the internal friction.

If bullets are fired from a rifle or pistol, with velocities not less than 600 or 700 m/sec, into a bank of clay, *ABKJ*, the only possible explosive effect will be in the direction towards the rifle: the resistance in all other directions is too great. A hole will be formed, as shown in the figure. At the mouth, the hole is of the shape of a crater: its diameter is much larger than that of the projectile, except in the case when there is some sort of support along *AB* in the form of a board or a piece of sheet metal. Behind this there is a considerable excavation: at this point most of the energy is given off and is dissipated at a point where the resistance is high. With rubber, the excavation closes almost entirely owing to the elasticity: with water, it closes by gravity. If sections of the clay bank are made through *CD*, *EF*, and *GH*, the sections are similar in shape. Careful experiments have been made with clay. Thiel has found with certain types of bullet, which have the tip slightly filed away, that the hole at the point of entry into the clay is fairly smooth, and that the edges seem to be pulled inwards. He considers this to be a secondary effect: the edges of the hole are at first turned up on the outside: in a short time there is a rarefaction of the air owing to the explosive effect of the bullet: this sucks the air inwards and so alters the shape of the edge of the hole. Schatte has confirmed this view by photographic methods, which show that the edge is at first bent outwards and then inwards: for the purpose of his experiments he used the S-bullet at short ranges, firing with the normal initial velocity.



It may be of interest to give some results which were obtained in the author's laboratory in 1909 by firing the S-bullet into damp clay: the specific gravity of the clay was 1·8.



(*a*) Rectangular plates served as targets for the bullet M. 98 S: height, breadth, and thickness of plates, 60, 60, and 10 cm respectively. The velocity of impact, v_0, was varied: diameter of hole of perforation, at front d_v, in the middle d_i, at the back d_h: depth of projection at front a_v, and at back a_h. See table on opposite page.

(*b*) Clay plates, 10 cm thick, $v_0 = 870$ m/sec. The experiment was to determine the maximum superficial area that could be completely shattered by the explosion. It was found that plates the sides of which measured 20 centimetres, were just on the border line, and were generally broken into fragments.

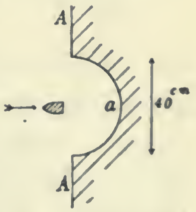
v_0	d_v	d_i	d_h	a_v	a_h
	cm	cm	cm	cm	cm
870	10	8	11	4	4
710	10	8	11	4	3·5
633	5·5	4·5	10	2	3
525	4	4	8	2	2·5
475	4	4	7·5	2	2·5
400	3·5	3·5	7	2	2
388	3	3	7	2	1·5
330	3	4	6	2	2
200	2	3	5	—	1·5
100	1	2	3	—	1

(*c*) Clay plates: length = breadth = 60 cm. Thickness variable: velocity of impact = 870 m/sec.

Thickness of plate	d_v	d_i	d_h	a_v	a_h
cm	cm	cm	cm	cm	cm
0·5	4·5	3·2	4·5	3·5	0·4
2·0	8	5·5	8·5	1·5	1·5
4·0	11	7·5	12	3	3
6·0	11	7·5	12·5	3·5	4·5
10·0	12	8	14	3·5	4·5
20	11	12	25	5	5
30	12	15	20	5	5

(*d*) Clay balls of variable diameters: velocity of impact = 870 m/sec. When the diameter was less than 30 cm, the balls were totally destroyed. With a ball, having a diameter of 45 cm, the bullet did not break it up: $d_v = 4$ cm, and $d_h = 8$ cm. Inside there was a nearly spherical hole, 25 cm in diameter. At the points both of entry and of exit, the edges of the holes were drawn somewhat inwards: this is the so-called after-effect: portions of the fragments from the edges were carried inside the ball. The bullet changed its course inside the ball: the point of exit was on a higher level than the point of entry, though the bullet was fired horizontally.

(*e*) The same clay plates as in (*a*). $v_0 = 870$ m/sec. The bullet was at first fired with the tip forwards, and then with the base forwards. In the first case, $d_i = 8$ cm: in the second case, when the bullet was reversed in the cartridge, $d_i = 26$ cm.

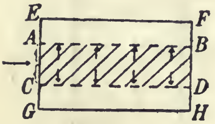


(*f*) A bullet was fired, with the base forwards, at a large lump of clay, which had a volume of about 1 cub metre. A hemispherical depression was formed on the surface of the block: the diameter of the depression was 40 cm, and the depth 20 cm: the edges were bent up outwards. At the deepest part of the depression, *a*, was found the broken steel covering of the bullet, completely detached from the lead core.

§ 75. Deflection of the projectile from the normal trajectory.

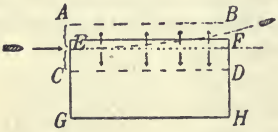
1. Deflection through the action of the target.

It has been already explained that there is a pressure-zone, *ABDC*, about the line of fire, when the bullet penetrates a block of clay: the projectile strikes the adjacent particles, which in turn react on their neighbours. There is consequently a considerable reaction on the bullet itself, which tends to deflect it from its course. If the bullet is fired from a smooth bore, there is no great deflecting tendency, if the mass of the clay is very great, and if it is homogeneous. In such a case, the pressures would be distributed symmetrically about the trajectory.



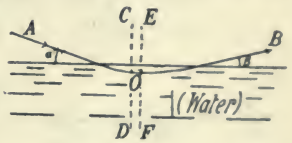
But with rotating bullets, things are different. Suppose the rifling to be right-handed: then looking at it from behind, the bullet rotates clockwise: consequently a very slight initial obliquity of the axis of the bullet with respect to the tangent to the trajectory, or even a lack of complete homogeneity in the clay, would suffice to produce very marked oscillations in the clockwise direction. If at the moment of entering the clay, the tip of the bullet is very slightly pointed upwards, the tip will be lifted by the resistance of the material and then goes to the right. The bullet, as a whole, will then go upwards, and afterwards turn to the right, if the mass of the clay is sufficient in amount. The opposite directions of motion will hold, if the tip is pointed downwards on entry. These effects have been often noticed with bullets and shells. Sometimes a boomerang-like path has been observed within a target, and a bullet has been even known to come out again on the side of entry.

But there are other causes besides rotation which may deflect a projectile; a target, like a muddy bank, may be struck on the one side, so that the pressure-zone, *ABDC*, extends to the surface, or even falls partially outside the body. In such a case the reaction on the lower side of the projectile is greater than that on the upper side. The bullet is therefore deflected upwards, moving along the path of least resistance. With grazing shots of this character, it is easy to understand that there may be a considerable deflection: a shell striking the water in a more or less horizontal direction may be deflected upwards from the surface, if the velocity is sufficiently great in amount.



2. Ricochets.

A ricochet can also occur when the impact takes place at a small angle with the surface of the object struck: the conditions depend on the circumstances of the case. von Chrismar states that French shells, with diameters of 10 cm, ricochet if the acute angle of descent on a sandy surface is less than 10°: 32 cm shells ricochet up to angles of 28°: while on the Krupp firing range, 26 cm shells strike the frozen ground at a distance of 1500 metres and then bound further over distances of 8000 metres. von Chrismar further reports that shells ricochet from the surface of a smooth sea, if the angle of descent is less than 25°: mostly they experience a considerable lateral deflection. French 32 cm shells make in this way a number of ricochets, the total ranges of which may measure from 1500 m to 11000 m. Ramsauer has made some interesting tests on this point. He fired some brass balls, weighing 5·85 grammes, and 11 mm in diameter, from a smooth bore with velocities between 621 and 625 m/sec.



The angles of impact were different in the various cases: the firing took place into a large tank of water. The angle at which the bullet emerged was measured, and also its velocity. If the angle of impact, α, was greater than 7°, the bullet did not come to the surface. The results were as follows.

$\alpha = 1°\ 1'\ 23''$,	$\beta = 1°\ 0'\ 17''$,	$\alpha - \beta = 1'\ 6''$
1 58 12	1 54 17	3 55
3 2 55	2 51 34	11 21
4 0 34	3 47 32	13 2
5 0 49	4 39 12	21 37
5 59 40	5 33 51	25 49
6 40 13	5 52 3	48 10

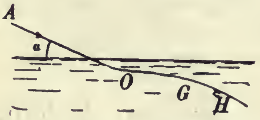
The figures show that β is always less than α. This explains why a number of springs on the surface of the water are possible, and why the well-known phenomenon with flat stones takes place.

The trajectories beneath the surface of the water were examined by means of vertical screens, placed at suitable distances below the surface. Bircher's procedure was used to prove that the screens were ruptured by the passage of the balls. Paths, similar to AOB, have often been observed with shells, when they strike a soft grass-grown surface.

The second part of the trajectory, OB, is therefore quite independent of the first part, OA. If a thin layer of air, $CDFE$, were supposed to exist at the lowest point, O, this would not affect the matter in any way. This is in agreement with what has been said about firing horizontally through water at no great distance below the surface. Ramsauer also showed that a similar lifting of the bullet takes place when it is fired through 10 parallel lead plates, 3 mm thick, placed vertically at distances of 2·5 cm, passing horizontally at a maximum distance of 9 mm from their upper edges.

The velocity of egress was measured for different angles of impact by Pouillet's procedure: thus we have

$\alpha =$	1° 2′ 13″	2° 0′ 44″	...	6° 2′ 31″	6° 49′ 27″
$v_e =$	608·3	571·5	...	221·5	67·5 m/sec.



In these tests $v_0 = 625{\cdot}3$ m/sec. If $\alpha = 7°$, the velocity of the projectile is diminished to such an extent by the resistance of the water, that it no longer rises to the surface, but continues to fall downwards along the path OGH.

NOTES AND APPENDIX

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techn. Zeitschr. 10 (1907), p. 14 on the law of Chapel-Vallier. On the Mayevski-Sabudski zone-laws, see Mayevski, p. 41 (1872); Sabudski, *Petersburg Art. Journ.* 1894, 4, p. 299, and Klussmann, *Arch. f. Art.- u. Ing.-Off.* 102 (1895), p. 18. For Siacci's law, see *Riv. d' artigl. e gen.* 1896, vol. I. pp. 5, 195, 341, and *Arch. f. Art.- u. Ing.-Off.* 103 (1896), pp. 5, 195 and specially p. 341; K. Becker and C. Cranz, *Artill. Monatshefte*, 1912, 69, p. 189, and 1912, 71, p. 833; O. von Eberhard, ditto, 1912, 69, p. 196; J. H. Hardcastle, *Arms and Explosives*, 1913, 249, vol. XXI. p. 86, translation of the paper by K. Becker and C. Cranz; M. Thiesen, *Ann. Phys. Chem.* 2, 26 (1885), p. 314; von Lössl, *Laws of air-resistance*, Vienna, 1896. Patent for smoke-emitting shells, by Semple, D. R. P. 190,051, class I. 72 d, Aug. 26, 1905. H. Rohne, *Artillerist. Monatshefte*, 1908, p. 347, and *Deutsch. Waffenzeitung*, 1908, p. 17; Ölker, *Zeitschr. f. d. ges. Schiess- u. Sprengstoff-Wesen*, 11 (1916), pp. 145, 167 and 185.

For projectiles showing lights, see the German patents, class 72 d, group 19, Nos. 242,554: 265,383: 268,324: 271,095: 272,070: 272,115. If smoke-emitting shells or light-emitting shells are used for tests on air-resistance, care must be taken to see that the trajectory is not affected by the devices that are employed.

For the measurement of air-resistance, see J. Didion, *Lois de la résistance de l'air*, Paris, 1857, and C. E. Page, *De la résistance de l'air*, Paris, 1878. For the determination of laws of air-resistance from experiment or by means of the method of least squares, see Siacci, p. 313; Sabudski, *Petersb. Art. Journ.* 1894, 4, p. 299: 1892, 6, p. 601, and Klussmann, *Arch. f. Art.- u. Ing.-Off.* 97 (1890), p. 546; Siacci, *Riv. d' art. e gen.* 1889, vol. III. p. 227, and 1891, vol. I. p. 199; *Arch. f. Art.- u. Ing.-Off.* 99 (1892), p. 172; J. Schatte, *Kriegstechn. Zeitschr.* 16 (1913), pp. 1, 57, and 111; A. Hamilton, *Journ. of the United Stat. Artill.* 24 (1905), pp. 31 and 99, and 30 (1908), p. 363; S. Finsterwalder, article on aerodynamics in *Encykl. d. mathem. Wissensch.* Leipsic, 1903, vol. IV. 17, 4, p. 163; C. F. Close, *Proc. Roy. Art. Inst.* 1905, January; G. Greenhill, *Journ. of the Roy. Artill.* 1906, February, and 1909, February, p. 473; C. E. Wolff, ditto, 1908, April.

§§ 12 and 13. Components of the force of air-resistance when the shell lies obliquely, and the calculation of form-values: Mayevski, p. 40; Mayevski-Klussmann, p. 58; St Robert, 1, pp. 251–276; Rutzki, p. 68; Siacci, p. 378; M. de Sparre, *Sur le mouvement des projectiles dans l'air*, Paris, 1891, p. 64; von Wuich, pp. 70–101, specially p. 92 with table; Cranz, *Zeitschr. Math. Phys.* 43 (1898), pp. 169 and 133; E. Kummer, *Berl. Abh.* 1875, p. 1, with experimental appendix, 1876, p. 1; Gauthier, *Ann. éc. norm.* 5 (1868), pp. 7–65; G. Wellner, *Zeitschr. f. Luftschiff.* 12 (1897), p. 237, and *Zeitschr. d. österr. Ing. u. Arch.-Ver.* 45 (1893), pp. 25–28; H. Résal, *Nouv. ann.* 2, 12 (1873), pp. 561–565; J. M. Ingalls, *Journ. of Un. Stat. Art.* 4 (1895), p. 191; A. von Obermayer, *Wien. Ber.* 104 (1895), p. 963; Duchemin, *Mémor. de l'art. marine*, 5 (1842), p. 65; P. Touche, *Rev. d'art.* 36 (1890), p. 131; E. Vallier, p. 10, and *Rev. d'art.* 36 (1890), p. 160; Ingalls, *Journ. of Un. Stat. Art.* 4 (1895), p. 208, and *Mitt. üb. Geg. d. Art.- u. Gen.-Wes.* 1896, p. 411; Siacci, p. 7; Sabudski, 1, pp. 57–90; *Lehre vom Schuss* by Heydenreich, 1908, II. p. 116; A. Hamilton, *Journ. of the Un. Stat. Art.* 1908, vol. III. and *Artillerist. Monatshefte*, 1909, 26, p. 133; A. Frank, *Zeitschr. d. Ver. Deutsch. Ing.* 1906.

The rules governing the construction of models are discussed by H. von Helmholtz, *Wissensch. Abhandl.* Leipsic, 1882, I. p. 158; H. Lorenz, *Zeitschr. d. Ver. Deutsch. Ing.* 1907, p. 1824; V. von Niesiolowski-Gawin, *Ausgew. Kapitel der Technik*, Vienna, 1908, p. 326.

§ 14. Regarding the best shape for the tip of the shell, see A. M. Legendre, Paris, *Mém. de l'Acad.* 1788, pp. 7–37; G. von Lamezan, *Arch. f. Art.- u. Ing.-Off.* 87 (1880), p. 485; Rutzki, pp. 30–51; F. August, *J. f. Math.* 103 (1888), pp. 1–24, and *Arch. f. Art.- u. Ing.-Off.* 94 (1887), p. 1; von Wuich, 1, p. 128; R. Benzivenga, *Riv. d' art. e gen.* 1897, vol. III. p. 123; B. von Lefèvre, *Rev. d'art.* 57 (1900), p. 221; A. Bassani, *La corrisp.* 1 (1900), p. 485; L. Decepts, *Rev. d'art.* 57 (1901), p. 425, and *La corrisp.* 2 (1901), p. 63; E. Armanini, *Ann. di mat.* 3, 4 (1900), pp. 131–149; E. Lampe, *Berlin. Verhandl. d. deutsch. phys. Ges.* 3 (1901), pp. 119 and 151; Kneser, *Arch. f. Math. u. Phys.* (3), 2 (1902), p. 267; L. Decepts, *Rev. d'art.* 57 (1900), p. 221; Lacroix, *Traité du calcul diff. et intégr.* 2nd edit. Paris, 1814, part 2, p. 791; S. von Kobbe, *Artill. Monatshefte*, 1911, II. 58, p. 283; M. de Masson d'Autume, *Mémorial de l'artill. nav.* series 3, vol. VII. (1913), 22, p. 481; Finsterwalder, *Encykl. d. math. Wissensch.* IV. 17, footnote 90.

Some shells taper at the rear, and are more or less torpedo-shaped. These have been examined by d'Alembert, Piobert, Dreyse, and Whitworth; they include the German D-shell and the Z-shell. See also *Zeitschr. f. d. ges. Schiess- u. Sprengstoffwes.* 5 (1910), No. 9, pp. 161–163, and Selter, *Zeitschr. f. d. ges. Schiess- u. Sprengstoffwes.* 10 (1915), pp. 125 and 142. For pointed shells, including those of the torpedo type, see Ayrolles, *Rev. d'artill.* 38 (1910), vol. LXXV. pp. 214 and 274: 38 (1910), vol. LXXVI. pp. 98, 148, and 275: 39 (1910–1911), vol. LXXVII. p. 356.

§ 15. See observations of the meteorological station in Bavaria, published in Munich in 1907, *Beiblätt. zu d. Annal. d. Physik* (1908), p. 558: during the lowest 3000 m, the temperature decreases at the rate of 0·57 degrees Centigrade per 100 m; at a height between 6 and 8 km, the temperature decreases by 0·71 degrees Centigrade per 100 m; at heights between 9 and 13 km, the temperature is constant at values between $-48°$ and $-60°$. See also § 111 and P. Charbonnier, *Traité de balistique extérieure*, Paris, 1904, p. 329. Also Linke, *Aeronautische Meteorologie*, Berlin, 1913; Fr. Fischli, *Aeronautische Meteorologie*, Berlin, 1913. For Siacci's formula (II), see *Balistique extérieure*, Paris, 1892, p. 14.

P. Charbonnier proposes to apply the linear formula, given in the text, for the decrease of atmospheric density with the height up to heights of about 2000 m, and has given a more complete formula. See also *Balistique extérieure rationnelle*, Paris, 1907, pp. 12 and 13. According to Schubert's figures (see § 111), the linear formula holds at considerable heights. It is quite easy to use Schubert's figures directly.

§§ 17 to 19. Some ranges are said to be longer than they would be in a vacuum. See von Minarelli, p. 37, and Darapsky, *Arch. f. d. Art.- u. Ing.-Off.* 69 (1871), p. 256, and the disc-like shells, proposed by St Robert, 2, pp. 1 and 49. With regard to the conception of the sectional load see Galileo Galilei, *Dialoghi delle nuove scienze*, Leiden, 1638, and Ostwald's *Klassiker der exakten Wissenschaften*; Galilei, *Unterredungen*, edited by A. J. von Ottingen, vol. XI. 24, 25.

St Robert first stated the general equations in § 17, and the propositions in § 19, viz. 1 to 8 and 11; the propositions 9 and 10 were added by Sabudski. See

St Robert, 1, pp. 50 and 336, *Tor. Mem.* 2, 16 (1855), pp. 434 and 498; Mayevski, pp. 52 and 71; Siacci, 1, p. 25, and for similar trajectories, p. 97; Sabudski, 1, p. 118, and *La corrisp.* 1 (1900), p. 293 and 2 (1901), p. 3; Siacci, *Riv. d' art. e gen.* 1901, vol. I. p. 287, and vol. II. p. 21; M. de Brettes, Paris, *C. R.* 67 (1868), p. 896; 68 (1869), p. 1336; 69 (1870), pp. 394 and 1239. For the proof of proposition 3, see Hjalmar Aner, *Artill. Monatshefte*, 1916, No. 118, p. 147.

The following theoretical discussions refer to the angle of departure, corresponding to the maximum range: F. Astier, *Rev. d'art.* 9 (1877), p. 313. He arrives at the result that it depends on the law of air-resistance as to whether the angle is greater or less than 45°. Siacci, p. 42 and p. 393, and *Mitteil. üb. Geg. d. Art.- u. Gen.-Wes.* 1888, p. 49; E. Vallier, *Rev. d'art.* 31 (1888), p. 362; Guébhard, *Nouv. ann.* (2), 13 (1874), pp. 436–438; R. Radau, Paris, *C. R.* 66 (1868), pp. 1032–1034; M. de Brettes, Paris, *C. R.* 66 (1868), p. 896; 68 (1869), pp. 1336–1338; 69 (1870), pp. 394–397 and pp. 1239–1242; N. Sabudski, *Über die Lösung des Problems des indirekten Schiessens u. d. Winkel grösster Schussweite* (Russian), Petrograd (1888), p. 83; Klussmann, *Arch. f. Art.- u. Ing.-Off.* 96 (1889), p. 376. E. Vallier gives the following rules: the angle is probably greater than 45° for a shell with a considerable sectional load, and a calibre greater than 24 cm; but in the case of any shell on which the air-resistance exercises a relatively great effect, the angle is less than 45°. There are, as yet, no sufficiently extensive practical tests on which to base conclusions, and figures with regard to bullets must be treated with great caution, seeing that they very rarely depend on exact measurements.

With reference to Piton-Bressant's results, see an anonymous article in the *Rev. d'art.* 8 (1876), p. 219, and G. Greenhill, *Mém. de l'artill. navale*, Paris, 1909, 2.

With reference to the integrability of the main equation, see J. L. d'Alembert, *Traité de l'équilibre et du mouvement des fluides*, Paris, 1744, p. 359; Siacci, *C. R.* 132 (1901), p. 1175, and 133 (1901), p. 381; *Riv. d' art. e gen.* 1901, vol. III. p. 5, and vol. IV. p. 5; P. Appell, *Arch. der Math. u. Phys.* 3, vol. V. (1903), p. 177; E. Ouivet, *C. R.* vol. CL. (1910), p. 1229; T. Hayashi, *Giorn. di Matematiche di Battaglini*, 3, vol. XLIX. (1911), p. 231; C. Cranz and R. Rothe, *Artill. Monatshefte*, 1917, who illustrate the procedure by two examples, and at the same time a method is indicated by which the change in the density of the air can be taken into account in a perfectly general way without using mean values.

The mechanical integration of the main equation by the planimeter is described by L. Filloux, *Rev. d'art.* 72 (1908), 6, p. 345, and see also E. Pascal, *I miei integrafi per equazioni differenziali*, Naples, 1914, published by L. C. Pellerano.

§ 20. J. Bernoulli, *Act. erudit.* 1719, p. 216, or *Collected Works*, vol. II. p. 394; A. M. Legendre, *Dissertation sur la question de balistique, proposée par l'Académie Roy. des Sciences et belles lettres de Prusse*, Berlin, 1782, partially reprinted in *Journ. école. polyt.* 4, Cah. 11 (1802), p. 204, and *Journ. des armes spéciales*, 1845, pp. 537 and 600, and 1846, p. 32; C. G. J. Jacobi, *J. f. Math.* 24 (1842), p. 25, or *Collected Works*, 4, p. 286. The matter is further investigated with the use of elliptic integrals by A. G. Greenhill, *Woolwich Roy. Art. Inst. Proc.* 11 (1881), pp. 131 and 589: 12 (1882), p. 17: 17 (1890), p. 181, and by Sabudski in the book previously mentioned, and in his *Äussere Ballistik*, 1, 550; see also L. Austerlitz, *Wien. Ber.* 84 (1882), p. 794. P. A. MacMahon's tables are included in Greenhill's paper. See also Th. Vahlen, *Archiv d. Math. u. Phys.* 25 (1916), vol. III. p. 209. For similar trajectories see St Robert, *Mémoir. scientifiques*, Paris, 1872, I. p. 313, and F. Siacci, *Bal. extér.* Paris, 1892, p. 97; E. Röggla, *Mitteil. üb. Geg. d. Art.- u. Gen.- Wes.* 1908, p. 224.

§ 21. L. Euler, *Berl. Ber.* 1753, p. 348; S. D. Poisson, *Traité de mécanique*, 2 vols. 2nd ed. Paris, 1833. Tables are given by Didion in an appendix to his *Traité de balistique.* With reference to Otto's tables, see J. C. F. Otto, *Tafeln für den Bombenwurf*, Berlin, 1842, *Gebrauchsanweisung*, p. 40; Vallier, p. 111; Siacci, *Rivista d' Artigleria e Genio*, 1885, vol. I. and *Revue d'artillerie*, 1885, vol. XXVI. p. 431; S. Braccialini, *Rev. d'art.* 27 (1885), p. 237; Ingalls, *Exterior Ballistics in the Plane of Fire*, New York, 1886, and *Journ. of Un. St. Art.* 51 (1896), pp. 52–74; von Scheve, *Arch. f. Art.- u. Ing.-Off.* 92 (1893), p. 529; 93 (1886), pp. 97, 271; 103 (1896), p. 236; these articles give extensions of Otto's tables. F. Mola, *Riv. d' art. e gen.* 1892, vol. III. p. 253, and *Arch. f. Art.- u. Ing.-Off.* 100 (1893), p. 1. Sabudski, 1, p. 239 and p. 252, takes into account the decrease of air-density with the altitude, and *Rev. d'art.* 34 (1889), p. 427; 38 (1891), p. 46; Mayevski-Klussmann, p. 34; A. Bassani, *La Corrisp.* 1 (1900), p. 116, and 1 (1900), p. 275.

§ 22. Bashforth, p. 45; Mayevski-Klussmann, p. 28; Vallier, p. 49.

§ 22*a*. Didion, p. 162; Ligowski, *Arch. f. Art.- u. Ing.-Off.* 81 (1877), pp. 79, 163, 178, and 83 (1878), p. 203; also Neumann, *Arch. f. Art.- u. Ing.-Off.* 6 (1838), p. 213; 14 (1842), p. 49; 29 (1851), p. 93; J. H. Lambert, *Berl. Abh.* 1767, pp. 102–188; J. C. Borda, Paris, *Hist. de l'Acad.* 1769, pp. 247–271; G. F. von Tempelhof, *Berl. Abh.* 1788–1789, pp. 216–299, separately printed as *Der preussische Bombardier*, Berlin, 1791. Français's work is mentioned by Didion, p. 168. Heim, p. 205; von Pfister, *Arch. f. Art.- u. Ing.-Off.* 88 (1881), p. 489; St Robert, 1, p. 125; Denecke, *Arch. f. Art.- u. Ing.-Off.* 90 (1883), p. 231 and p. 405; von Zedlitz, ditto, 103 (1896), p. 388.

With regard to the assumption of a definite curve-form with an experimental determination of the coefficients, see M. Prehn, *Ballistik der gezogenen Geschütze*, Berlin, 1864, and *Arch. f. Art.- u. Ing.-Off.* 74 (1873), p. 189; A. Mieg, *Theoretische äussere Ballistik*, Berlin, 1884; O. Dolliak, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1879, p. 3 of the notes; Hélie, 2, p. 267; ditto, p. 262, and Vallier, p. 186; see also *Rev. d'art.* 8 (1876), p. 219; E. Ökinghaus, *Die Hyperbel als ballistische Kurve*, *Arch. f. Art.- u. Ing.-Off.* 100 (1893), p. 241, with other articles between 1894 and 1896; F. Chapel, Paris, *C.R.* 120 (1895), p. 677; J. Stauber, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1897, p. 118, and 1909, p. 575; R. G. Fernandez, *Jahrbücher f. d. deutsche Armee u. Marine*, 1907, I. p. 206; F. Affolter, *Allgemein. Schweiz. Militärzeitung*, 1905, 52, p. 424, and 1906, 9, p. 67; P. Haupt, *Artill. Monatshefte*, 1915, II. Nos. 103–104, p. 1; C. Veithen, *Artill. Monatshefte*, 1917.

§ 23. J. C. Borda, Paris, *Hist. de l'Acad.* 1769, pp. 247–271, and *Journ. des armes spéciales*, 1846, p. 49; Besout, *Mouvement des projectiles*, Paris, 1788, pp. 138–197; Legendre, *Dissert. sur la question de balistique proposée par l'académie roy. des sciences*, Berlin, 1782: partly reprinted in the *J. éc. polyt.* 4, Cah. 11 (1802), p. 204, and *Journ. des armes spéciales*, 1845, pp. 537 and 600, and 1846, p. 32; Didion, p. 159 and p. 168; Didion, p. 59; St Robert, *Mémoires scientifiques*, I. Paris, 1872, p. 119 and p. 124; Siacci, *Riv. d' art. e gen.* 1897, vol. IV. p. 5; von

Wuich, p. 215, and *Mitt. üb. Geg. d. Art.- u. Gen.-Wes.* 1894, p. 424, and 1902, pp. 651 and 893; R. v. Portenschlag-Ledermayer, ditto, 1903, p. 563.

For the method, described as Siacci I, see *Rev. d'art.* 17 (1880), p. 45.

For the method, described as Siacci II, see Siacci, *Balistique extérieure*, Paris, 1892, p. 34, and *Rev. d'art.* 27 (1886), p. 315; also Heydenreich, *Lehre vom Schuss*, 1st ed. Berlin, 1898, 2, p. 90: for value of β, see Siacci, p. 36, and *Riv. d' art. e gen.* 1896, vol. I. p. 341, and 1897, vol. IV. p. 5.

For Vallier's method, see Vallier, p. 45; Sabudski, *Rev. d'art.* 34 (1889), p. 427: 29 (1886–87), p. 11: 36 (1890), pp. 42 and 153: 37 (1890), p. 273: *Balistique extérieure* (a portion of the *Encycl. scientif. des Aides-Mémoire*, Paris, no date), and *Balist. expériment.* Paris, 1894; E. Vallier, *Artill. Monatshefte*, 1912, No. 70, p. 253.

P. Charbonnier's method is described in his *Traité de balistique extérieure*, 2nd ed. 1904, p. 221; S. Takeda, *Artill. Monatshefte*, 1914, 89, p. 321, which connects the procedures of Didion and Siacci; J. Schatte, *Kriegstechn. Zeitschr.* 12 (1909), 9, p. 416; G. Bianchi, *Riv. d' art. e gen.* 27 (1910), vol. I. p. 175.

§ 24. J. Didion, *Traité de balistique*, Paris, 1848 and 1860; Paul de St Robert, *Mémoires scientifiques*, vol. I. *Balistique*, Paris, 1872; N. Mayevski, *Traité de bal. extér.* Paris, 1872.

§§ 25 and 26. St Robert and N. Mayevski, as in § 24. Siacci, *Balistique extérieure*, Paris, 1892 and *Ballistik u. Praxis*, Berlin, 1882; J. Bernoulli, *Acta erudita*, Leipsic, 1719, p. 1453: *Opera*, 2, pp. 393–402 and p. 513; von Wuich, p. 199. Siacci, pp. 86 and 455; Chapel, *Rev. d'art.* 17 (1881), p. 437, and 18 (1881), p. 484.

von Zedlitz, *Arch. f. Artill.- u. Ing.-Off.* 103 (1896), p. 388, and *Mitt. üb. Geg. d. Art.- u. Gen.-Wes.* 1898, p. 881; G. Ronca and A. Bassani, *Riv. maritim.* 1895, p. 569; Siacci, *Riv. d' art. e gen.* 1896, vol. II. p. 5; Ronca and Bassani, *Riv. marit.* 1897, p. 217.

§ 27. Krupp's earlier method, see *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1891, p. 1; W. Gross, *Die Berechnung der Schusstafeln*, Leipsic, 1901; W. Olsson, *Ballistike tabeller for beregning af skydetabeller*, Christiania, 1904: *Artill. Monatshefte*, 1908, p. 112.

§ 28. For Siacci II, see § 23. F. Pouchelon, *Rev. d'art.* 26 (1885), p. 467; W. C. Hojel, *Rev. d'art.* 24 (1884), p. 262; E. Vallier, p. 45A, and Paris, *C. R.* 115 (1892), p. 648. For Siacci III, see F. Siacci, *Riv. d' art. e gen.* 1896, vol. I. p. 341: *Riv. d' art. e gen.* 1897, vol. IV. p. 5 and *Rev. d'art.* 35 (1890), p. 493, and E. Fasella, *Tavole balistiche secondarie*, Genoa, 1901; Parodi, *Balistica esterna*, Turin, 1901, p. 105 and p. 314.

§ 29. Vallier, p. 45 and *Rev. d'art.* 29 (1888), p. 11; Sabudski, *Rev. d'art.* 34 (1889), p. 427; Vallier, *Rev. d'art.* 36 (1890), pp. 42 and 153, and 37 (1890), p. 273.

§ 30. P. Charbonnier, *Traité de bal. extér.* 2nd ed. Paris, 1904, p. 221: *Manuel de balistique extér.* Paris, 1908.

§ 31. Graphical methods are to be found in J. V. Poncelet, *Leçons de mécanique industrielle*, 2, Metz (1828), p. 55; Didion, p. 196; A. Indra, *Graphische Ballistik*, Vienna, 1876; C. Cranz, *Zeitschr. Math. Phys.* 42 (1897), p. 197. M. d'Ocagne's method is described by G. Pesci, *Riv. marit.* 1899, p. 113, and 1900, pp. 1–52 of the appendix; G. Ronca, *Riv. marit.* 1899, and *La corrisp.* 2 (1901), p. 278; R. von Portenschlag-Ledermayer, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1900, p. 796, and 1904, p. 769; G. Ronca, *Manuale del tiro*, Leghorn, 1901, p. 296; G. Ronca and

G. Pesci, *Abbachi per il tiro*, and *Abbachi generali della balistica*, Leghorn, 1901; Rothe, *Artill. Monatshefte*, 1911, II. 59, p. 371; Narath, ditto, 1915, I. p. 69; Rothe, ditto, 1915, 102, p. 314; A. Nowakowski, *Mitteil. üb. Geg. d. Art.- u. Gen.-Wes.* 1913, 7, p. 547, and 1913, 5, p. 383; Garbasso, *Riv. d' art. e gen.* 20 (1903), vol. II. p. 387; A. H. Barker, *Graphical Calculus*, London, 1908; J. E. Mayer, *Das Rechnen in der Technik*, Leipsic, 1908; R. Mehmke, *Numerisches Rechnen*, *Encykl. d. math. Wissensch.* vol. I. F, Leipsic, Teubner; A. Morley and W. Inchley, *Elementary Applied Mechanics*, London, 1911; M. d'Ocagne, *Coordonnées parallèles et axiales*, Paris, 1885: *Traité de nomographie*, Paris, 1899: *Calcul graphique et nomographie*, Paris, 1908; J. B. Peddle, *The Construction of Graphical Charts*, New York, 1910; J. Perry, *Praktische Mathematik*, Vienna, 1903; M. v. Pivani, *Graphische Darstellung* (Göschen), Berlin and Leipsic, 1914; F. Schilling, *Über die Nomographie von M. d'Ocagne*, Leipsic, 1900; L. von Schrutka, *Theorie u. Praxis des logarithm. Rechenschiebers*, Leipsic, 1911; E. Schultz, *Mathem. u. technische Tabellen*, edn. 2 B, Essen, 1911; Soreau, *Contribution à la théorie et aux applications de la nomographie*, Paris, 1901: *Nouveaux types d'abaques*, Paris, 1906.

§ 32. For Cauchy's process, see P. de St Robert, *Mémoires scient.* I. Paris, 1872, p. 160; Moigno, *Leçons sur le calcul diff. et intég.* II. 1844, leç. 26–28, and 33; Coriolis, *Journ. de Math. de Liouville*, 2 (1837), p. 229; Lipschitz, *Lehrb. d. Analysis*, II. (1880), p. 504; Picciati, *Il polytechnico*, Milan, 1893, vol. XLI. pp. 493 and 537; Runge, *Math. Ann.* 44 (1894), p. 437, and 46 (1895), p. 437; K. Heun, *Jahrb. d. deutsch. Math. Ver.* 9 (1900), p. 111, and *Zeitschr. f. Math. u. Phys.* 45 (1900), p. 23; Photogrammetric methods by O. Nowakowski, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1912, 3, p. 262.

§§ 35, 36 and 37. Cranz, *Schuss u. Waffe*, 2 (1909), 18, p. 413: *Artill. Monatshefte*, 1909, 30, pp. 412–415; Eckhardt, *Artill. Monatsh.* 1912, II. 61, p. 64; A. von Burgsdorff, *Zeitschr.*, *Schuss u. Waffe*, 2 (1908–9), 8, p. 179; H. Rohne, ditto, 2 (1908–9), 7, p. 152; the maximum vertical ascent of a shell is discussed in St Robert, *Mém. scient.* Paris, 1872, p. 43.

Didion's process for calculating high-angle trajectories in various sections is given in his *Traité de balistique*, Paris, 1860, p. 127. Also J. Schmidt, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1908, p. 431; von Zedlitz, *Artill. Monatsh.* 1913, 79, p. 1 and 1914, 88, p. 274; F. E. Harris, *Journ. of the Un. St. Artill.* 23 (1905), p. 43; P. Charbonnier, *Rev. d'artill.* 40, p. 79, p. 133: 79, March 1912, p. 357: 80, April 1912, p. 45; v. Portenschlag-Ledermayer, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1911, 7, p. 616.

§§ 38–40. With regard to tilting the trajectory, see Heydenreich, *Lehre vom Schuss*, Berlin, 1908, I. p. 106; Gouin, *Rev. d'art.* 35 (1907), p. 121; Percin, ditto, 19 (1882), p. 281 and 27 (1885), p. 118; A. von Burgsdorff, *Zeitschr. f. d. ges. Schiess.- u. Sprengstoff.- Wes.* 1 (1906), 18, p. 332, and *Schuss u. Waffe*, 2 (1907), 8, p. 179; Kerkhof, *Artill. Monatsh.* 1908, 13, p. 44. For Fernandez's and Gonzalez's methods, see *Jahrb. f. deutsche Armee u. Marine*, 1905, December; Kolarski, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1906, p. 301; Pucherna, ditto, 1908, p. 809.

§ 41. E. Vallier, *Balist. expér.* Paris, 1894, p. 45 for the factor m. For the change of the density of the air with the altitude, see Charbonnier, *Traité de bal. extér.* Paris, 1904, p. 329; A. Hamilton, *Journ. of the Un. St. Artill.* 100, Dec. 1909, p. 257; Cranz, *Artill. Monatsh.* 1909, 34, p. 241 for the accuracy of formula (23).

§ 42. For the primary tables for Siacci III, see F. Siacci, *Riv. d' art. e gen.* 1896, 1, p. 341: *Riv. d' art. e gen.* 1897, 4, p. 5: *Rev. d'art.* 25 (1890), p. 493; E. Fasella, *Tavole balistiche secondarie*, Genoa, 1901; F. Siacci, *Bal. extér.* 1892, p. 454.

§ 45. For the effect of small changes of v_0, ϕ, or c on X, see Siacci, p. 105; Vallier, p. 67; Denecke, *Arch. f. Art.- u. Ing.-Off.* 93 (1886), p. 1 and 94 (1887), p. 226: *Arch. f. Art.- u. Ing.-Off.* 97 (1890), p. 274; v. Pfister, ditto, 93 (1886), p. 73; Rohne, *Kriegstechn. Zeitschr.* 3 (1900), pp. 129 and 201: 4 (1901), p. 326; Sabudski, *Petrog. Art. Journ.* 1889, 11, p. 941; Charbonnier, *Traité de bal. extér.* 1894, p. 175.

On daily differences, Heydenreich, 1 (1st ed.), p. 53, and 2, p. 39; H. Rohne, *Kriegstechn. Zeitschr.* 3 (1900), pp. 129, 201: 4 (1901), p. 326; v. Minarelli, p. 61, and p. 53. Many figures are given by K. Exler, *Zeitschr. f. d. ges. Schiess.- u. Sprengstoff.- Wes.* 1 (1906), pp. 107, 127, 376, 399; M. de Sparre, *C. R.* 161, p. 767: 162, pp. 33, 496. He gives a method of calculating the trajectory in zones, taking into account the decrease of air-density. The velocity of the shell thus passes through a minimum and then through a maximum: finally it decreases again. The assumptions about the gun, with which the tests were made, are merely guesswork.

§ 46. Didion, p. 364; *Arch. f. Art.- u. Ing.-Off.* 93 (1886), p. 45; von Minarelli, p. 54; von Eberhard, *Das Wesen der mod. Visiervorrichtungen der Landartillerie*, published by Krupp, Berlin, 1908.

§ 47. The effect of the wind is considered by Didion, 1, p. 311; von Wuich, p. 474; Siacci, p. 113; Résal, 2, appendix, p. 409; Heydenreich, 1st ed. 1, p. 57; Sabudski, p. 302; Denecke, *Arch. f. Art.- u. Ing.-Off.* 93 (1886), p. 1 and 94 (1887), p. 226; von Minarelli, p. 57; Rohne, *Kriegstechn. Zeitschr.* 3 (1900), pp. 129 and 201: 4 (1901), p. 326. Rohne calculates that with a mean wind-velocity of 5·5 m/sec (Potsdam observations, 1893–1897), a horizontal breeze, coming obliquely forwards at an angle of 45°, shortens a range of 2000 m by 31 m: the lateral drift with this mean wind-velocity amounts only to about 18 m in a range of 2000 m. Rohne considers that the daily changes in the weather exercise less effect than that arising from errors of observation, even when wind and temperature act, as it were, in the same direction, i.e. wind from behind and temperature high, or wind from the front and temperature low. This is at any rate the case for rifles and for distances less than 1000 m. With guns, the effects are more noticeable. For a field gun with initial velocity of 465 m/sec, range 6000 m, $\phi = 18° 11'$, $\omega = 28° 30'$, Rohne finds that at $-22{\cdot}5°$ C. the range is short by 1038 m. Heydenreich's general conclusions rest on the very extensive series of tests, carried out by the German Artillery Committee. Krause's experimental results, which were published in the *Kriegstechn. Zeitschr.* 5 (1902), p. 433, show a very satisfactory agreement between calculated and observed values, in so far as temperature and barometric pressure are concerned. The value of C on p. 289 and of c on p. 292 only apply when the shell is at relatively low altitudes. At greater altitudes, it must be remembered that these factors are proportional to the density of the atmosphere. The expressions, representing their values, must therefore be multiplied by a factor which takes the variation of air-density into account.

§ 48. As for the effect of the rotation of the earth, see G. Galilei, *Dialog über* *das Weltsystem*, translated by Strauss, Leipsic, 1891, pp. 189–192; S. D. Poisson, *J. éc. polyt.* 15 (1832), p. 187; *Recherches sur le mouvement des projectiles dans l'air*, Paris, 1839, pp. 41 and 63; C. E. Page, *Nouv. ann.* 2, 6 (1867), pp. 96, 387, 481; St Robert, 1, p. 357; F. Astier, *Rev. d'art.* 5 (1875), p. 272; R. Berger, *Über den Einfluss der Erdrotation auf den freien Fall der Körper und die Flugbahnen der Projektile*, Coburg, 1876; J. Finger, *Wien. Ber.* 76^2 (1878), p. 67, and R. Hoppe, *Arch. d. Math.* 64 (1879), p. 96; W. Schell, *Theorie der Bewegung und der Kräfte*, 1, p. 528; A. W. F. Sprung, *Arch. d. deutsch. Seewarte*, 1879, p. 27: *Deutsche meteor. Zeitschr.* 1 (1884), p. 250: *Arch. f. Art.- u. Ing.-Off.* 103 (1896), p. 13; Résal, 1, p. 107; Ökinghaus, *Wochenschrift f. Astron.* 1891, p. 89, and *Arch. f. Art.- u. Ing.-Off.* 103 (1896), p. 89; Sabudski, *Petrog. Art. Journ.* 1894, 2, p. 120, and *Rev. d'art.* 44 (1894), p. 467; A. von Obermayer, *Mitt. üb. Geg. d. Art.- u. Gen.-Wes.* 1901, p. 707.

Budde, *Allgemein. Mechanik*, I. p. 317, remarks that if a shell is fired vertically upwards, it deflects to the north and not to the south until $t=\frac{4v_0}{g}$.

§ 49. As for the effect of the bayonet, see Fr. Hentsch, *Ballistik der Handfeuerwaffen*, Leipsic, 1873, where on p. 312 he says it was found with all rifles that the following fact was true. If the rifle is so adjusted that the bullet hits the bull's eye when no bayonet is attached, then if the rifle is fired with the bayonet on the right-hand side of the bore, the bullet will deviate considerably to the left, even with short ranges. H. Weygand says the same in *Das Schiessen mit Handfeuerwaffen, eine vereinfachte Schiesslehre, mit bes. Berücksichtigung des deutschen Inf.-Gew. M.* 71, Berlin, 1876. On p. 184 he says that experience has shown that the bayonet causes a regular deviation to the opposite side to that to which the bayonet is attached. See also von Stachorowski's *Unterricht in der Waffenlehre an den Kgl. Kriegsschulen*, 1876, p. 150; von Neumann, 1886, p. 121; 1890, p. 58; C. Cranz, *Civil-Ingenieur*, 21 (1885), vol. II.; F. Kötter, *Verhandl. d. Phys. Gesell. zu Berlin*, 7 (1888), p. 17; C. Cranz and K. R. Koch, *Münch. Akad. Ber.* 21 (1901), p. 572: *Jahresber. d. deutsch. Math. Vereinigung*, 6 (1899), p. 118; Minarelli-Fitzgerald, *Das moderne Schiesswesen*, Vienna, 1901, p. 55; F. Kötter, *Sitz.-Ber. d. Berl. Mathem. Gesell.* 2 (1903), p. 65; C. Cranz, ditto, 3 (1904), p. 11; J. C. F. Otto, *Hilfsmittel für ballist. Rechnungen*, Berlin, 1859, p. 266, who was probably the first to suggest that the vibration of the bore was the cause of the jump.

§§ 50–60. Drift through the rotation of the shell, with cannon balls, and the modern types of shell, together with vibratory movements of the projectile, are referred to in the following papers. Didion, pp. 304 and 318, and *J. éc. polyt.* 16 (1839), p. 51; G. Piobert, *Traité d'artillerie*, Paris, p. 169; Résal, 1, p. 375; G. Magnus, *Berl. Ber.* 1852, 1–24, and *Ann. Phys. Chem.* 88 (1853), p. 1; P. de St Robert, *Mém. scient.* Paris, 1872, I. p. 277; M. de Sparre, *Mouvement des projectiles oblongs dans le cas du tir de plein fouet*, Paris, 1875, and *Sur le mouvement des projectiles dans l'air*, Paris, 1891: *Archiv f. Math. Astron. u. Phys.* Stockholm, 1904, and *Ann. de la société de Bruxelles*, 35 (1911), p. 79; R. Timmerhans, *Essai d'un traité d'artillerie*, 2, Paris, 1846, p. 113; J. C. F. Otto, *Umdrehung der Artilleriegeschosse*, Berlin, 1843 and 1847: *Allgemein. Militärzeitung*, 1846, Nos. 64–65, and *Arch. f. Art.- u. Ing.-Off.* 6 (1840), p. 118; J. P. G. v. Heim, p. 169;

Neumann, *Arch. f. Art.- u. Ing.-Off.* 6 (1838), p. 213: 14 (1842), p. 49, and 17 (1845), p. 193; C. Mondo, *Derivation der Langgeschosse*, Munich, 1860; V. v. Vieth, *Flugbahn der Geschosse*, Dresden, 1861; C. H. Owen, *Woolwich Roy. Art. Inst. Proc.* 4 (1863), p. 180, and 23 (1869), p. 217; Brockhusen, *Arch. f. Art.- u. Ing.-Off.* 15 (1843), p. 93; Rutzki, p. 169; W. v. Rouvroy, *Theorie der Bewegung der Spitzgeschosse*, Berlin, 1862, and *Arch. f. Art.- u. Ing.-Off.* 18 (1845), p. 19; Darapsky, *Derivation der Spitzgeschosse*, Cassel, 1865; N. Mayevski, p. 178, and *Petrog. Art. Journ.* 1865, 3, p. 11, and *Rev. technol. milit.* 5, 1865, p. 1; P. Gauthier, *Mouvement d'un projectile dans l'air*, Paris, 1867; A. Paalzow, *Über die Drehung fester Körper, insbesondere der Geschosse und der Erde*, Berlin, 1867; Kummer, *Berlin. Akad. Abh.* 1875, p. 1 and 1876, p. 1; F. Astier, *Essai sur le mouvement des projectiles oblongs*, Paris, 1873; F. P. Jouffret, *Rev. d'art.* 4 (1874), pp. 245, 547; P. Haupt, *Mathematische Theorie der Flugbahnen gezogener Geschosse*, Berlin, 1876; J. Märker, *Über das ballistische Problem*, Gymn. Prog., Hersford, 1876; Muzeau, *Rev. d'art.* 12 (1878), pp. 422 and 495, continued till 14, p. 38; Ingalls, *Handbook of Problems in Exterior Ballistics*, New York, 1900: *Arch. f. Art.- u. Ing.-Off.* 85 (1879), p. 134, and 87 (1880), p. 180; K. B. Bender, *Bewegungserscheinungen d. Langgeschosse*, Darmstadt, 1888; Jansen, *Arch. f. Art.- u. Ing.-Off.* 97 (1890), p. 424; Sabudski, *Petrog. Art. Journ.* 1890, 7, p. 649, and 1891, 1, p. 1: *Äussere Ballistik*, 1895, pp. 323–393; A. Brix, *Marine Abh.* (Russian), 1891, No. 1, p. 25: No. 2, p. 61: No. 3, p. 41; Engelhardt, *Arch. f. Art.- u. Ing.-Off.* 100 (1893), pp. 403 and 449; P. G. Tait, *Nature*, 48 (1893), p. 202; H. Müller, *Entwicklung der Feldartillerie*, Berlin, 1894; E. Ökinghaus, *Arch. f. Art.- u. Ing.-Off.* 103 (1896), p. 185; J. Altmann, *Erklärung u. Berechnung d. Seitenabweichungen*, Vienna, 1897; A. v. Obermayer, *Organ der militärwissenschaftlichen Vereine*, Vienna, 1898, and *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1899, p. 869; A. G. Greenhill, *Woolwich Roy. Art. Inst. Proc.* 11 (1882), pp. 119 and 124; v. Minarelli, p. 43; Ludwig, *Studien über Ballistik*, Carlsruhe, 1853; P. G. Tait, *Trans. Roy. Soc. Edinb.* 37 (2), (1893), p. 427, and *Beiblätt. zu d. Annal. d. Phys. u. Chem.* 4 (1895), p. 288: *Proc. Roy. Soc.* 21 (1896), p. 116: *Beiblätt. zu d. Annal. d. Phys. u. Chem.* 21 (1897), p. 389; E. Röggla, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1912, 4, p. 317; Cranz, *Zeitschr. f. mathem. Phys.* 43 (1898), pp. 133 and 169: *Jahresber. d. deutschen Math. Ver.* 6 (1899), p. 110.

§ 50. Deflections of cannon balls. J. Didion and Saulcy, *Cours d'artillerie, partie théorique, rédigé après Piobert*, Paris, 1841; F. Otto, *Über die Umdrehung der Artill. Geschosse*, Berlin, 1843, p. 109, continued by Neisse, 1847; S. D. Poisson, *Recherches sur le mouvement des projectiles*, Paris, 1839, p. 69: *Über die Luftreibung*, p. 74; A. Winkelmann, *Handbuch der Physik*, Breslau, 1891, 1, p. 600. For tests with eccentric projectiles, see Heim, p. 169; Rouvroy, *Arch. f. Art.- u. Ing.-Off.* 18 (1845), p. 19; H. Müller, *Die Rotation der runden Artilleriegeschosse*, Berlin, 1862. Magnus's experiments, *Berlin. Akad. Abh.* 1852, and *Über die Abweichung der Geschosse*, Berlin, 1860. For golf balls and boomerangs, see G. T. Walker, *Encykl. d. math. Wissensch.* IV. 9: referred to under the heading "Spiel und Sport," p. 135. Lanchester's explanation will be found in his book on Aerodynamics.

§ 51. M. Hélie, *Traité de balistique*, 1884, II. p. 310. For observations of projectiles with the naked eye, see Heydenreich, 1, p. 7, and 2, pp. 95-98; Rutzki, *Theorie u. Praxis der Geschoss- und Zünderkonstruktionen*, Vienna, 1871; H. Müller, *Die Entwicklung der preuss. Festungs- und Belagerungsartillerie*, Berlin, 1876,

p. 162. For indirect observations, see Jansen, *Arch. f. Art.- u. Ing.-Off.* 97 (1890), pp. 425 and 497. A photographic method of registration in the shell itself is given by Neesen, *Arch. f. Art.- u. Ing.-Off.* 96 (1889), p. 68: 99 (1892), p. 476: and 101 (1894), p. 253. Observations at low velocities, C. Cranz, *Zeitschr. f. math. Phys.* 43 (1898), pp. 133 and 169.

§§ 52–57. For the theory of the gyroscope, see F. Klein and A. Sommerfeld, *Über die Theorie des Kreisels*, Leipsic, 1897–1910, specially vol. IV. section C, *Ballistics*, 8, p. 317 and P. Stäckel, *Encyklop. d. math. Wissensch.* IV. 6, which includes a bibliography; C. Cranz, *Zeitschr. math. Phys.* 43 (1898), pp. 133 and 169; H. Putz, *Rev. d'art.* 24 (1884), p. 293; K. B. Bender, *Bewegungserscheinungen der Langgeschosse*, Darmstadt, 1888; Jansen, *Arch. f. Art.- u. Ing.-Off.* 97 (1890), p. 424; H. Müller, *Die Entwickl. d. Preuss. Festungs- u. Belag. Artill.* Berlin, 1876, specially pp. 162 and 175. Gyroscopic experiments with models of shells, A. von Obermayer, *Mitt. üb. Geg. d. Art.- u. Gen.-Wes.* 1899, p. 869; K. F. Harris, *Journ. of Un. St. Artill.* 10 (1901), p. 63, and pp. 189 and 303; Magnus, *Pogg. Ann.* 88 (1853), p. 1; J. Altmann, *Erklärung u. Berechnung d. Seitenabweichung rotierender Geschosse*, Vienna, 1897; Krall, *Mitt. üb. Geg. d. Art.- u. Gen.-Wes.* 1888, p. 118; C. V. Boys, ditto, 1897, p. 836.

Photographic measurements of vibratory movements are described by F. Neesen, *Verhandl. d. deutsch. physikal. Gesell.* 11 (1909), 24, pp. 441 and 724.

§ 52. Layriz, *Zeitschr. f. d. ges. Schiess.- u. Sprengstoff.-Wes.* 10 (1915), p. 303; Jansen, *Arch. f. Art.- u. Ing.-Off.* 97 (1890), pp. 424 and 497, and *Mitt. üb. Geg. d. Art.- u. Gen.-Wes.* 1871, p. 85; A. Rutzki, p. 62 and *Grundlagen für neue Geschoss.- u. Waffensysteme*, Teschen, 1876.

§ 53. A. Dahne, *Neue Theorie der Flugbahnen von Langgeschossen*, Berlin, 1888: *Bausteine zur Flugbahn- u. Kreiseltheorie*, Berlin, 1914, by the same author, as also *Kriegstechn. Zeitschr.* 10 (1907), pp. 65 and 265, and 12 (1909), p. 58.

§ 55. A. v. Rutzki, *Bewegung u. Abweichung der Spitzgeschosse*, Vienna, 1861, and *Theorie u. Praxis der Zünderkonstruktionen*, Vienna, 1871; Jansen, *Arch. f. Art.- u. Ing.-Off.* 97 (1890), pp. 425 and 497; N. Sabudski, *Untersuchungen üb. d. Bewegungen des Langgeschosses*, Petrograd, 1908; T. Terada and M. Okochi, *Artill. Monatshefte*, 1909, p. 301; A. Dittli, *Artill. Monatshefte*, 1916, 116, p. 49.

§ 56. Hélie, 2, pp. 94 and 309; E. Vallier, *Bal. exp.* Paris, 1894, pp. 40 and 178; G. v. Gleich, *Zeitschr. f. Math. u. Phys.* 55 (1907), p. 363; F. H. Lanchester, *Aerodynamics*, and E. Bravetta, *Zeitschr. f. d. ges. Schiess.- u. Sprengstoff.-Wes.* 6, 1911, pp. 81 and 107: ditto, 1914, p. 291; A. Hamilton, *Ballistics*, Fort Monroe, 1908, I. p. 155; P. Haupt, *Mathem. Theorie der Flugbahnen*, Berlin, 1876, p. 101; Charbonnier, *Traité de bal. ext.* Paris, 1894, p. 238.

For lateral deviations, see E. Thiel, *Das Inf. Gew.* Bonn, 1883, p. 20; H. Rohne, *Schiesslehre für Infanterie*, Berlin, 1906, p. 182. Krause says it amounts to 1 metre in a range of 1000 m with the rifle M. 88, *Militär. Wochenblatt*, 1904, 113, p. 2737; Wille, *Waffenlehre*, Berlin, 1908, IV. p. 231.

§ 57. Demonstration apparatus, M. J. Perrodon, *Sur un appareil destiné...*, Paris, 1875. Pfaundler's apparatus is described in Klimpert's *Dynamik*, Stuttgart, 1889; Majneri-Kempen, *Artill. Monatshefte*, 1913, 76, p. 299.

§§ 58–70. For the theory of probabilities, see E. Czuber, *Wahrsch.-Rechnung u. ihre Anwendung auf Fehlerausgleichung*, Leipsic, 1903: ditto, *Theorie d. Beobachtungsfehler*, Leipsic, 1891: *Jahresber. d. deutsch. Math. Verein.* 7 (1899), 2, pp. 1–279, with bibliography; N. Sabudski, *Die Wahrscheinlichkeitsrechnung*, translated by Eberhard, Stuttgart, 1906; J. Kozák, *Grundprobleme der Ausgleichsrechnung*, Leipsic, 1907–08. For the characteristic errors, see S. Wellisch, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1908, pp. 889 and 975; J. Kozák, ditto, 1910, p. 47.

§ 60. S. D. Poisson, *Mém. de l'art. de la marine*, 8 (1830), p. 141; J. Didion, *Calcul des probabilités appliqué au tir des projectiles*, Paris, 1858, and *J. écol. polyt.* 16, cah. 27 (1839), p. 51; Hélie, 2, p. 95; v. Wuich, p. 481; Eschler, *Vorträge a. d. Artill.-Lehre*, Vienna, 1898; Fischer, *Kriegstechn. Zeitschr.* 1909, pp. 164 and 209; B. Schöffler, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1901, p. 823, and 1902, pp. 97 and 366; J. U. van Loon, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1914, pp. 249, 735, 875.

Cranz, *Komp. d. Ball.*, Leipsic, 1896, p. 297; W. Dyck, *Katalog mathem. u. math. phys. Modelle*, Munich, 1892, p. 154; A. v. Obermayer, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1899, p. 130, and 1900, 2, notes.

§ 62. For the method of successive differences, see E. Czuber, *Jahresber. d. deutsch. Math. Ver.* 7 (1899), 2, p. 205; E. Vallier, p. 166; Sabudski's work on probabilities refers to Eberhard's proposals; W. Heydenreich, *Zeitschr. f. d. ges. Schiess.- u. Spreng.- Wes.* 1 (1906), p. 272.

§ 65. Vallier, p. 160 and *Rev. d'art.* 9 (1877), p. 222; E. Czuber, *Jahresber. d. deutsch. Math. Verein.* 7 (1899), 2, p. 212; W. Heydenreich, *Kriegstechn. Zeitschr.* 6 (1903), p. 253; A. Mazzuoli, *Rivista maritima*, 1908, January; H. Rohne, *Artill. Monatshefte*, 1907, 9, p. 235, and 32, p. 129; J. Kozák, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1910, p. 47.

§ 66. F. Siacci, *Rev. d'art.* 22 (1883), p. 521 and ditto, 24 (1884), p. 445; H. Putz, ditto, 24 (1884), pp. 5 and 105: 32 (1888), pp. 213 and 313.

§ 67. Krause, *Die Gestaltung der Geschossgarbe*, Berlin, 1904; von Zedlitz, *Kriegstechn. Zeitschr.* 6 (1903), p. 129; H. Rohne, *Schiesslehre f. Infanterie*, Berlin, 1906; v. Minarelli, pp. 65 and 82; K. Endres, *Arch. f. Art.- u. Ing.-Off.* 90 (1883), p. 113; A. Percin, *Rev. d'art.* 20, 1882, p. 5; Giletta, *Riv. d' art. e gen.* 1884, p. 218; Parst, *Kriegstechn. Zeitschr.* 4 (1901), p. 330, and 7 (1902), p. 235; H. Rohne, ditto, 4 (1901), p. 119; *Artillerist. Monatshefte*, 1907, pp. 232, 257, 397; *Schiesslehre für Infanterie*, Berlin, 1906, specially p. 139.

The theory of finding the range for artillery is described by Rohne, *Arch. f. Art.- u. Ing.-Off.* 100 (1894), pp. 385 and 481, and 102 (1895), pp. 64 and 257, and specially 104 (1897), p. 172: *Kriegstechn. Zeitschr.* 1 (1898), pp. 209 and 399, and 2 (1899), p. 115; Callenberg, *Über die Grundlagen des Schrapnellschiessens bei der Feldartillerie*, Berlin, 1898, and *Kriegstechn. Zeitschr.* 2 (1899), pp. 27 and 93; Preiss, *Kriegstechn. Zeitschr.* 3 (1900), p. 81; E. Strnad, *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1892, p. 879: 1887, p. 375; A. Weigner, ditto, 1898, p. 821; Schöffler, ditto, 1902, p. 97: 1900, p. 429 and 1901, p. 823; N. Sabudski, *Wahrsch.-Rechn.* Stuttgart, 1906; J. Kozák, *Theorie der Schiesswesens*, vol. II. part 2, Vienna, 1900; E. Eschler, *Vorträge aus der Artillerielehre*, Vienna, 1898.

Vallier, *Rev. d'art.* 30 (1887), p. 106; V. Gandolfi, *Riv. d' art. e gen.* 1896, vol. IV. p. 231, and *Mitt. üb. Geg. d. Art.- u. Gen.- Wes.* 1897, p. 645; E. Strnad, ditto, 1897, p. 763; Indra, ditto, 1897, pp. 163 and 291; A. Ludwig, ditto, 1901, pp. 91 and 189; A. Calichiopulo, *Riv. d' art. e gen.* 1893, vol. I. pp. 245 and 411; Dragas, *Streffleur's österr. milit. Zeitschr.* Vienna, 1890, p. 184.

§ 69. R. Rothe, *Artill. Monatshefte*, 1916, 110, p. 65, and 111, p. 125; G. Scheffers, *Berlin. Akad. Ber., Phys.-math. Kl.* 42 (1915), p. 733.

§ 70. Kohlrausch, *Prakt. Phys.* Leipsic, 1901, p. 17; E. Vallier, *Bal. expér.* Paris, 1894, pp. 138 and 151.

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R. Robins, *Nouveaux principes d'artillerie* (French translation by Lombard), Paris, 1873, p. 365; Poncelet, *Introduction à la mécanique industrielle*, Brussels, 1839, p. 619; Résal, Paris, *C. R.* 120 (1895), p. 397; T. Levi-Civita, *Atti del reale istituto Veneto di Scienze*, 65 (1905), II. p. 1149.

§ 72. v. Minarelli, p. 143, and Wille, *Waffenlehre*, Berlin, 1900, p. 173; Wernicke, *Zeitschr. f. d. ges. Schiess.- u. Sprengst.-Wes.* 1910, p. 201.

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§ 75. E. de Jonquières states that balls of a calibre of 0·16 m, having a velocity of 455 m/sec, ricochetted on an average 22 times on the surface of the water: range = 2470 m; Persy, *Cours de balistique*, Metz, 1827, p. 61; A. Preuss, *Schuss u. Waffe*, 3 (1909–10), 10, p. 217, and 4 (1910–11), 11, p. 213; C. Ramsauer, *Über den Ricochetschuss*, Kiel dissertation, 1903.

GENERAL INDEX

INDEX OF NAMES

PHOTOGRAPHIC RECORDS

THE INSTANTANEOUS PHOTOGRAPHIC RECORDS

Record No. 1.

This shows a bullet of a calibre of 8 mm, together with the surrounding waves and eddies: $v = 880$ m/sec. This was taken with a concave mirror, and a vertical stop, which was placed at right angles to the path of the bullet. The method is described in vol. III. chap. VIII. and by B. Glatzel in *Die elektrischen Methoden der Momentphotographie*, Brunswick, 1915, published by Vieweg. This method shows the variations of air-density, in directions at right angles to the stop. The photograph can be regarded as a model, illuminated in a direction at right angles to the stop, and a model of this kind was made in the author's laboratory in plaster of Paris.

Record No. 2.

As in No. 1, but the edge of the stop is parallel to the line of fire, corresponding to an illumination of the model in a direction at right angles to the path of the bullet.

Record No. 3.

Bullet of 8 mm calibre: $v = 880$ m/sec. This was taken by the shadow process.

Record No. 4.

As in No. 1, but the bullet was fired with the base forward. The line of flight is slightly oblique to the position of the bullet.

Record No. 5.

Cylindrical projectile, flying through calm air: $v = 880$ m/sec. This was taken with concave mirror, objective and stop: the position of the stop corresponds to an illumination in a direction parallel to the line of fire.

Record No. 6.

Cylindrical projectile with rounded tip: $v = 640$ m/sec. The angle of the head-wave is increased accordingly.

Record No. 7.

This shows the beginning of the wave with a pointed bullet of 8 mm calibre: $v = 880$ m/sec. The gases, which escaped from the muzzle, had initially had a greater velocity than the bullet, but at the moment the exposure was made, they had been overtaken by the bullet. The bounding line of the escaping gases is here seen to be somewhat irregular. The head-wave begins to form just outside the explosive gases, because, relatively to these latter gases, the bullet has a velocity less than that of sound. This was taken by the shadow process.

Record No. 8.

Pointed bullet, 8 mm calibre: $v = 340$ m/sec. This is the velocity of sound. As the velocity of the bullet decreases, the points of the head-wave and tail-wave move with regard to the bullet forwards or backwards. The angle of the waves increases at the same time. And when the velocity of the bullet has fallen below that of sound, the waves become separated from the bullet. The head-wave becomes a kind of spherical condensation.

Record No. 9.

Bullet, 8 mm calibre: $v = 880$ m/sec. The bullet has grazed a wooden board, and is vibrating noticeably. Splinters of wood are seen.

Record No. 10.

This proves that the head-wave and the tail-wave are the envelopes of infinitesimally small head-waves, which are produced by the impact of the bullet on the surrounding air. In the records 1–9, these elementary waves are scarcely visible, though they can sometimes be seen within the tail-wave.

The waves can be separated, as in the present record, by firing through a horizontal tube, which has holes at top and bottom: $v = 880$ m/sec. The bullet is seen to be emerging from the tube. Partial waves escape through the holes in the tube and their method of combining to form an envelope can be seen. Taken by the shadow process.

Record No. 11.

As in No. 10, taken by concave mirror, objective and stop. The bullet has already left the field of view.

Record No. 12.

Shows the beginning of the head-wave: $v = 880$ m/sec. By means of a sound-damping device, attached to the muzzle, the head-wave starts as soon as the bullet escapes from the muzzle. Taken by the shadow process.

Record No. 13.

This shows the reflection of the waves from two parallel plates, between which the bullet passes: $v = 880$ m/sec. Taken by the shadow process.

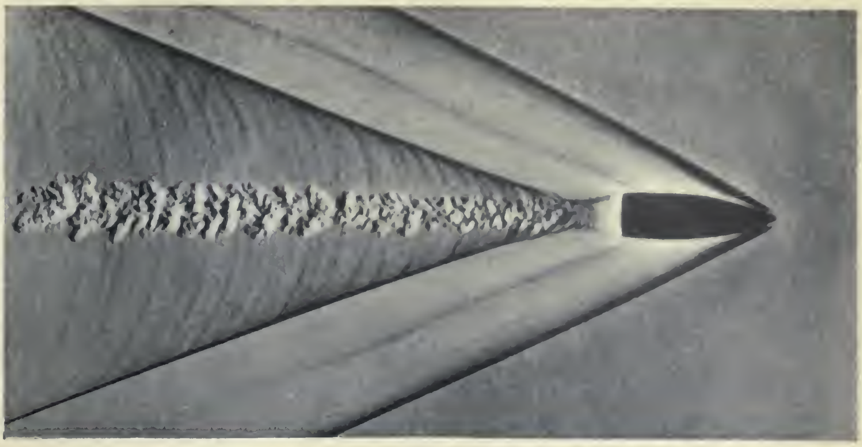
Record No. 14.

As in No. 12, but taken at a later instant. The bullet has already escaped at the right. There seems to be trace of periodicity in the eddies: these eddies would be helical, on account of the rotation of the bullet. Probably here it is a case of variations of air-density, due mainly to temperature.

Record No. 15.

A bullet fired through two candle-flames: $v = 880$ m/sec. Both flames were extinguished. The record shows (*a*) the inertia of the burning gases, seeing that the flame through which the bullet has already passed appears to be burning quietly, and the ascending column of hot gases is not disturbed. (*b*) No kind of perceptible pressure seems to precede the bullet, even when its velocity exceeds that of sound. (*c*) The waves, shot out from the flame, appear to be separate waves, just as though the bullet had passed through a solid body. These waves appear to be transplanted faster in the hot gases of the flame than in the cool surrounding air.

All these records were taken in the author's laboratory.



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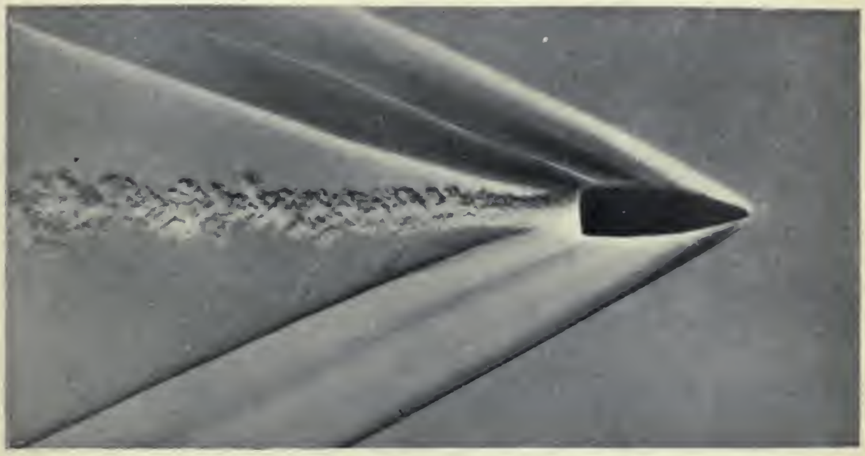

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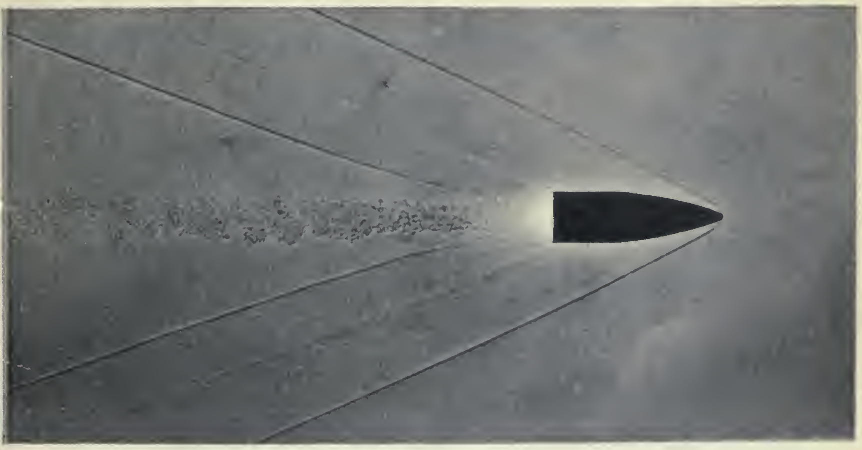

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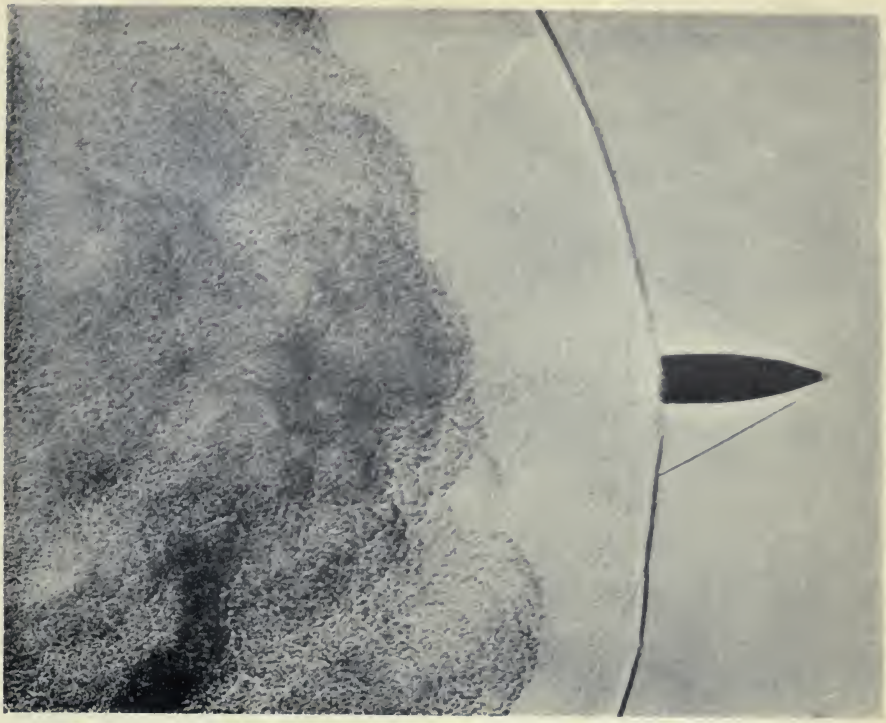

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PLATE IV

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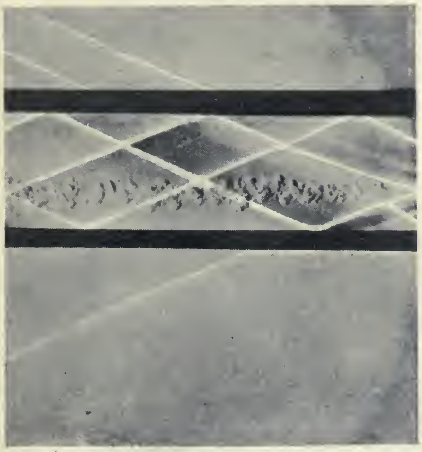

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